THE ERINDALE ACADEMY
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Course: MDM4U Teacher: Ms. Ella

Unit 1: Permutation

Lesson 2: Permutation and factorials

Part I: Factorial notation

As you can see from Lesson 1, many counting and probability calculations involve the product of a series of consecutive integers. You can use **factorial notation** to write such expressions more easily such as the following definition.

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$$

This expression is read as *n factorial*.

Example 1: Evaluate the following terms.

a) *3! + 2! + 4!*

b) $\frac{10\,!}{4\,!}$

c) $\frac{78!}{75!}$

Example 2: use factorial to do counting possibilities.

The senior choir has rehearsed five songs for an upcoming assembly. In how many different orders can the choir perform the songs?

Part II: Permutation Formula

Permutation: all possible arrangements of a collection of things where the order is important.

A permutation of n distinct items is an arrangement of all the items in a definite order. The total number of such permutations is denoted by $_nP_n$ *or* $P\ (n, n)$.

There are n possible ways of choosing the first item, n – 1 ways of choosing the second, n – 2 ways of choosing the third, and so on. Applying the fundamental counting principle:

$$_nP_n = n\ \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$$

Example 3: In how many ways could a president and a vice-president be chosen from a group of eight nominees?

A permutation of n distinct items taken r at a time is an arrangement of r of the n items in a definite order. Such permutations are sometimes called r-arrangements of n items. The total number of possible arrangements of r items out of a set of n is denoted by ${}_nP_r$ *or* $P\,(n, r)$.

There are n ways of choosing the first item, n – 1 ways of choosing the second item, and so on down to n – r + 1 ways of choosing the r^{th} item. Using the fundamental counting principle:

$$\boldsymbol{{}_nP_r = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times (n-r+1) = \frac{n!}{(n-r)!}}$$

Example 4: In a card game, each player is dealt a face down "reserve" of 13 cards that can be turned up and used one by one during the game. How many different sequences of reserve cards could a player have?

Example 5: Use permutation notation to represent following factorial and numeral expressions.

a) $\frac{5!}{2!}$

b) $\frac{10!}{5!}$

c) $6 \times 5 \times 4$

d) $101 \times 100 \times 99 \times 98 \times 97 \times 96$

Example 6: Simplify each of the following in factorial form.

a) (n+4) (n+5) (n+3)!

b) 72 x 7!

c) $\dfrac{(n+2)!}{(n-1)!}$

Practice:

Simplify the expression

$\frac{(2n+1)!}{(2n-1)!}$ $\frac{(n-3)!}{(n-4)!}$ $\frac{(n+2)!}{n+2}$

Solve for n in the following equation:

$\frac{(n+2)!}{(n+1)!}=8$ $\frac{(n+1)!}{(n-1)!}=10n$ *(n+3)! =20(n+1)!*

${}_{n}P_{2} = 56$ ${}_{2n+2}P_{1} = \frac{1}{2}\,{}_{2n}P_{2}$

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Practice from Textbook:

Practise

A

1. Express in factorial notation.

a) $6 \times 5 \times 4 \times 3 \times 2 \times 1$

b) $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

c) $3 \times 2 \times 1$

d) $9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

2. Evaluate.

a) $\frac{7!}{4!}$ **b)** $\frac{11!}{9!}$

c) $\frac{8!}{5!\,2!}$ **d)** $\frac{15!}{3!\,8!}$

e) $\frac{85!}{82!}$ **f)** $\frac{14!}{4!\,5!}$

3. Express in the form ${}_nP_r$.

a) $6 \times 5 \times 4$

b) $9 \times 8 \times 7 \times 6$

c) $20 \times 19 \times 18 \times 17$

d) $101 \times 100 \times 99 \times 98 \times 97$

e) $76 \times 75 \times 74 \times 73 \times 72 \times 71 \times 70$

4. Evaluate without using technology.

a) $P(10, 4)$ **b)** $P(16, 4)$ **c)** ${}_5P_2$

d) ${}_9P_4$ **e)** $7!$

5. Use either a spreadsheet or a graphing or scientific calculator to verify your answers to question 4.

Apply, Solve, Communicate

6. a) How many ways can you arrange the letters in the word *factor*?

b) How many ways can Ismail arrange four different textbooks on the shelf in his locker?

c) How many ways can Laura colour 4 adjacent regions on a map if she has a set of 12 coloured pencils?

B

7. Simplify each of the following in factorial form. Do not evaluate.

a) $12 \times 11 \times 10 \times 9!$

b) $72 \times 7!$

c) $(n + 4)(n + 5)(n + 3)!$

8. **Communication** Explain how a factorial is an iterative process.

9. Seven children are to line up for a photograph.

a) How many different arrangements are possible?

b) How many arrangements are possible if Brenda is in the middle?

c) How many arrangements are possible if Ahmed is on the far left and Yen is on the far right?

d) How many arrangements are possible if Hanh and Brian must be together?

10. A 12-volume encyclopedia is to be placed on a shelf. How many incorrect arrangements are there?

11. In how many ways can the 12 members of a volleyball team line up, if the captain and assistant captain must remain together?

12. Ten people are to be seated at a rectangular table for dinner. Tanya will sit at the head of the table. Henry must not sit beside either Wilson or Nancy. In how many ways can the people be seated for dinner?

13. **Application** Joanne prefers classical and pop music. If her friend Charlene has five classical CDs, four country and western CDs, and seven pop CDs, in how many orders can Joanne and Charlene play the CDs Joanne likes?

14. In how many ways can the valedictorian, class poet, and presenter of the class gift be chosen from a class of 20 students?

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15. **Application** If you have a standard deck of 52 cards, in how many different ways can you deal out

a) 5 cards? **b)** 10 cards?

c) 5 red cards? **d)** 4 queens?

16. **Inquiry/Problem Solving** Suppose you are designing a coding system for data relayed by a satellite. To make transmissions errors easier to detect, each code must have no repeated digits.

a) If you need 60 000 different codes, how many digits long should each code be?

b) How many ten-digit codes can you create if the first three digits must be 1, 3, or 6?

17. Arnold Schoenberg (1874–1951) pioneered serialism, a technique for composing music based on a tone row, a sequence in which each of the 12 tones in an octave is played only once. How many tone rows are possible?

18. Consider the students' council described on page 223 at the beginning of this chapter.

Chapter Problem

a) In how many ways can the secretary, treasurer, social convenor, and fundraising chair be elected if all ten nominees are eligible for any of these positions?

b) In how many ways can the council be chosen if the president and vice-president must be grade 12 students and the grade representatives must represent their current grade level?

19. **Inquiry/Problem Solving** A student has volunteered to photograph the school's championship basketball team for the yearbook. In order to get the perfect picture, the student plans to photograph the ten players and their coach lined up in every possible order. Determine whether this plan is practical.

ACHIEVEMENT CHECK

Knowledge/Understanding	Thinking/Inquiry/Problem Solving	Communication	Application

20. Wayne has a briefcase with a three-digit combination lock. He can set the combination himself, and his favourite digits are 3, 4, 5, 6, and 7. Each digit can be used at most once.

a) How many permutations of three of these five digits are there?

b) If you think of each permutation as a three-digit number, how many of these numbers would be odd numbers?

c) How many of the three-digit numbers are even numbers and begin with a 4?

d) How many of the three-digit numbers are even numbers and do *not* begin with a 4?

e) Is there a connection among the four answers above? If so, state what it is and why it occurs.

C

21. TI-83 series calculators use the definition $\left(-\frac{1}{2}\right)! = \sqrt{\pi}$. Research the origin of this definition and explain why it is useful for mathematical calculations.

22. **Communication** How many different ways can six people be seated at a round table? Explain your reasoning.

23. What is the highest power of 2 that divides evenly into 100! ?

24. A committee of three teachers are to select the winner from among ten students nominated for special award. The teachers each make a list of their top three choices in order. The lists have only one name in common, and that name has a different rank on each list. In how many ways could the teachers have made their lists?

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Part 3: Permutation with identical items, always together, and always separated

Permutation with Identical items (Repetitions are not allowed): In many cases, some of the items we want to arrange are identical. For example, in the word TOOTH, if we exchange the places of the two O's, we still get TOOTH. Because of this we have to get rid of extraneous cases by dividing out repetitions.

Example 1: How many ways can you arrange the letters in the word THESE?

How many ways can you arrange the letters in the word REFERENCE?

How many ways can the letters in the word MISSISSIPPI be arranged?

Conclusion: the number of permutations of a set of *n* items containing *a* identical objects of one kind, *b* identical objects of a second kind, *c* identical objects of a third kind, and so on is $\dfrac{n!}{a!\mathrm{b}!c!\ldots\ldots}$

Practice:

1) If a multiple choice test has 10 questions, of which 1 is answered A, 4 are answered B, 3 are answered C, and 2 are answered D. How many answer sheets are possible?

2) Tanisha is laying out tiles for the edge of a mosaic. How many patterns can she make if she uses four yellow tiles and one each of blue, green, red, and grey tiles?

3) Barbara is hanging a display of clothing imprinted with the school's crest on a line on a wall in the cafeteria. She has five sweaters, three T-shirt, and four pairs of sweatpants. In how many ways can Barbara arrange the display?

Permutation with items always together: Frequently, certain items must always be kept together. To do these questions, you must treat the joined items as if they were only one object.

Example 2: How many ways can 3 math books, 5 chemistry books, and 7 physics books be arranged on a shelf if the books of each subject must be kept together?

Example 3: How many ways can the letters in OBTUSE be ordered if all the vowels must be kept together?

Permutation with items never together: If certain items must be kept apart, you will need to figure out how many possible positions the separate item can occupy.

Example 4: How many arrangements of the word ACTIVE are there if C&E must never be together?

Example 5: if 8 boys and 2 girls must stand in line for a picture, how many line-up's will have the girls separated from each other? (Use both direct and in-direct method)

Practice:

1) There are 9 switches on a fuse box. How many different arrangements are there?

2) How many three letter words can be formed, if repetitions are allowed? (How about the repetitions are not allowed? How about a word with 5 letters?)

3) How many three digit numbers can be formed? (Zero can't be the first digit, and consider both conditions with and without repetitions)

4) How many ways can three cars (red, green, blue) be parked in five parking stalls?

5) An electrical panel has five switches. How many ways can the switches be positioned up or down if three switches must be up and two must be down?

6) A phone number in British Columbia consist of one of four area codes (236,250, 604, and 778), followed by a 7-digit number that cannot begin with 0 or 1. How many unique phone numbers are there?

8) How many ways can 4 rock, 5 pop, and 6 classical albums be ordered if all albums of the same genre must be kept together?

9) Eight cars (3 red, 3 blue, and 2 yellow) are to be parked in a line. How many unique lines can be formed if the yellow cars must not be together? Assume that cars of each color are identical.

10) Six vehicles (3 different brands of cars and 3 different brands of trucks) are going to be parked in a line. How many unique lines can be formed if the row starts with at least two trucks?

Practise

A

1. Identify the indistinguishable items in each situation.

a) The letters of the word *mathematics* are arranged.

b) Dina has six notebooks, two green and four white.

c) The cafeteria prepares 50 chicken sandwiches, 100 hamburgers, and 70 plates of French fries.

d) Thomas and Richard, identical twins, are sitting with Marianna and Megan.

2. How many permutations are there of all the letters in each name?

a) Inverary **b)** Beamsville

c) Mattawa **d)** Penetanguishene

3. How many different five-digit numbers can be formed using three 2s and two 5s?

4. How many different six-digit numbers are possible using the following numbers?

a) 1, 2, 3, 4, 5, 6 **b)** 1, 1, 1, 2, 3, 4

c) 1, 3, 3, 4, 4, 5 **d)** 6, 6, 6, 6, 7, 8

Apply, Solve, Communicate

B

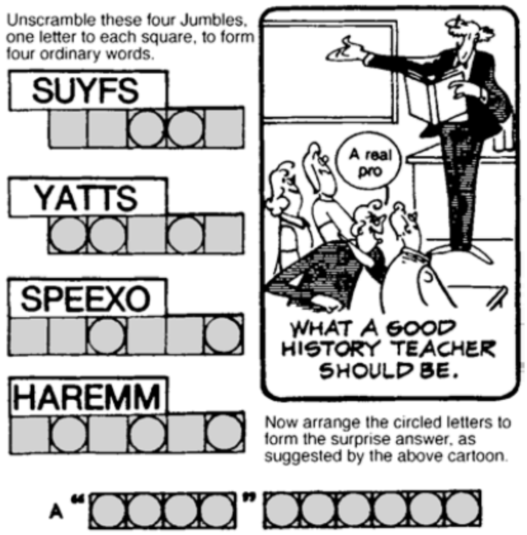
5. **Communication** A coin is tossed eight times. In how many different orders could five heads and three tails occur? Explain your reasoning.

6. **Inquiry/Problem Solving** How many 7-digit even numbers less than 3 000 000 can be formed using all the digits 1, 2, 2, 3, 5, 5, 6?

7. Kathryn's soccer team played a good season, finishing with 16 wins, 3 losses, and 1 tie. In how many orders could these results have happened? Explain your reasoning.

8. **a)** Calculate the number of permutations for each of the jumbled words in this puzzle.

b) Estimate how long it would take to solve this puzzle by systematically writing out the permutations.



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WEB CONNECTION

www.mcgrawhill.ca/links/MDM12

For more word jumbles and other puzzles, visit the above web site and follow the links. Find or generate two puzzles for a classmate to solve.

9. **Application** Roberta is a pilot for a small airline. If she flies to Sudbury three times, Timmins twice, and Thunder Bay five times before returning home, how many different itineraries could she follow? Explain your reasoning.

10. After their training run, six members of a track team split a bag of assorted doughnuts. How many ways can the team share the doughnuts if the bag contains

a) six different doughnuts?

b) three each of two varieties?

c) two each of three varieties?

11. As a project for the photography class, Haseeb wants to create a linear collage of photos of his friends. He creates a template with 20 spaces in a row. If Haseeb has 5 identical photos of each of 4 friends, in how many ways can he make his collage?

12. **Communication** A used car lot has four green flags, three red flags, and two blue flags in a bin. In how many ways can the owner arrange these flags on a wire stretched across the lot? Explain your reasoning.

13. **Application** Malik wants to skateboard over to visit his friend Gord who lives six blocks away. Gord's house is two blocks west and four blocks north of Malik's house. Each time Malik goes over, he likes to take a different route. How many different routes are there for Malik if he only travels west or north?

ACHIEVEMENT CHECK

Knowledge/Understanding	Thinking/Inquiry/Problem Solving	Communication	Application

14. Fran is working on a word puzzle and is looking for four-letter "scrambles" from the clue word *calculate*.

a) How many of the possible four-letter scrambles contain four different letters?

b) How many contain two *a*s and one other pair of identical letters?

c) How many scrambles consist of any two pairs of identical letters?

d) What possibilities have you not yet taken into account? Find the number of scrambles for each of these cases.

e) What is the total number of four-letter scrambles taking all cases into account?

15. Chapter Problem Ten students have been nominated for the positions of secretary, treasurer, social convenor, and fundraising chair. In how many ways can these positions be filled if the Norman twins are running and plan to switch positions on occasion for fun since no one can tell them apart?

16. **Inquiry/Problem Solving** In how many ways can all the letters of the word *CANADA* be arranged if the consonants must always be in the order in which they occur in the word itself?

C

17. Glen works part time stocking shelves in a grocery store. The manager asks him to make a pyramid display using 72 cans of corn, 36 cans of peas, and 57 cans of carrots. Assume all the cans are the same size and shape. On his break, Glen tries to work out how many different ways he could arrange the cans into a pyramid shape with a triangular base.

a) Write a formula for the number of different ways Glen could stack the cans in the pyramid.

b) Estimate how long it will take Glen to calculate this number of permutations by hand.

c) Use computer software or a calculator to complete the calculation.

18. How many different ways are there of arranging seven green and eight brown bottles in a row, so that exactly one pair of green bottles is side-by-side?

19. In how many ways could a class of 18 students divide into groups of 3 students each?