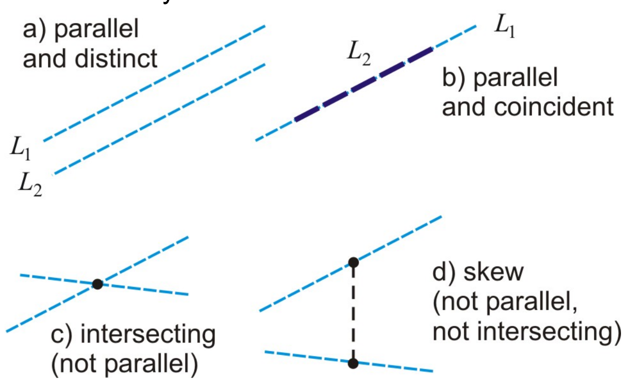
9.1 Intersection of two Lines

A Relative Position of two Lines

Two lines may be:



B Intersection of two Lines (Algebraic Method)

The point of intersection of two lines $L_1 : \vec{r} = \vec{r}_{01} + t\vec{u}_1, \quad t \in R$ and $L_2 : \vec{r} = \vec{r}_{02} + s\vec{u}_2, \quad s \in R$ is given by the *solution* of the following system of equations (if exists):

$$\begin{cases} x_{01} + tu_{x1} = x_{02} + su_{x2} \\ y_{01} + tu_{y1} = y_{02} + su_{y2} \\ z_{01} + tu_{z1} = z_{02} + su_{z2} \end{cases} \quad (*)$$

Hint: Solve by *substitution* or *elimination* the system of two equations and *check* if the third is satisfied.

C Unique Solution

If by solving the system (*), you end by getting a *unique* value for t and s satisfying this system, then the lines have a *unique point of intersection*.
To get this point, substitute either the t value into the line L_1 equation or substitute the s value into the line L_2 equation.

Ex 1. Find the point(s) of intersection of the following two lines. Show that this point is unique.

$L_1 : \vec{r} = (0,1,2) + t(1,-1,2), \quad t \in R$

$L_2 : \vec{r} = (-3,4,-4) + s(0,1,2), \quad s \in R$

$$\begin{cases} t = -3 & (1) \\ 1 - t = 4 + s & (2)\,\&\,(1) \Rightarrow 1 - (-3) = 4 + s \quad (4) \\ 2 + 2t = -4 + 2s & (3) \end{cases}$$

$(4) \Rightarrow 4 = 4 + s \Rightarrow s = 0 \quad (5)$

Let plug-in (1) and (5) into (3) to check it:

$2 + 2(-3) = -4 + 2(0) \Rightarrow -4 = -4 \quad (true)$

The solutions $t = -3$ and $s = 0$ satisfy the system. There is a unique point of intersection. To get it, let plug-in $s = 0$ into $L_2 :\ \vec{r} = (-3,4,-4) + 0(1,-1,2) = (-3,4,-4)$. We get the same point if we plug-in $t = -3$ into line L_1 equation.

$\therefore$ The lines are intersecting at the point $P(-3,4,-4) = L_1 \cap L_2$.

D Infinite Number of Solutions

If by solving the system (*), you end by getting two true statements (like $2 = 2$) and one equation in s and t, then there exist an *infinite number of solutions* of the system (*).
Therefore the lines intersect into an *infinite number of points*.
In this case the lines are *parallel and coincident*.

Ex 2. Find the point(s) of intersection of the following two lines. Show that there are an infinite number of points of intersections and therefore the lines are parallel and coincident.

$L_1 : \vec{r} = t(0,-1,2), \quad t \in R$

$L_2 : \vec{r} = (0,-6,12) + s(0,3,-6), \quad s \in R$

$$\begin{cases} 0 = 0\ (true) & (1) \\ -t = -6 + 3s & (2) \Rightarrow t = 6 - 3s \quad (4) \\ 2t = 12 - 6s & (3)\,\&\,(4) \Rightarrow 2(6 - 3s) = 12 - 6s \quad (5) \end{cases}$$

$(5) \Rightarrow 12 - 6s = 12 - 6s \Rightarrow 0 = 0 \quad (true)$

For any number s, there exists a number t given by (4) satisfying the system of equations. So, there are an infinite number of solutions. Therefore, the lines are parallel and coincident.
You may double check the fact that the lines are parallel by noticing that $\vec{u}_2 = -3\vec{u}_1$.

E No Solution (Parallel Lines)

If by solving the system (*) you get at least one *false* statement (like $0=1$) then the system has *no solution*.
Therefore, the lines have *no point of intersection*.
If, in addition, the lines are *parallel* ($\vec{u}_1 \times \vec{u}_2 = \vec{0}$), then the lines are *parallel and distinct*.

Ex 3. Find the point(s) of intersection of the following two lines. Show that there is no point of intersection and the lines are parallel and distinct.

$L_1 : \vec{r} = (-2,3,1) + t(1,-2,1), \quad t \in R$

$L_2 : \vec{r} = (0,2,1) + s(-2,4,-2), \quad s \in R$

$$\begin{cases} -2+t=-2s & (1) \Rightarrow t = 2-2s \quad (4) \\ 3-2t = 2+4s & (2)\,\&\,(4) \Rightarrow 3-2(2-2s) = 2+4s \quad (5) \\ 1+t = 1-2s & (3) \end{cases}$$

$(5) \Rightarrow 3-4+4s = 2+4s \Rightarrow -1 = 2 \quad (\mathit{false}) \Rightarrow \mathit{no\ solution}$

$\vec{u}_2 = (-2,4,-2) = -2(1,-2,1) = -2\vec{u}_1 \quad (L_1 \parallel L_2)$

The system has no solution and the lines are parallel. Therefore, the lines are parallel and distinct.

F No Solutions (Skew Lines)

If by solving the system (*) you get at least one *false* statement (like $0=1$) then the system has *no solution*. Therefore, the lines have *no point of intersection*.
If, in addition, the lines are *not parallel* ($\vec{u}_1 \times \vec{u}_2 \neq \vec{0}$), then the lines are *skew*.

Ex 4. Find the point(s) of intersection of the following two lines. Show that there is no point of intersection and the lines not parallel, therefore the lines are skew.

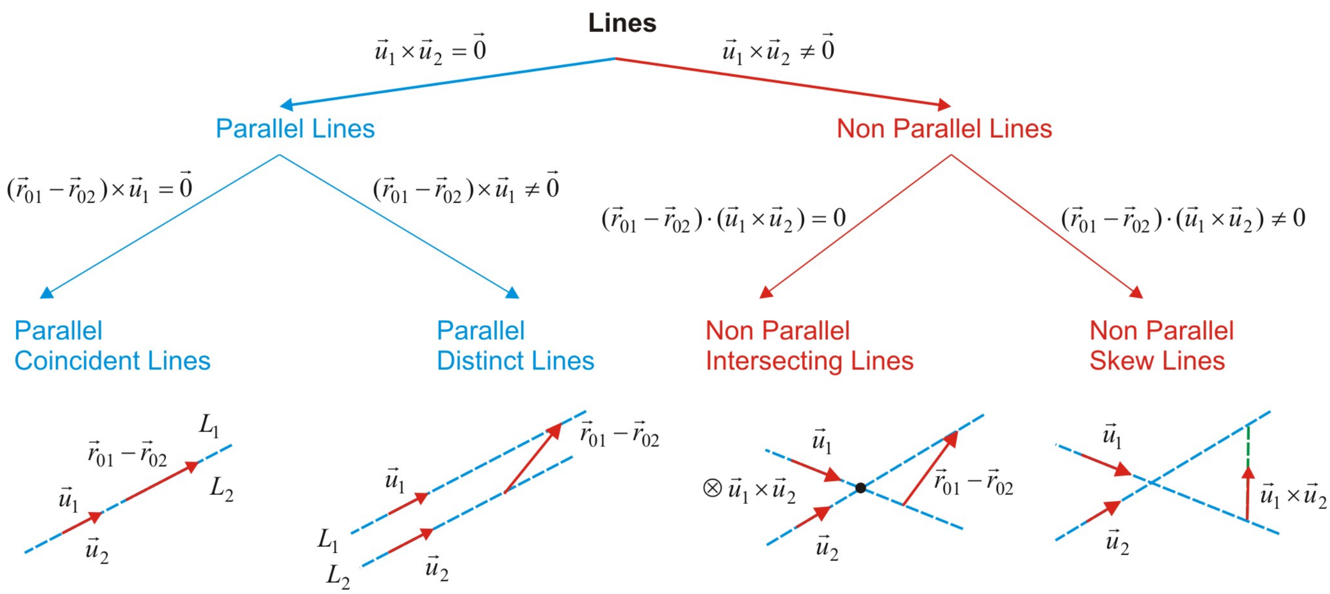
$L_1 : \vec{r} = (1,-1,0) + t(0,0,1), \quad t \in R$

$L_2 : \vec{r} = (-2,1,0) + s(1,0,0), \quad s \in R$

$$\begin{cases} 1 = -2 & (1) \quad (\mathit{false}) \\ -1 = 1 & (2) \quad (\mathit{false}) \\ t = 0 & (3) \end{cases}$$

There is no solution. The lines are not parallel ($\vec{u}_1 \times \vec{u}_2 = \vec{k} \times \vec{i} = \vec{j} \neq \vec{0}$).
Therefore the lines are skew.

G Classifying Lines (Vector Method)



Ex 5. Use the vector method presented above to classify each pair of lines as parallel and distinct, parallel and coincident, not parallel and intersecting or not parallel and skew.

a) $L_1 : \vec{r} = (0,1,2) + t(1,2,3), \quad t \in R$
$L_2 : \vec{r} = (-2,-1,0) + s(-2,-4,-6), \quad s \in R$

$\vec{u}_1 = (1,2,3), \quad \vec{u}_2 = (-2,-4,-6) \quad \Rightarrow \quad \vec{u}_2 = -2\vec{u}_1 \quad (L_1 \parallel L_2)$

$\vec{r}_{01} - \vec{r}_{02} = (0,1,2) - (-2,-1,0) = (2,2,2) \quad \Rightarrow \quad \vec{r}_{01} - \vec{r}_{02} \neq \lambda \vec{u}_1$

Therefore, the lines L_1 and L_2 are parallel and distinct.

b) $L_1 : \vec{r} = t(1,-1,0), \quad t \in R$
$L_2 : \vec{r} = (-4,4,0) + s(3,-3,0), \quad s \in R$

$\vec{u}_1 = (1,-1,0), \quad \vec{u}_2 = (3,-3,0) \quad \Rightarrow \quad \vec{u}_2 = 3\vec{u}_1 \quad (L_1 \parallel L_2)$

$\vec{r}_{01} - \vec{r}_{02} = (0,0,0) - (-4,4,0) = (4,-4,0) \quad \Rightarrow \quad \vec{r}_{01} - \vec{r}_{02} = 4\vec{u}_1$

Therefore, the lines L_1 and L_2 are parallel and coincident.

c) $L_1 : \vec{r} = (2,1,3) + t(0,1,2), \quad t \in R$
$L_2 : \vec{r} = (0,1,-1) + s(1,0,2), \quad s \in R$

$\vec{u}_1 \neq \lambda \vec{u}_2$

$$\vec{u}_1 \times \vec{u}_2 = \begin{matrix} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 0 & 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 & 0 \end{matrix} = (2,2,-1)$$

$\vec{r}_{01} - \vec{r}_{02} = (2,1,3) - (0,1,-1) = (2,0,4)$

$(\vec{r}_{01} - \vec{r}_{02}) \cdot (\vec{u}_1 \times \vec{u}_2) = 4 - 4 = 0$

Therefore, the lines L_1 and L_2 are not parallel and intersecting.

d) $L_1 : \vec{r} = (1,0,0) + t(0,0,1), \quad t \in R$
$L_2 : \vec{r} = (0,1,0) + s(1,0,0), \quad s \in R$

$\vec{u}_1 \neq \lambda \vec{u}_2$

$$\vec{u}_1 \times \vec{u}_2 = \begin{matrix} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{matrix} = (0,1,0)$$

$\vec{r}_{01} - \vec{r}_{02} = (1,0,0) - (0,1,0) = (1,-1,0)$

$(\vec{r}_{01} - \vec{r}_{02}) \cdot (\vec{u}_1 \times \vec{u}_2) = -1 \neq 0$

Therefore, the lines L_1 and L_2 are not parallel and skew.

Ex 6. Prove that (if exists) the point of intersection between two lines $L_1 : \vec{r} = \vec{r}_{01} + t\vec{u}_1, \quad t \in R$ and $L_2 : \vec{r} = \vec{r}_{02} + s\vec{u}_2, \quad s \in R$ is given by the vector formula:

$$\vec{r} = \vec{r}_{01} + \frac{[(\vec{r}_{02} - \vec{r}_{01}) \times \vec{u}_2] \cdot (\vec{u}_1 \times \vec{u}_2)}{\| \vec{u}_1 \times \vec{u}_2 \|^2} \vec{u}_1$$

Indeed:

$\vec{r}_{01} + t\vec{u}_1 = \vec{r}_{02} + s\vec{u}_2 \quad | \times \vec{u}_2$

$t(\vec{u}_1 \times \vec{u}_2) = (\vec{r}_{02} - \vec{r}_{01}) \times \vec{u}_2 \quad | \cdot (\vec{u}_1 \times \vec{u}_2)$

$t(\vec{u}_1 \times \vec{u}_2) \cdot (\vec{u}_1 \times \vec{u}_2) = [(\vec{r}_{02} - \vec{r}_{01}) \times \vec{u}_2] \cdot (\vec{u}_1 \times \vec{u}_2)$

$$t = \frac{[(\vec{r}_{02} - \vec{r}_{01}) \times \vec{u}_2] \cdot (\vec{u}_1 \times \vec{u}_2)}{\| \vec{u}_1 \times \vec{u}_2 \|^2}$$

$$\therefore \vec{r} = \vec{r}_{01} + \frac{[(\vec{r}_{02} - \vec{r}_{01}) \times \vec{u}_2] \cdot (\vec{u}_1 \times \vec{u}_2)}{\| \vec{u}_1 \times \vec{u}_2 \|^2} \vec{u}_1$$

Reading: Nelson Textbook, Pages 489-496
Homework: Nelson Textbook: Page 497 #8, 9, 10, 11, 12, 13, 14, 15, 18

