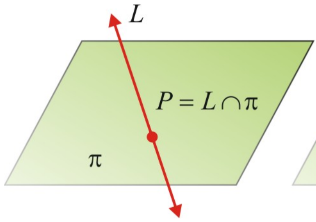
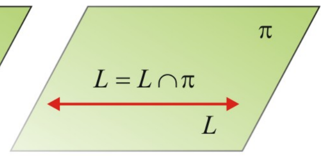
## 9.1 Intersection of a Line with a Plane

**A Relative Position of a Line and a Plane**
There are three possible situations as represented below:




a) The line *intersects* the plane at a single point.




b) The line *lies* on the plane. There are an infinite number of points of intersections.


$L \cap \pi = \varnothing$

c) The line is *parallel* to the plane but *distinct*. There is no point of intersection.

**B Intersection of a Line and a Plane (Algebraic Method)**
To get the intersection between a line $L$ and a plane $\pi$:
a) *Substitute* the parametric equations of the line

$$L:\begin{cases} x = x_0 + tu_x \\ y = y_0 + tu_y \\ z = z_0 + tu_z \end{cases} \quad (1)$$

into the Cartesian equation of the plane

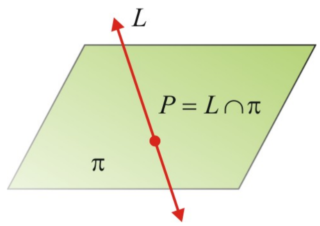
$$\pi : Ax + By + Cz + D = 0 \quad (2)$$

to get the equation:

$$A(x_0 + tu_x) + B(y_0 + tu_y) + C(z_0 + tu_z) + D = 0 \quad (*)$$

b) *Solve* (if possible) the equation (*) for the parameter $t$.
c) *Substitute* the value of the parameter $t$ into the parametric equations of the line (1) to get the point of intersection.

**C Unique Solution (Point Intersection)**
In this case, by solving the equation (*) you get a *unique value* for the parameter $t$. Therefore, there is a unique *point of intersection* between the line and the plane.




The line *intersects* the plane at a unique point.

Ex 1. Find the point(s) of intersection between the line $L: \vec{r} = (-6,9,-1) + t(-2,3,1), \quad t \in R$ and the plane $\pi: -x + 2y + z + 4 = 0$.

$$L:\begin{cases} x = -6 - 2t \\ y = 9 + 3t \\ z = -1 + t \end{cases}$$

$-(-6-2t) + 2(9+3t) + (-1+t) + 4 = 0$

$6 + 2t + 18 + 6t - 1 + t + 4 = 0$

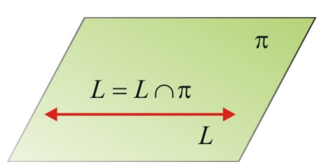
$9t = -27 \Rightarrow t = -3$

$$\begin{cases} x = -6 - 2(-3) = 0 \\ y = 9 + 3(-3) = 0 \\ z = -1 + (-3) = -4 \end{cases} \Rightarrow \ni P(0,0,-4) = L \cap \pi$$

**D Infinite Number of Solutions (Line Intersection)**
In this case, by solving the equation (*) you get the equation:

$$0t = 0$$

which has an *infinite number of solutions*. Therefore, there are an *infinite number of points of intersection*.




The line is *lies* on the plane.

Ex 2. Find the point(s) of intersection between the line $L: \vec{r} = (3,0,0) + t(0,2,-3), \quad t \in R$ and the plane $\pi: -2x + 3y + 2z + 6 = 0$.

$$L:\begin{cases} x = 3 \\ y = 2t \\ z = -3t \end{cases}$$

$-2(3) + 3(2t) + 2(-3t) + 6 = 0$

$-6 + 6t - 6t + 6 = 0$

$0t = 0$

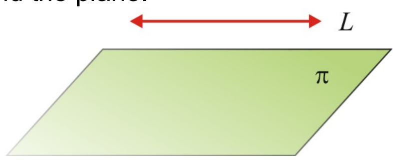
There are an infinite number of solutions and therefore an infinite number of points of intersection. The line is lies on the plane.

**E No Solution (No Intersection)**
In this case, by solving the equation (*) you get a false statement like:

$$0t = 1$$

The equation *does not have any solution* and therefore there is *no point of intersection* between the line and the plane.




The line is *parallel* to the plane and *does not lie* on the plane.

Ex 3. Ex 2. Find the point(s) of intersection between the line $L: \vec{r} = (1,2,3) + t(0,1,1), \quad t \in R$ and the plane $\pi : x + y - z - 3 = 0$.

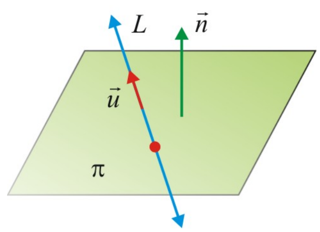
$$L: \begin{cases} x = 1 \\ y = 2 + t \\ z = 3 + t \end{cases}$$

$$1 + (2 + t) - (3 + t) - 3 = 0$$
$$1 + 2 + t - 3 - t - 3 = 0$$
$$-3 = 0$$

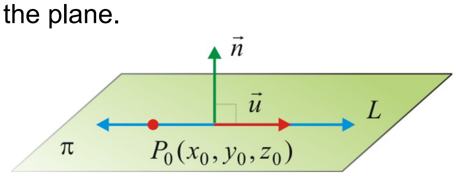
There is no solution. Therefore there is no point of intersection between the line and the plane. The line is parallel to the plane and does not lie on the plane.

**F Classifying Lines**
Let consider the line $L: \vec{r} = \vec{r}_0 + t\vec{u}, \quad t \in R$, where $P_0(x_0, y_0, z_0)$ is a specific point on the line, and the plane $\pi : Ax + By + Cz + D = 0$, where $\vec{n} = (A, B, C)$ is a normal vector to the plane.

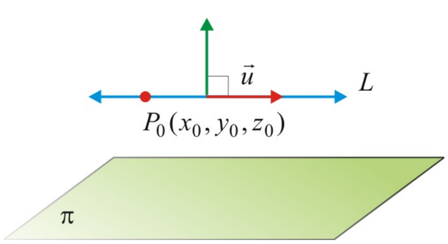
a) If $\vec{n} \cdot \vec{u} \neq 0$ the line *intersects* the plane at a unique point.




b) If $\vec{n} \cdot \vec{u} = 0$ and $Ax_0 + By_0 + Cz_0 + D = 0$ then the line *lies* on the plane.




c) If $\vec{n} \cdot \vec{u} = 0$ and $Ax_0 + By_0 + Cz_0 + D \neq 0$ then the line is *parallel* to the plane but *does not lie* on the plane.




Note. By solving the equation (*) for $t$ you will end by getting the same cases and conditions as above. Try this as an exercise.

Ex 4. Consider the plane $\pi : 4x + 3y - 2z + 12 = 0$. Classify each line as intersecting the plane, contained by the plane, or distinct from the plane. Do not find the point(s) of intersection using the algebraic method.

a) $L: \vec{r} = (-3,0,0) + t(0,2,3), \quad t \in R$

$(-3,0,0) \in \pi \text{ and } \vec{n} \cdot \vec{u} = (4,3,-2) \cdot (0,2,3) = 0 + 6 - 6 = 0$

Therefore the line lies on the plane.

b) $L: \vec{r} = (1,0,-2) + t(1,-2,0), \quad t \in R$

$\vec{n} \cdot \vec{u} = (4,3,-2) \cdot (1,-2,0) = 4 - 6 = -2 \neq 0$

Therefore the line intersects the plane at a single point.

c) $L: \vec{r} = t(1,0,2), \quad t \in R$

$(0,0,0) \notin \pi \text{ and } \vec{n} \cdot \vec{u} = (4,3,-2) \cdot (1,0,2) = 4 - 4 = 0$

Therefore the line is parallel to the plane but does not lie on the plane.

Ex 5. Show that the point $P$ of intersection between the plane $\pi : \vec{r} = \vec{p}_0 + s\vec{u} + t\vec{v}$ and the line $L: \vec{r} = \vec{l}_0 + q\vec{w}$ is given by: $\vec{r}_P = \vec{l}_0 + \dfrac{(\vec{p}_0 - \vec{l}_0) \cdot (\vec{u} \times \vec{v})}{\vec{w} \cdot (\vec{u} \times \vec{v})} \vec{w}$. Explain.

$$\vec{p}_0 + s\vec{u} + t\vec{v} = \vec{l}_0 + q\vec{w} \quad | \cdot (\vec{u} \times \vec{v})$$
$$(\vec{p}_0 - \vec{l}_0) \cdot (\vec{u} \times \vec{v}) = q\vec{w} \cdot (\vec{u} \times \vec{v})$$
$$q = \frac{(\vec{p}_0 - \vec{l}_0) \cdot (\vec{u} \times \vec{v})}{\vec{w} \cdot (\vec{u} \times \vec{v})}$$
$$\vec{r}_P = \vec{l}_0 + \frac{(\vec{p}_0 - \vec{l}_0) \cdot (\vec{u} \times \vec{v})}{\vec{w} \cdot (\vec{u} \times \vec{v})} \vec{w}$$

If $\vec{w} \cdot (\vec{u} \times \vec{v}) \neq 0$ then there is a unique point of intersection. The line intersects the plane.

If $(\vec{p}_0 - \vec{l}_0) \cdot (\vec{u} \times \vec{v}) = 0 \text{ and } \vec{w} \cdot (\vec{u} \times \vec{v}) = 0$ then there are an infinite points of intersections. The line lies on the plane.

If $(\vec{p}_0 - \vec{l}_0) \cdot (\vec{u} \times \vec{v}) \neq 0 \text{ and } \vec{w} \cdot (\vec{u} \times \vec{v}) = 0$ then there is no intersection point. The line is parallel to the plane but does not lie on the plane.

**Reading**: Nelson Textbook, Pages 489-496
**Homework**: Nelson Textbook: Page 497 #1, 5, 7, 17