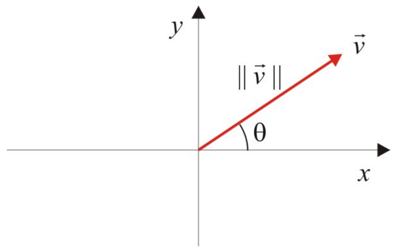
6.5 Vectors in R^2 and R^3

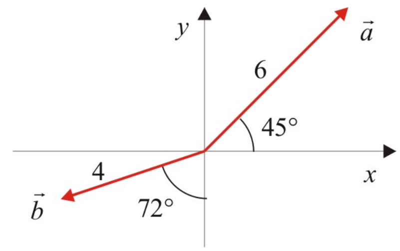
A Polar Coordinates

Given a *Cartesian system of coordinates*, a 2D vector $\vec{v}$ may be defined by its *magnitude* $\|\vec{v}\|$ and the counter-clockwise *angle* θ between the positive direction of the x-axis and the vector.



The pair $(\|\vec{v}\|,\theta)$ determines the *polar coordinates* of the 2D vector and $\vec{v}=(\|\vec{v}\|,\theta)$.

Ex 1. Express each vector in polar coordinates in the form $\vec{v}=(\|\vec{v}\|,\theta)$.

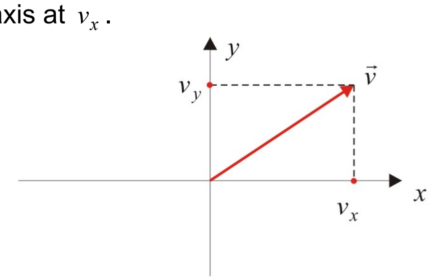


a) $\|\vec{a}\|=6,\quad \theta=45° \Rightarrow \vec{a}=(6,45°)$

b) $\|\vec{b}\|=4,\quad \theta=270°-72°=198° \Rightarrow \vec{b}=(4,198°)$

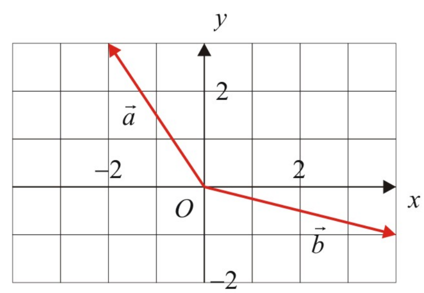
B Scalar Components for a 2D Vector

Let consider a 2D vector with the tail in the origin of the Cartesian system. Parallels through its tip to the coordinates axes intersect the x-axis at v_x and the y-axis at v_x.



The pair (v_x,v_y) determines the *scalar coordinates* of the 2D vector and $\vec{v}=(v_x,v_y)$.

Ex 2. Express each vector in scalar coordinates in the form $\vec{v}=(v_x,v_y)$.



a) $\vec{a}=(-2,3)$

b) $\vec{b}=(4,-1)$

C Link between the Polar Coordinates and Scalar Components

To convert a vector from the *polar coordinates* $\vec{v}=(\|\vec{v}\|,\theta)$ to the *scalar components* $\vec{v}=(v_x,v_y)$, use the formulas:

$$v_x=\|\vec{v}\|\cos\theta$$
$$v_y=\|\vec{v}\|\sin\theta$$

To convert a vector from the *scalar components* $\vec{v}=(v_x,v_y)$ to the *polar coordinates* $\vec{v}=(\|\vec{v}\|,\theta)$, use the formulas:

$$\|\vec{v}\|=\sqrt{v_x^{\,2}+v_y^{\,2}}$$ (to get the magnitude)

$$\tan\theta=\frac{v_y}{v_x}$$ (to get the direction)

Ex 3. Do the required conversions.

a) Convert $\vec{a}=(10,120°)$ to the scalar coordinates.

$\|\vec{a}\|=10,\quad \theta=120°$

$a_x=\|\vec{a}\|\cos\theta=10\cos120°=-5$

$a_y=\|\vec{a}\|\sin\theta=10\sin120°=5\sqrt{3}$

$\therefore \vec{a}=(-5,5\sqrt{3})$

b) Convert $\vec{b}=(-4,-7)$ to the polar coordinates.

$\|\vec{b}\|=\sqrt{b_x^{\,2}+b_y^{\,2}}=\sqrt{(-4)^2+(-7)^2}=\sqrt{65}$

$\tan\theta=\dfrac{a_y}{a_x}=\dfrac{-7}{-4}=\dfrac{7}{4}\Rightarrow\theta=\tan^{-1}(7/4)+180°=240.26°$

$\therefore \vec{b}=(\sqrt{65},240.26°)$

D Magnitude of a 2D Algebraic Vector

The *magnitude* of a 2D algebraic vector $\vec{v} = (v_x, v_y)$ is given by:

$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2}$$

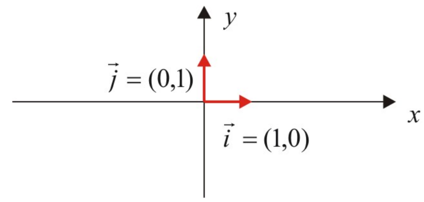
Ex 4. Find the magnitude of the following 2D vector: $\vec{v} = (4, -3)$.

$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2} = \sqrt{4^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$

$\therefore \|\vec{v}\| = 5$

E Standard Unit Vectors

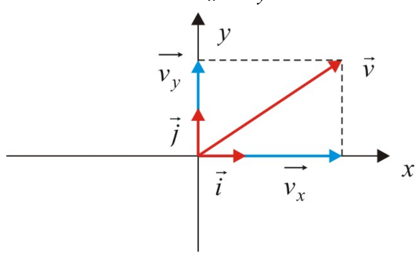
The unit vectors $\vec{i} = (1,0)$ and $\vec{j} = (0,1)$ are called the *standard unit vectors* in 2D space. See the figure to the right.



F Vector Components for a 2D Vector

Any vector $\vec{v}$ may be decomposed into two perpendicular *vector components* $\overrightarrow{v_x}$ and $\overrightarrow{v_y}$, parallel to each of the standard unit vectors.

$$\vec{v} = \overrightarrow{v_x} + \overrightarrow{v_y}$$



The link between the *scalar components* and the *vector components* is given by:

$$\overrightarrow{v_x} = v_x \vec{i}$$

$$\overrightarrow{v_y} = v_y \vec{j}$$

A 2D vector may be written in *algebraic form* as:

$$\vec{v} = \overrightarrow{v_x} + \overrightarrow{v_y} = v_x \vec{i} + v_y \vec{j} = (v_x, v_y)$$

Ex 5. Convert the vector $\vec{v} = -2\vec{i} + 5\vec{j}$ into the form $\vec{v} = (v_x, v_y)$.

$\therefore \vec{v} = -2\vec{i} + 5\vec{j} = (-2, 5)$

Ex 6. Convert the vector $\vec{v} = (4, -6)$ into the form $\vec{v} = v_x \vec{i} + v_y \vec{j}$.

$\therefore \vec{v} = (4, -6) = 4\vec{i} - 6\vec{j}$

Ex 7. Find the vector components for $\vec{a} = (-3, -5)$.

$\therefore \overrightarrow{a_x} = -3\vec{i} = (-3, 0), \quad \overrightarrow{a_y} = -5\vec{j} = (0, -5)$

G Position 2D Vector

The *directed line segment* $\overrightarrow{OP}$, from the origin O to a generic point $P(x, y)$ determines a vector called the *position vector* and:

$$\overrightarrow{OP} = (x, y) = x\vec{i} + y\vec{j}$$

Ex 8. Find the algebraic position vector $\overrightarrow{OA}$, where $A(-2, -3)$.

$\overrightarrow{OA} = (-2, -3) = -2\vec{i} - 3\vec{j}$

H Displacement 2D Vector

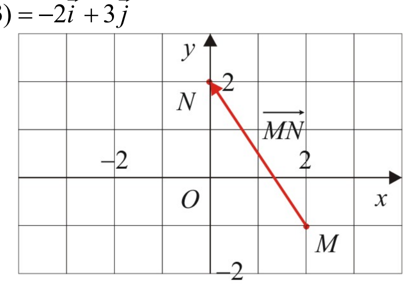
The *directed line segment* $\overrightarrow{AB}$ from the point $A(x_A, y_A)$ to the point $B(x_B, y_B)$ determines a vector called the *displacement vector* and:

$$\overrightarrow{AB} = (x_B - x_A, y_B - y_A) = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j}$$

Ex 9. Find the algebraic displacement vector $\overrightarrow{MN}$, where $M(2, -1)$ and $N(0, 2)$. Draw a diagram.

$\overrightarrow{MN} = (x_N - x_M, y_N - y_M) = (0 - 2, 2 - (-1)) = (-2, 3)$

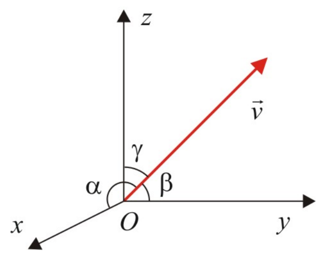
$\therefore \overrightarrow{MN} = (-2, 3) = -2\vec{i} + 3\vec{j}$



I Direction Angles

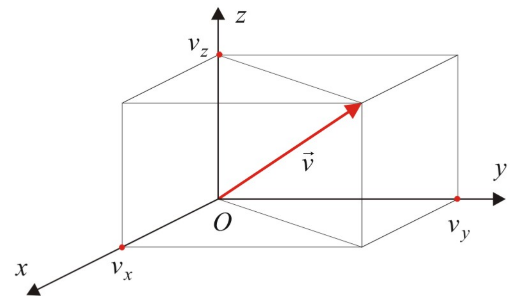
Let consider a 3D coordinate system and a 3D vector $\vec{v}$ with the tail in the origin O.

Direction angles are the angles α, β, and γ between the vector and the positive directions of the coordinates axes:



J Scalar Components of a 3D Vector

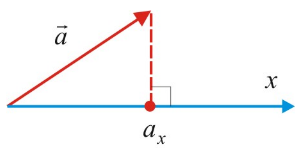
Let consider a 3D coordinate system and a 3D vector $\vec{v}$ with the tail in the origin O. Parallel planes through its tip to the coordinates planes intersect the x-axis at v_x the y-axis at v_y, and the y-axis at v_z.



The triple (v_x, v_y, v_z) determines the *scalar components* of the 3D vector and $\vec{v} = (v_x, v_y, v_z)$.

K Link between the Direction Angles and the 3D Scalar Coordinates

The link between the *direction angles* (α, β, and γ) and the *scalar components* of a vector (v_x, v_y, and v_z) is given by:



$$v_x = \|\vec{v}\|\cos\alpha$$
$$v_y = \|\vec{v}\|\cos\beta$$
$$v_z = \|\vec{v}\|\cos\gamma$$

and by:

$$\|\vec{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

$$\cos\alpha = \frac{v_x}{\|\vec{v}\|}$$

$$\cos\beta = \frac{v_y}{\|\vec{v}\|}$$

$$\cos\gamma = \frac{v_z}{\|\vec{v}\|}$$

Note that:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Ex 10. The magnitude of a vector $\vec{a}$ is $\|\vec{a}\| = 20$ and the direction angles are $\alpha = \angle(\vec{a}, Ox) = 60°$, $\beta = \angle(\vec{a}, Oy) = 45°$, and $\gamma = \angle(\vec{a}, Oz) = 60°$. Write the vector $\vec{a}$ in the algebraic form (using the scalar components).

$$a_x = \|\vec{a}\|\cos\alpha = 20\cos 60° = 10$$
$$a_y = \|\vec{a}\|\cos\beta = 20\cos 45° = 10\sqrt{2}$$
$$a_z = \|\vec{a}\|\cos\gamma = 20\cos 60° = 10$$
$$\therefore \vec{a} = (10, 10\sqrt{2}, 10) = 10\vec{i} + 10\sqrt{2}\vec{j} + 10\vec{k}$$

Ex 11. Find the direction angles for the vector $\vec{u} = -2\vec{i} + 3\vec{j} - \vec{k}$.

$$\|\vec{u}\| = \sqrt{u_x^2 + u_y^2 + u_z^2} = \sqrt{(-2)^2 + 3^2 + (-1)^2} = \sqrt{14}$$

$$\cos\alpha = \frac{u_x}{\|\vec{u}\|} = \frac{-2}{\sqrt{14}} \Rightarrow \alpha = \cos^{-1}\left(\frac{-2}{\sqrt{14}}\right) = 122.31°$$

$$\cos\beta = \frac{u_y}{\|\vec{u}\|} = \frac{3}{\sqrt{14}} \Rightarrow \beta = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) = 36.70°$$

$$\cos\gamma = \frac{u_z}{\|\vec{u}\|} = \frac{-1}{\sqrt{14}} \Rightarrow \gamma = \cos^{-1}\left(\frac{-1}{\sqrt{14}}\right) = 105.50°$$

L Magnitude of a 3D Algebraic Vector

The magnitude of a 3D algebraic vector $\vec{v} = (v_x, v_y, v_z)$ is given by:

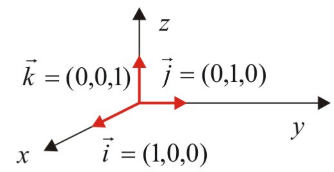
$$\|\vec{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

Ex 12. Find the magnitude for the vector $\vec{v} = (2, -3, 4)$.

$$\|\vec{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{2^2 + (-3)^2 + 4^2} = \sqrt{29}$$
$$\therefore \|\vec{v}\| = \sqrt{29} \cong 5.39$$

M 3D Standard Unit Vectors

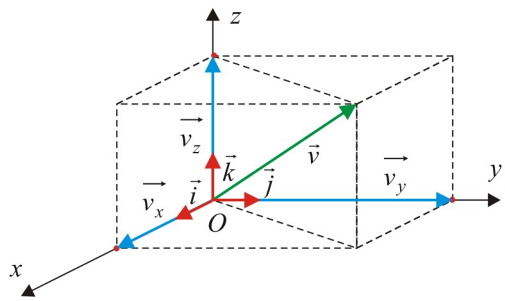
The unit vectors $\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$, $\vec{k} = (0,0,1)$ and are called the *standard unit vectors* in 3D space. See the figure to the right.



N Vector Components for a 3D Vector

Any 3D vector $\vec{v}$ may be decomposed into three perpendicular *vector components* $\overrightarrow{v_x}$, $\overrightarrow{v_y}$ and $\overrightarrow{v_z}$, parallel to each of the 3D standard unit vectors.

$$\vec{v} = \overrightarrow{v_x} + \overrightarrow{v_y} + \overrightarrow{v_z}$$



The link between the *scalar components* and the *vector components* is given by:

$$\overrightarrow{v_x} = v_x\vec{i}, \quad \overrightarrow{v_y} = v_y\vec{j}, \quad \overrightarrow{v_z} = v_z\vec{k}$$

A 3D vector may be written in *algebraic form* as:

$$\vec{v} = \overrightarrow{v_x} + \overrightarrow{v_y} + \overrightarrow{v_z} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k} = (v_x, v_y, v_z)$$

Ex 13. Convert the vector $\vec{v} = -3\vec{i} - 4\vec{j} + 2\vec{k}$ into the form $\vec{v} = (v_x, v_y, v_z)$.

$\therefore \vec{v} = -3\vec{i} - 4\vec{j} + 2\vec{k} = (-3,-4,2)$

Ex 14. Convert the vector $\vec{v} = (-3,4,-5)$ into the form $\vec{v} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$.

$\therefore \vec{v} = (-3,4,-5) = -3\vec{i} + 4\vec{j} - 5\vec{k}$

Ex 15. Find the vector components for $\vec{a} = (4,0,-3)$.

$\therefore \overrightarrow{a_x} = 4\vec{i} = (4,0,0), \quad \overrightarrow{a_y} = \vec{0} = (0,0,0), \quad \overrightarrow{a_z} = -3\vec{k} = (0,0,-3)$

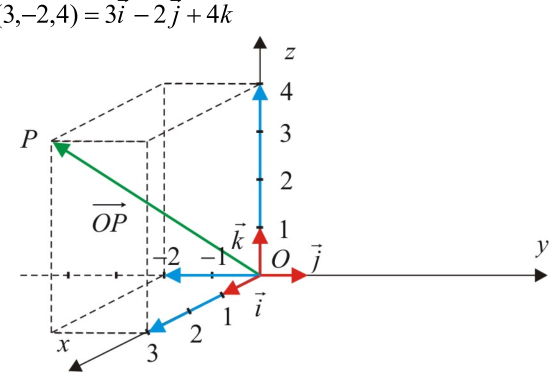
O Position 3D Vector

The *directed line segment* $\overrightarrow{OP}$, from the origin O to a generic point $P(x,y,z)$ determines a vector called the position vector and:

$$\overrightarrow{OP} = (x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$$

Ex 16. Find the algebraic position vector $\overrightarrow{OP}$, where $P(3,-2,4)$. Draw a diagram.

$\overrightarrow{OP} = (3,-2,4) = 3\vec{i} - 2\vec{j} + 4\vec{k}$



P Displacement 3D Vector

The *directed line segment* $\overrightarrow{AB}$ from the point $A(x_A, y_A, z_A)$ to the point $B(x_B, y_B, z_B)$ determines a vector called the displacement vector and:

$$\overrightarrow{AB} = (x_B - x_A, y_B - y_A, z_B - z_A)$$
$$= (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} + (z_B - z_A)\vec{k}$$

Ex 17. Find the algebraic displacement vector $\overrightarrow{PQ}$, where $P(1,-2,3)$ and $Q(-2,3,-4)$.

$\overrightarrow{PQ} = (x_Q - x_P, y_Q - y_P, z_Q - z_P)$

$= (-2-1, 3-(-2), -4-3) = (-3,5,-7) = -3\vec{i} + 5\vec{j} - 7\vec{k}$

Reading: Nelson Textbook, Pages 310-316
Homework: Nelson Textbook: Page 316 #10cb, 15, 16ad, 18, 19