6.6 Operations with Algebraic Vectors in R^2

A 2D Algebraic Vectors
A 2D Algebraic Vector may be written in *components form* as:

$$\vec{v} = (v_x, v_y)$$

or in *terms of unit vectors* as:

$$\vec{v} = v_x \vec{i} + v_y \vec{j}$$

and has a *magnitude* given by:

$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2}$$

Ex 1. Consider the vector $\vec{a} = -3\vec{i} + 4\vec{j}$.

a) Write the vector in components form.

$\vec{a} = -3\vec{i} + 4\vec{j} = (-3, 4)$

b) Find the magnitude of the vector $\vec{a}$.

$\|\vec{a}\| = \sqrt{{a_x}^2 + {a_y}^2} = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5$

B Addition of 2D Algebraic Vectors
The sum of two 2D algebraic vectors $\vec{a} = (a_x, a_y) = a_x \vec{i} + a_y \vec{j}$ and $\vec{b} = (b_x, b_y) = b_x \vec{i} + b_y \vec{j}$ is a 2D algebraic vector given by:

$$\vec{a} + \vec{b} = a_x \vec{i} + a_y \vec{j} + b_x \vec{i} + b_y \vec{j}$$
$$= (a_x + b_x)\vec{i} + (a_y + b_y)\vec{j} = (a_x + b_x, a_y + b_y)$$

Ex 2. Find the sum of the vector $\vec{a} = 3\vec{i} - 5\vec{j}$ and $\vec{b} = (-2, 7)$.

$\vec{s} = \vec{a} + \vec{b} = (3 - 2, -5 + 7) = (1, 2) = \vec{i} + 2\vec{j}$

C Subtraction of 2D Algebraic Vectors
The difference of two 2D algebraic vectors $\vec{a} = (a_x, a_y) = a_x \vec{i} + a_y \vec{j}$ and $\vec{b} = (b_x, b_y) = b_x \vec{i} + b_y \vec{j}$ is a 2D algebraic vector given by:

$$\vec{a} - \vec{b} = a_x \vec{i} + a_y \vec{j} - (b_x \vec{i} + b_y \vec{j})$$
$$= (a_x - b_x)\vec{i} + (a_y - b_y)\vec{j} = (a_x - b_x, a_y - b_y)$$

Ex 3. Find the difference $\vec{d} = \vec{a} - \vec{b}$ between the vector $\vec{a} = 3\vec{i} - 5\vec{j}$ and $\vec{b} = (-2, 7)$.

$\vec{d} = \vec{a} - \vec{b} = (3 - (-2), -5 - 7) = (5, -12) = 5\vec{i} - 12\vec{j}$

D Multiplication of 2D Algebraic Vector by a Scalar
The multiplication of a 2D algebraic vector $\vec{a} = (a_x, a_y) = a_x \vec{i} + a_y \vec{j}$ by a scalar λ is a 2D algebraic vector given by:

$$\lambda \vec{a} = \lambda(a_x \vec{i} + a_y \vec{j}) = \lambda a_x \vec{i} + \lambda a_y \vec{j} = (\lambda a_x, \lambda a_y)$$

Ex 4. For each case, multiply the given vector by the given scalar.

a) $\vec{u} = (-1, 3), \quad \lambda = -2$

$\lambda \vec{u} = -2(-1, 3) = ((-2)(-1), (-2)(3)) = (2, -6)$

b) $\vec{v} = 4\vec{i} - 2\vec{j}, \quad \mu = \frac{1}{2}$

$\mu \vec{v} = \frac{1}{2}(4\vec{i} - 2\vec{j}) = \frac{1}{2}(4)\vec{i} + \frac{1}{2}(-2)\vec{j} = 2\vec{i} - \vec{j}$

Ex 5. Consider $\vec{a} = (3, -5)$, $\vec{b} = (0, -2)$, and $\vec{c} = (-1, 0)$. Find the vector $\vec{v} = -2\vec{a} + 3\vec{b} - 4\vec{c}$.

$\vec{v} = -2\vec{a} + 3\vec{b} - 4\vec{c} = -2(3, -5) + 3(0, -2) - 4(-1, 0)$

$= (-6, 10) + (0, -6) + (4, 0) = (-6 + 0 + 4, 10 - 6 + 0) = (-2, 4)$

$\therefore \vec{v} = (-2, 4)$

E Vector Equations
Use backward operations to solve equations involving vectors:

$$\vec{x} + \vec{a} = \vec{b} \quad \Rightarrow \quad \vec{x} = \vec{b} - \vec{a}$$
$$\vec{a} - \vec{x} = \vec{b} \quad \Rightarrow \quad \vec{x} = \vec{a} - \vec{b}$$
$$\lambda \vec{x} = \vec{a} \quad \Rightarrow \quad \vec{x} = \frac{1}{\lambda}\vec{a}$$

Ex 6. Given $\vec{a} = (-2, 1)$, $\vec{b} = (1, -3)$ solve for $\vec{x}$ the following vector equation $2\vec{a} - 3\vec{x} = \vec{b}$.

$2\vec{a} - 3\vec{x} = \vec{b} \Rightarrow 2\vec{a} - \vec{b} = 3\vec{x} \Rightarrow \vec{x} = \frac{1}{3}(2\vec{a} - \vec{b})$

$\vec{x} = \frac{1}{3}[2(-2, 1) - (1, -3)] = \frac{1}{3}(-4, 2) + (-1, 3)]$

$= \frac{1}{3}(-5, 5) = \left(\frac{-5}{3}, \frac{5}{3}\right) \quad \Rightarrow \quad \therefore \vec{x} = \left(\frac{-5}{3}, \frac{5}{3}\right)$

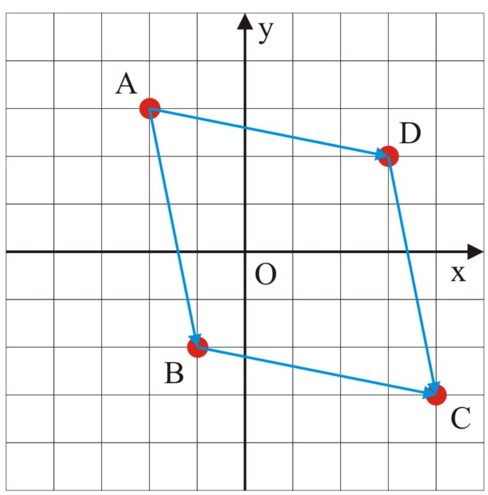
Ex 7. Find the perimeter of the triangle ΔABC where $A(0,1)$, $B(2,3)$, and $C(-1,-2)$.

$$\overrightarrow{AB} = (2-0,3-1) = (2,2)$$
$$\|\overrightarrow{AB}\| = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$$
$$\overrightarrow{BC} = (-1-2,-2-3) = (-3,-5)$$
$$\|\overrightarrow{BC}\| = \sqrt{(-3)^2+(-5)^2} = \sqrt{34}$$
$$\overrightarrow{AC} = (-1-0,-2-1) = (-1,-3)$$
$$\|\overrightarrow{AC}\| = \sqrt{(-1)^2+(-3)^2} = \sqrt{10}$$
$$perimeter = \|\overrightarrow{AB}\| + \|\overrightarrow{BC}\| + \|\overrightarrow{AC}\|$$
$$= 2\sqrt{2}+\sqrt{34}+\sqrt{10} \cong 11.82$$

Ex 8. Find an unit vector parallel to the vector $\vec{a} = (3,-4)$.

$$\|\vec{a}\| = \sqrt{3^2+(-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$
$$\vec{u} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{(3,-4)}{5} = \left(\frac{3}{5},\frac{-4}{5}\right)$$
$$\therefore \vec{u} = \left(\frac{3}{5},\frac{-4}{5}\right)$$

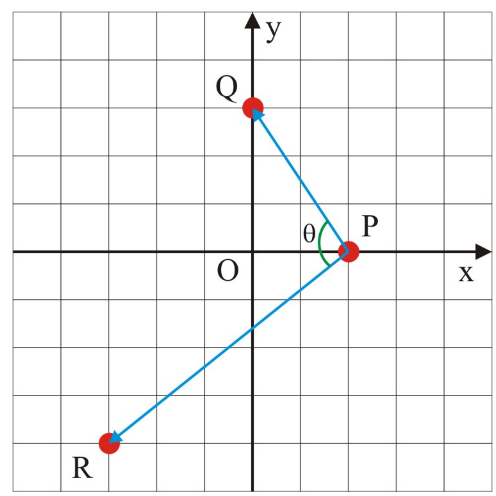
Ex 9. Given $A(-2,3)$, $B(-1,-2)$, and $D(4,2)$, find the point C such that the polygon $ABCD$ is a parallelogram.



Let C be the point $C(x,y)$. Then $\overrightarrow{BC} = \overrightarrow{AD}$. So:

$$(x+1,y+2) = (4+2,2-3)$$
$$x+1 = 6 \Rightarrow x = 5$$
$$y+2 = -1 \Rightarrow y = -3$$
$$\therefore C(5,-3)$$

Ex 10. Find the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ where $P(2,0)$, $Q(0,3)$, and $R(-3,-4)$.



$$\overrightarrow{PQ} = (-2,3) \Rightarrow \|\overrightarrow{PQ}\| = \sqrt{13}$$
$$\overrightarrow{PR} = (-5,-4) \Rightarrow \|\overrightarrow{PR}\| = \sqrt{41}$$
$$\overrightarrow{PQ}+\overrightarrow{PR} = (-7,-1) \Rightarrow \|\overrightarrow{PQ}+\overrightarrow{PR}\| = \sqrt{50}$$
$$\theta = \angle(\overrightarrow{PQ},\overrightarrow{PR})$$
$$\|\overrightarrow{PQ}+\overrightarrow{PR}\|^2 = \|\overrightarrow{PQ}\|^2 + \|\overrightarrow{PR}\|^2 + 2\|\overrightarrow{PQ}\|\|\overrightarrow{PR}\|\cos\theta$$
$$\cos\theta = \frac{\|\overrightarrow{PQ}+\overrightarrow{PR}\|^2 - \|\overrightarrow{PQ}\|^2 - \|\overrightarrow{PR}\|^2}{2\|\overrightarrow{PQ}\|\|\overrightarrow{PR}\|}$$
$$= \frac{50-13-41}{2\sqrt{13}\sqrt{41}} \Rightarrow \theta = \cos^{-1}\frac{-4}{2\sqrt{13}\sqrt{41}} = 94.97°$$

Reading: Nelson Textbook, Pages 319-324
Homework: Nelson Textbook: Page 324 #1, 3, 5, 7c, 11, 13b, 17

