THE ERINDALE ACADEMY
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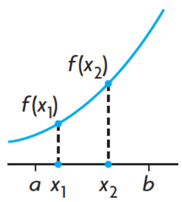
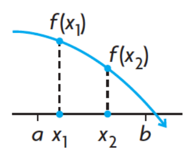
Unit 3: Curve Sketching, Optimization, and Related rates

Lesson 3.1: Interval of increase/decrease and critical points

Part I: Interval of increase/decrease

We say that a function *f* is increasing on an interval if, for any value of $x_1 < x_2$ on the interval, $f(x_1) < f(x_2)$. Similarly, a function *f* is decreasing on an interval if, for any value of $x_1 < x_2$ on the interval, $f(x_1) > f(x_2)$.



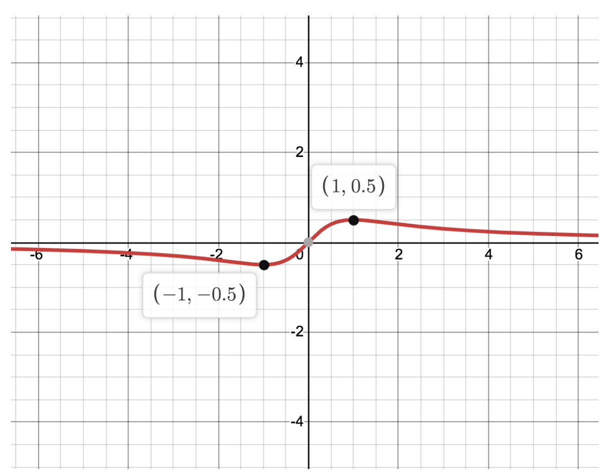


For a function *f* that is continuous and differentiable on an interval *I*,

- $f(x)$ is increasing on *I* if $f'(x) > 0$ for all values of *x*.
- $f(x)$ is decreasing on *I* if $f'(x) < 0$ for all values of *x*.

Example 1: Use derivative to reason about intervals of increasing and decreasing

$$y = \frac{x}{x^2 + 1}$$



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Part II: Critical points

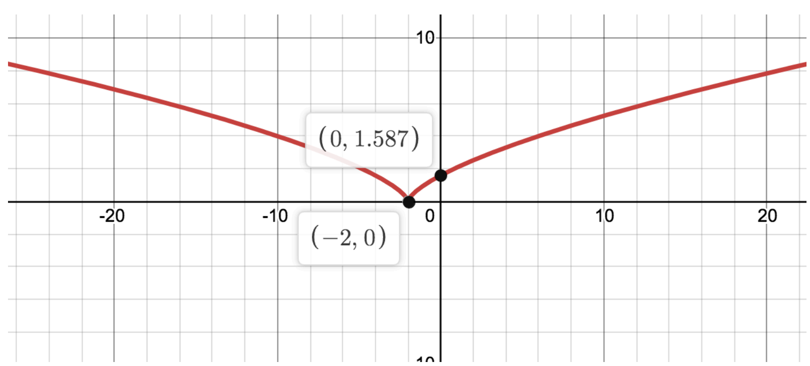
Critical point and extreme values: For a function *f(x)*, a critical number is a number, *x*, in the domain of *f(x)* such that $f'(x) = 0$ or $f'(x)$ is undefined. As a result, (c, f(c)) is called a critical point and usually corresponds to local or absolute extrema.

More specifically, maximum happens as the sign of $f'(x)$ has a transition from positive to negative; whereas minimum happens when sign changes from negative to positive.

Example 2: Find critical point, extrema, and intervals of increase and decrease.

a) $y = x^4 - 8x^3 + 18x^2$

b) $y = (x+2)^{\frac{2}{3}}$



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Practice: Find out local max/min, abs max/min, and interval of increase and decrease.

a) $f(x) = x + \frac{4}{x}, [1, 10]$

b) $f(x) = 4\sqrt{x} - x, [2, 9]$

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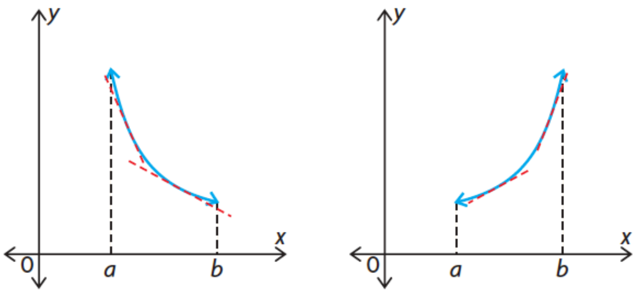
Unit 3: Curve Sketching, Optimization, and Related rates

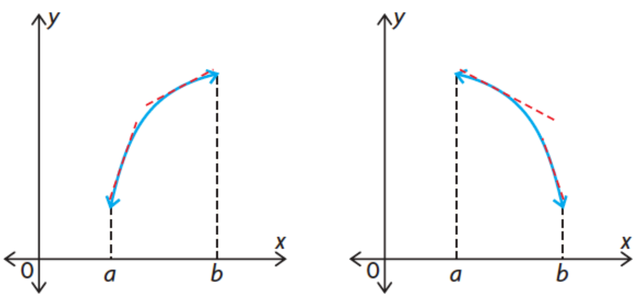
Concavity, second derivative, and curve sketching

Concavity and the Second Derivative

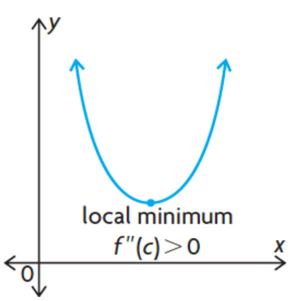
1. The graph of $y = f(x)$ is **concave up** on an interval $a \leq x \leq b$ in which the slopes of $f(x)$ are increasing. On this interval, $f''(x)$ exists and $f''(x) > 0$. The graph of the function is above the tangent at every point on the interval.



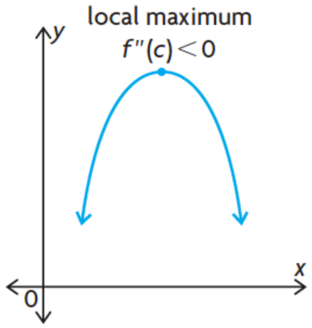
2. The graph of $y = f(x)$ is **concave down** on an interval $a \leq x \leq b$ in which the slopes of $f(x)$ are decreasing. On this interval, $f''(x)$ exists and $f''(x) < 0$. The graph of the function is below the tangent at every point on the interval.



3. If $y = f(x)$ has a critical point at $x = c$, with $f'(c) = 0$, then the behaviour of $f(x)$ at $x = c$ can be analyzed through the use of the **second derivative test** by analyzing $f''(c)$, as follows:
   a. The graph is concave up, and $x = c$ is the location of a local minimum value of the function, if $f''(c) > 0$.

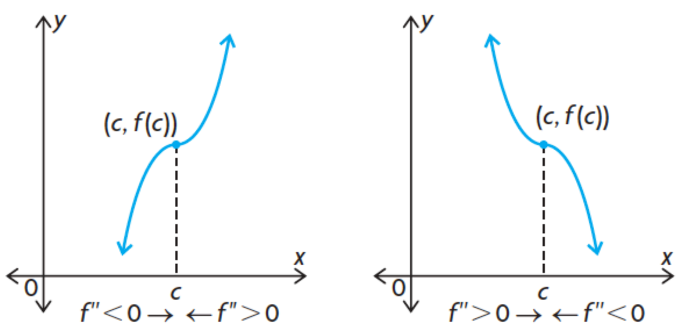


b. The graph is concave down, and $x = c$ is the location of a local maximum value of the function, if $f''(c) < 0$.



c. If $f''(c) = 0$, the nature of the critical point cannot be determined without further work.

4. A **point of inflection** occurs at $(c, f(c))$ on the graph of $y = f(x)$ if $f''(x)$ changes sign at $x = c$. That is, the curve changes from concave down to concave up, or vice versa.



5. All points of inflection on the graph of $y = f(x)$ must occur either where $\frac{d^2y}{dx^2}$ equals zero or where $\frac{d^2y}{dx^2}$ is undefined.

In the following examples, we will use these properties to sketch graphs of other functions.

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Example 1: Sketch the graph of $y = x^3 - 3x^2 - 9x + 10$

Algorithm:
Step 1: x-intercept and y-intercept
Step 2: Asymptotes and end behavior
Step 3: First derivative – Interval of increase and decrease & Point of local maximum and minimum (absolute maximum/minimum if necessary)
Step 4: Second derivative – interval of concavity & Point of inflection
Step 5: Put all together and do curve sketching

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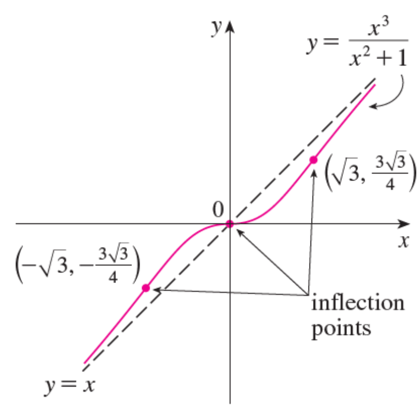
Example 2: Sketch the graph of $y = \frac{x^3}{x^2+1}$ with slant asymptotes in a form of $y = mx + b$, getting from long division.



Practice: Sketch the graph of $f(x) = \frac{4x^2-3}{x^3}$.

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Example 3: Sketch the graph of $f(x) = \frac{sinx}{1-sinx}$, $[-\pi, \pi]$

Practice: Sketch the graph of $f(x) = 4sin^2x - 1$, $[0, 2\pi]$