8.2 Cartesian Equation of a Line

A Symmetric Equation

The parametric equations of a line in R^2:

$$\begin{cases} x = x_0 + tu_x \\ y = y_0 + tu_y \end{cases} \quad t \in R$$

may be written as:

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} = t, \quad t \in R$$

The *symmetric equation* of the line is (if exists):

$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y}$$

Note. The symmetric equation does exist if $u_x \neq 0$ and $u_y \neq 0$.

Ex 1. Convert the vector or parametric equations to the symmetric equation (if exists).

a) $\begin{cases} x = -3 + 2t \\ y = -5t \end{cases} \quad t \in R$

$$\frac{x+3}{2} = \frac{y-0}{-5} = t, \quad t \in R$$

$$\therefore \frac{x+3}{2} = \frac{y}{-5}$$

b) $\vec{r} = (0,1) + t(1,-2), \quad t \in R$

$$\begin{cases} x = t \\ y = 1 - 2t \end{cases} \quad t \in R \quad \Rightarrow \quad \therefore \frac{x}{1} = \frac{y-1}{-2}$$

c) $\vec{r} = (-2,3) + t(2,0), \quad t \in R$

$$\begin{cases} x = -2 + 2t \\ y = 3 \end{cases} \quad t \in R \quad \Rightarrow \frac{x+2}{2} = \frac{y-3}{0} \Rightarrow \therefore \textit{does not exist}$$

Ex 2. Convert the symmetric equation of a line L:

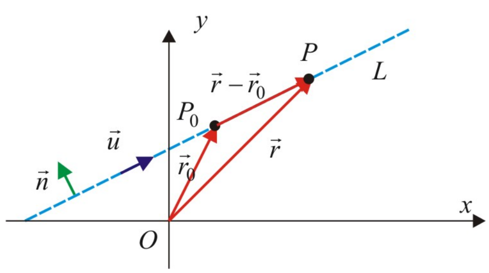
$\frac{x+2}{-1} = \frac{y-3}{2}$ to the vector equation.

$$\frac{x+2}{-1} = \frac{y-3}{2} = t, \quad t \in R$$

$$\begin{cases} x + 2 = -t \\ y - 3 = 2t \end{cases} \Rightarrow \begin{cases} x = -2 - t \\ y = 3 + 2t \end{cases} \Rightarrow \quad \therefore \vec{r} = (-2,3) + t(-1,2), \quad t \in R$$

B Normal Equation

Let consider a line L that passes through the specific point $P_0(x_0, y_0)$ and has the *direction vector* $\vec{u} = (u_x, u_y)$.



The vectors $\vec{n} = (-u_y, u_x) = (A, B)$ or $\vec{n} = (u_y, -u_x) = (A, B)$ are perpendicular to the vector $\vec{u}$ and so they are perpendicular to the line L.

These are called *normal vectors* to the line L.

Let $P(x, y)$ be a generic point on the line L. So:

$$\overrightarrow{P_0P} \parallel \vec{u} \Rightarrow \overrightarrow{P_0P} \perp \vec{n} \Rightarrow \overrightarrow{P_0P} \cdot \vec{n} = 0$$

$$(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0$$

The *normal equation* of a line is given by:

$$(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0$$

C Cartesian Equation

The normal equation can be written as:

$$\vec{r} \cdot \vec{n} - \vec{r}_0 \cdot \vec{n} = 0$$

$$(x, y) \cdot (A, B) - (x_0, y_0) \cdot (A, B) = 0$$

$$Ax + By - Ax_0 - By_0 = 0$$

$$Ax + By + C = 0 \textit{ where } C = -Ax_0 - By_0$$

The *Cartesian equation* of a line is given by:

$$Ax + By + C = 0$$

where

$\vec{n} = (A, B)$ is a *normal vector* and the constant C depends on a specific point of the line.

Ex 3. Convert the vector equation of the line $L: \vec{r} = (1,2) + t(3,-2), \quad t \in R$ to the Cartesian equation.

$$\vec{u} = (3,-2) \quad \Rightarrow \quad \vec{n} = (2,3) = (A,B)$$
$$2x + 3y + C = 0$$
$$P_0(1,2) \in L \Rightarrow 2(1) + 3(2) + C = 0 \Rightarrow C = -8$$
$$\therefore L: 2x + 3y - 8 = 0$$

Ex 4. Convert the Cartesian equation to the parametric equations and then to the vector equation.
$L: -x + 2y + 3 = 0$

$$\begin{cases} x = 3 + 2t \\ y = t \end{cases}, \quad t \in R$$
$$\vec{r} = (3,0) + t(2,1), \quad t \in R$$

D Slope y-intercept Equation
Let solve the symmetric equation of a line:

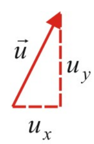
$$\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} = t, \quad t \in R$$

for y:

$$y - y_0 = u_y \frac{x - x_0}{u_x}$$
$$y = \frac{u_y}{u_x} x + y_0 - \frac{u_y}{u_x} x_0$$

The *slope y-intercept equation* of a line in R^2 is given by:

$$y = mx + n$$
$$m = \frac{u_y}{u_x}$$



$m = \frac{u_y}{u_x}$

where m is the *slope* and n is the *y-intercept* which depends on a specific point of the line.

Ex 5. Convert the vector equation to the slope y-intercept equation: $L: \vec{r} = (-2,3) + t(1,-2)$

$$\vec{u} = (1,-2) \Rightarrow m = \frac{u_y}{u_x} = \frac{-2}{1} = -2$$
$$y = -2x + n$$
$$(-2,3) \in L \Rightarrow 3 = (-2)(-2) + n \Rightarrow n = -1$$
$$\therefore y = -2x - 1$$

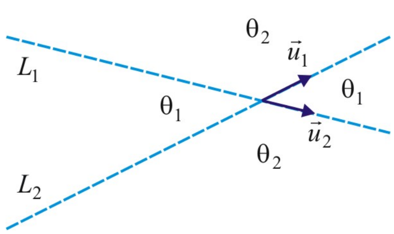
Ex 6. Convert the slope y-intercept equation to the vector equation.
$L: y = -\frac{2}{3}x + 4$.

$$m = \frac{-2}{3} = \frac{u_y}{u_x} \Rightarrow \vec{u} = (3,-2)$$
$$x = 0 \Rightarrow y = 4 \Rightarrow (0,4) \in L$$
$$L: \vec{r} = (0,4) + t(3,-2), \quad t \in R$$

E Angle between two Lines
The *angle* between two lines is determined by the angle between the *direction vectors*:

$$\cos\theta = \frac{\vec{u}_1 \cdot \vec{u}_2}{\|\vec{u}_1\| \|\vec{u}_2\|}$$

Note. There are two pairs of equal angles between two lines (see the figure below).



Note:

$$\theta_1 + \theta_2 = 180°$$

Ex 7. Find the acute angle between each pair of lines.

a) $L_1: \vec{r} = (0,1) + t(1,-3), \quad t \in R; \quad L_2: \frac{x-2}{3} = \frac{y+1}{-2}$

$$\vec{u}_1 = (1,-3), \quad \vec{u}_2 = (3,-2)$$
$$\cos\theta = \frac{\vec{u}_1 \cdot \vec{u}_2}{\|\vec{u}_1\| \|\vec{u}_2\|} = \frac{3+6}{\sqrt{1+9}\sqrt{9+4}} = \frac{9}{\sqrt{10}\sqrt{13}}$$
$$\therefore \theta = \cos^{-1}\frac{9}{\sqrt{10}\sqrt{13}} \cong 37.87°$$

b) $L_1: 2x - 3y + 6 = 0; \quad L_2: y = -\frac{1}{3}x + 2$

$$\vec{n}_1 = (2,-3) \Rightarrow \vec{u}_1 = (3,2)$$
$$m_2 = \frac{1}{-3} = \frac{u_{2y}}{u_{2x}} \Rightarrow \vec{u}_2 = (-3,1)$$
$$\cos\theta = \frac{\vec{u}_1 \cdot \vec{u}_2}{\|\vec{u}_1\| \|\vec{u}_2\|} = \frac{-7}{\sqrt{13}\sqrt{10}} \Rightarrow \theta = \cos^{-1}\frac{-7}{\sqrt{13}\sqrt{10}} \cong 127.87°$$
$$\therefore \theta_1 = 180° - \theta = 180° - 127.87° \cong 52.13°$$

Reading: Nelson Textbook, Pages 435-442
Homework: Nelson Textbook: Page 443 #1, 3, 4, 5, 6, 7, 8, 10ab, 11, 14

