Unit 3 – Rational Functions
Chapter 5.2 – 5.3: Quotients of Polynomial functions

Rational functions can be expressed as $f(x) = \frac{p(x)}{q(x)}$, where *p(x)* and *q(x)* are polynomial functions.

With the function *q(x)* in the denominator, we need to consider any **discontinuities** where *q(x) = 0*.

- A **hole** will occur at $x = a$ if both *p(x)* and *q(x)* have a common factor of *(x - a).*

  For example: $y = \frac{(x-1)}{(x-1)(x+2)}$ has a hole when $x = 1$

- A **vertical asymptote** (V.A.) will occur at *x = a* when $q(x) = 0$

  For example: $y = \frac{(x+1)(x-2)}{(x-1)(x+2)}$ has vertical asymptote at $x = 1$ *and* $x = -2$

There are also **horizontal asymptotes**, they tells the end behavior of the rational functions.

The function $f(x) = \frac{p(x)}{q(x)}$ has a horizontal asymptote (H.A.) if order of $p(x) \leq order\ of\ q(x)$:

- x – axis (or y = 0) is the equation of H.A if order of $p(x) < order\ of\ q(x)$

  For example: $y = \frac{1}{(x+2)}$ *and* $y = \frac{(x-1)}{(x+1)(x+2)}$ both have H.A at $y = 0$

- The ratio of leading coefficient is the equation of H.A if order of $p(x) = order\ of\ q(x)$

  For example: $y = \frac{2(x-1)(x+3)}{(x+1)(x+2)}$ has H.A at $y = 2$

Other than vertical and horizontal asymptote, **oblique/slant asymptote** also tells how does rational functions look like.

- Oblique asymptote only happens when order of $p(x) - 1 = order\ of\ q(x)$, the quotient of long division is the equation of O.A.

  (i.e., order of p(x) is greater than the order of q(x) by exactly 1)

  For example: $y = \frac{2(x-1)(x+3)}{(x+1)}$, O.A is: ________________________

Practice:

Analyze each function and predict the location of any VERTICAL asymptotes, HORIZONTAL asymptotes, HOLES (points of discontinuity), *x*- and *y*-INTERCEPTS, DOMAIN, and RANGE.

Characteristic	$y = \frac{2x-1}{x-7}$	$y = \frac{x^2+5x}{x^2+7x+10}$	$y = \frac{x^2-7x+12}{x^2-9}$	$y = \frac{2x^2+5x-3}{x+3}$
Vertical Asymptote(s) *Analyze Denominator*				
Horizontal Asymptote(s) *Analyze Degrees of Polynomial (num/den) (m<n, m=n, m>n)*				
HOLES Point(s) of Discontinuity *Simplify the Rational Function by factoring*				
x-intercept(s) *Set y=0*				
y-intercept *Set x=0*				
Domain				
Range				

Example 1:

Let $f(x) = \frac{3x - 1}{x^2 - 2x - 3}$, $g(x) = \frac{x^3 + 8}{x^2 + 9}$, $h(x) = \frac{x^3 - 3x}{x + 1}$, and $m(x) = \frac{x^2 + x - 12}{x^2 - 4}$.

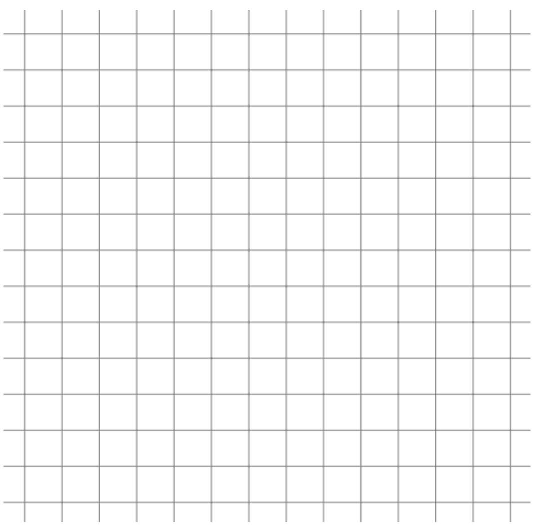
a) Which of these rational functions has a horizontal asymptote?

b) Which has an oblique asymptote?

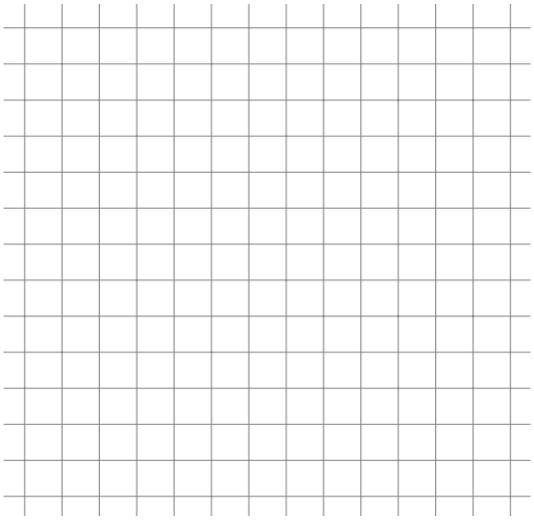
c) Which has no vertical asymptote?

d) Graph $y = m(x)$, showing the asymptotes and intercepts.

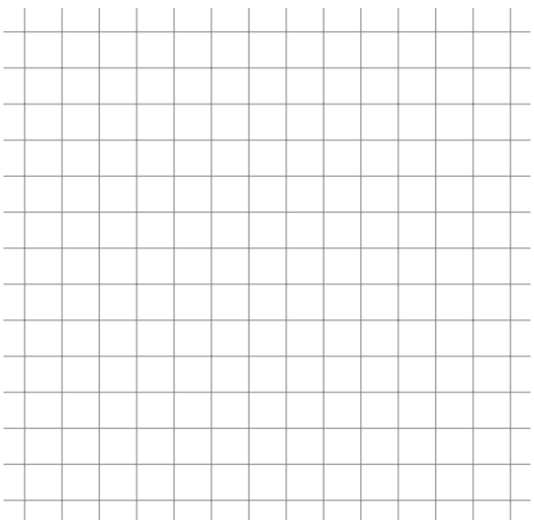
Example 2: Sketch $y = \frac{2x}{x+1}$.



Example 3: Sketch $y = \frac{x-3}{2x-6}$.



Example 4: Sketch $y = \frac{3x\ -1}{x^2-2x-3}$.



Example 5: Sketch $y = \frac{x^2+4}{x+1}$.

Steps to determine oblique/slant asymptote:
1) perform long division,
2) regardless of the remainder, take quotient as the equation of oblique asymptote.

Practice:

$$y = -\frac{4}{x^2 - 3x}$$

$$y = -\frac{x - 4}{-4x - 16}$$

$$y = -\frac{3x^2 - 12x}{x^2 - 2x - 3}$$

$$y = -\frac{x^3 - 9x}{3x^2 - 6x - 9}$$

Practice: Write equations and matching

Write a rational function with the given characteristics.

1. There are no zeros, a hole exists at $x = -3/2$, vertical asymptote is at $x = 1$, and horizontal asymptote is at $y = 0$.
2. The zeros are at -1 and 3 and the vertical asymptote is at $x = 0$.
3. The zero is at 2, vertical asymptote is at $x = -2$ and $x = 0$, and horizontal asymptote is at $y = 0$.

Match the following graphs with the equation

$f(x) = \dfrac{x^3 - 9x}{3x^2 - 6x - 9}$

factor ________________

HA: ________________

VA: ________________

roots: ________________

holes: ________________

graph: A B C

10. $f(x) = \dfrac{x^2 - x - 12}{x^2 - 2x - 8}$

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graph: A B C

11. $f(x) = \dfrac{x^3 + 1}{x^2 - 1}$

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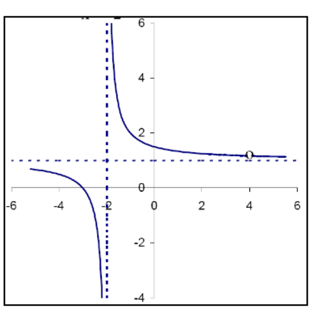
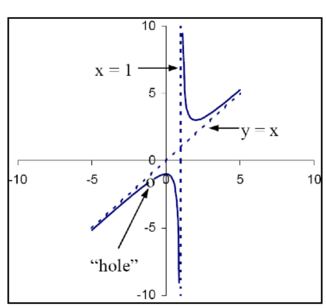
graph: A B C

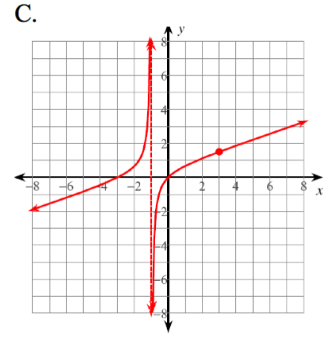
A.



B.

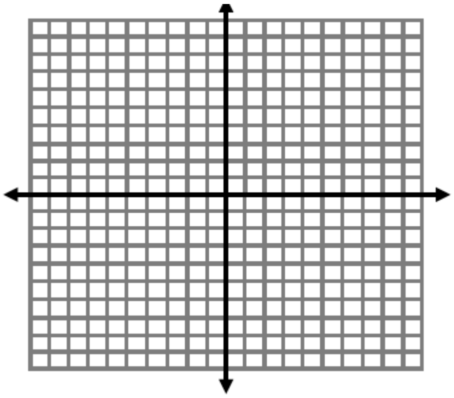


C.



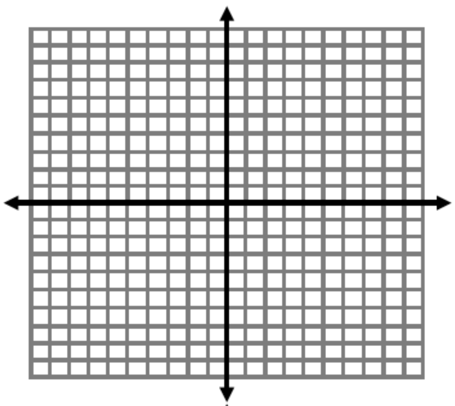
Extra practice:

1. $y = \dfrac{1}{x+2}$



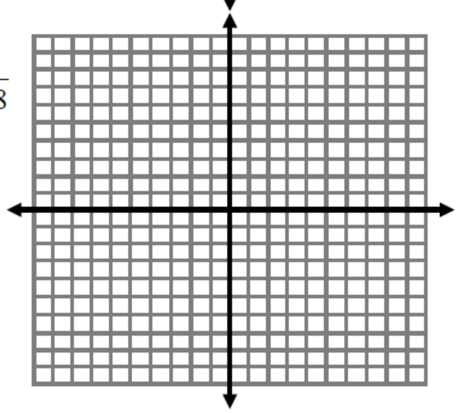
x-intercepts:	
Vertical Asymptotes:	
Horizontal or Slant Asymptotes:	
Holes:	
y-Intercept(s):	
Domain:	
Range	

2. $y = \dfrac{x^2+5x+6}{x^2-9}$



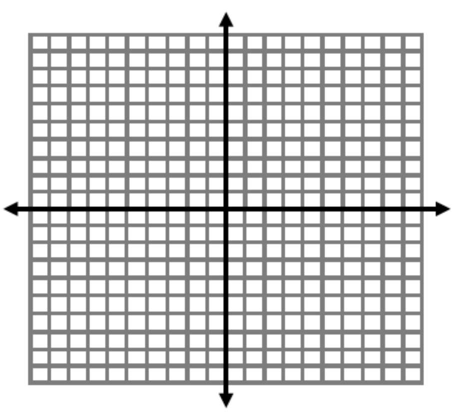
x-intercepts:	
Vertical Asymptotes:	
Horizontal or Slant Asymptotes:	
Holes:	
y-Intercept(s):	
Domain:	
Range	

3. $y = \dfrac{x^2-4}{3x^2-15x+18}$



x-intercepts:	
Vertical Asymptotes:	
Horizontal or Slant Asymptotes:	
Holes:	
y-Intercept(s):	
Domain:	
Range	

4. $y = \dfrac{5}{(x-2)^2}$

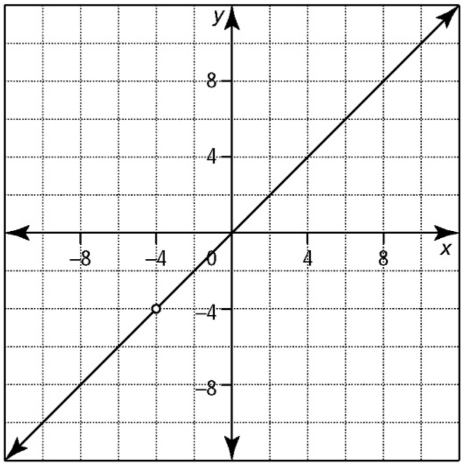


x-intercepts:	
Vertical Asymptotes:	
Horizontal or Slant Asymptotes:	
Holes:	
y-Intercept(s):	
Domain:	
Range	

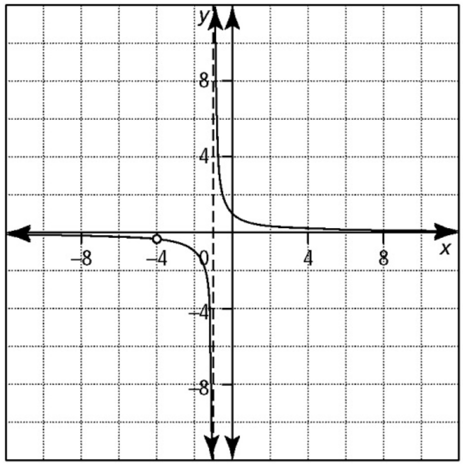
Match the equation of each rational function with the most appropriate graph. Explain your reasoning.

$y = \dfrac{x+4}{x^2 - 3x - 4}$ $\qquad$ $y = \dfrac{x+4}{x^2 + 5x + 4}$ $\qquad$ $y = \dfrac{x^2 + 4x}{x + 4}$

A



B



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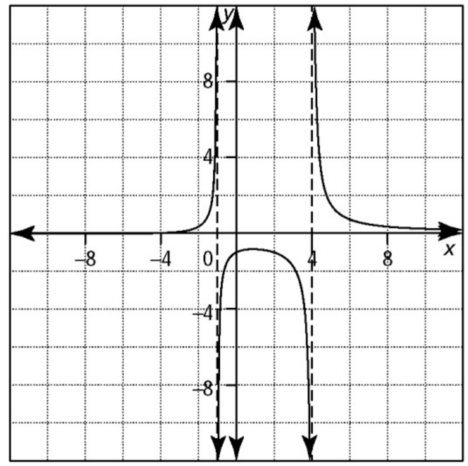

Write the equation for each graphed rational function.