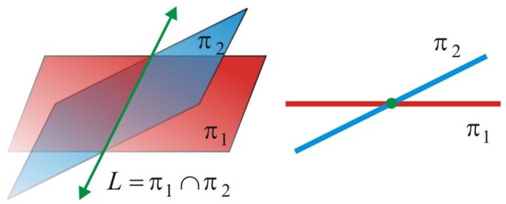
9.3 Intersection of two Planes

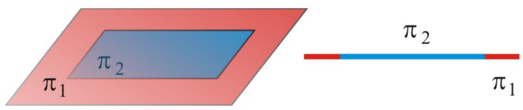
A Relative Position of two Planes

Two planes may be:

a) *intersecting* (into a line)

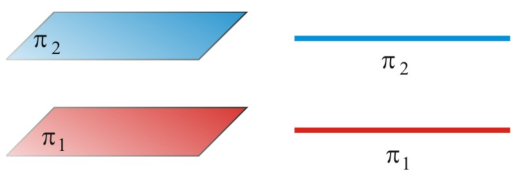


b) *coincident*



$\pi_1 = \pi_1 \cap \pi_2 = \pi_2$

c) *distinct*



$\pi_1 \cap \pi_2 = \varnothing$

B Intersection of two Planes

Let consider two plane given by their Cartesian equations:

$$\pi_1 : A_1x + B_1y + C_1z + D_1 = 0$$

$$\pi_2 : A_2x + B_2y + C_2z + D_2 = 0$$

To find the point(s) of intersection between two planes, *solve* the system of equations formed by their Cartesian equations:

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases} \quad (*)$$

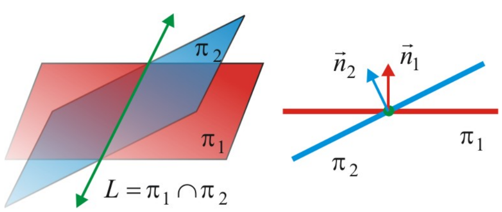
There are *two* equations are *three* unknowns.

Notes:

1. A normal vector to the plane π_1 is $\vec{n}_1 = (A_1, B_1, C_1)$ and a normal vector to the plane π_2 is $\vec{n}_2 = (A_2, B_2, C_2)$.
2. If the planes are *parallel* then coefficients A, B, C are *proportional*.
3. If the planes are *coincident* then coefficients A, B, C, D are *proportional*.
4. A system of equations is called *compatible* if there is *at least* one solution. A system of equations is called *incompatible* if there is *no solution*.

C Non Parallel Planes (Line Intersection)

In this case:



- ⇨ The coefficients A, B, C in the scalar equations are *not proportional*.
- ⇨ The normal vectors are *not parallel*: $\vec{n}_1 \times \vec{n}_2 \neq \vec{0}$.
- ⇨ By solving the system (*) you will be able to find two variables in terms of the third variable.
- ⇨ There are an *infinite number of solutions* and therefore an *infinite number of points of intersection*.
- ⇨ The intersection is a *line* and a *direction vector* for this line is $\vec{u} = \vec{n}_1 \times \vec{n}_2$.

Ex 1. Find the point(s) of intersection of the following two planes. Give a geometric representation of the solution(s).

$\pi_1 : -2x + 3y + z + 6 = 0$

$\pi_2 : 3x - y + 2z - 2 = 0$

D Coincident Planes
(Plane Intersection)

In this case:

$\pi_1 = \pi_1 \cap \pi_2 = \pi_2$

- ⇨ The planes are *parallel* and *coincident*.
- ⇨ The coefficients A, B, C, D in the scalar equations are *proportional*.
- ⇨ One equation in the system (*) is a *multiple* of the other equation and does not contain additional information (the equations are equivalent).
- ⇨ By solving the system of equations (*), you get a *true* statement (like $0 = 0$).
- ⇨ There are an *infinite number of solutions* and therefore an *infinite number of points of intersection*.
- ⇨ The intersection is a *plane*.

Ex 2. Find the point(s) of intersection of the following two planes. Give a geometric representation of the solution(s).

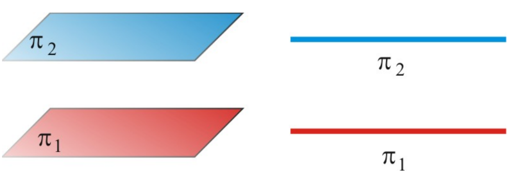
$\pi_1 : x - 2y + 3z + 1 = 0$

$\pi_2 : -3x + 6y - 9z - 3 = 0$

$$\begin{cases} x - 2y + 3z + 1 = 0 & (1) \\ -3x + 6y - 9z - 3 = 0 & (2) \end{cases}$$

E Parallel and Distinct Planes
(No Intersection)

In this case:



- ⇨ The planes are *parallel* and *distinct*.
- ⇨ The coefficients A, B, C in the scalar equations are *proportional* but the coefficients A, B, C, D are *not proportional*.
- ⇨ By solving the system (*) you get a *false* statement (like $0 = 1$).
- ⇨ There is *no solution* and therefore *no point of intersection* between the two planes.

Ex 3. Find the point(s) of intersection of the following two planes. Give a geometric representation of the solution(s).

$\pi_1 : x - y - 2z + 1 = 0$

$\pi_2 : -4x + 4y + 8z - 3 = 0$

$$\begin{cases} x - y - 2z + 1 = 0 & (1) \\ -4x + 4y + 8z - 3 = 0 & (2) \end{cases}$$

Ex 4. Classify each pair of planes as distinct, coincident, or intersecting. Do not attempt to solve algebraically the system of equations.

a) $\pi_1 : 2x - 3y + z - 1 = 0$, $\pi_2 : 4x - 6y + 2z - 2 = 0$

b) $\pi_1 : 3x + 6y - 9z - 3 = 0$, $\pi_2 : 2x + 4y - 6z - 4 = 0$

c) $\pi_1 : x + 2y + 3z + 1 = 0$, $\pi_2 : 3x + 2y + z + 2 = 0$

Reading: Nelson Textbook, Pages 510-515
Homework: Nelson Textbook: Page 515 # 6abc, 8, 10, 11, 12

