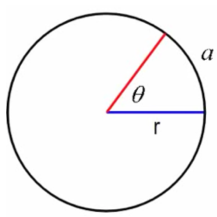
Unit 4 – Trigonometry

Lesson 1: Radian measure for angles and Radian angles in Cartesian plane

Part I: Radian measure for angles

Radian Measure:



An angle measurement can be defined as the ratio of the arc length to the radius of a circle: $\theta = \frac{a}{r}$

For a full circle, the arc length is the circumference: $C = 2\pi r$,

Therefore, the angle described by a full circle, 360° is: $360° = 2\pi$

$\because 360° = 2\pi$ and $180° = \pi$

$\therefore \mathbf{1° = \frac{\pi}{180°}}$ ***or 1 radian*** $\mathbf{= \frac{180°}{\pi}}$

Example 1: Convert each of the following angle

a) $20°$

b) $225°$

c) $\frac{5\pi}{6}$

d) 1.75 radian

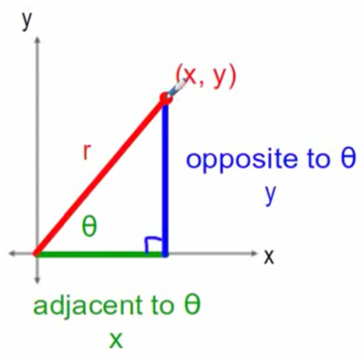
Part II: Radian angles in Cartesian plane

Recall: For any angle of interest (θ), there are three primary trigonometric ratios and three reciprocal trigonometric ratios.

$sine\ of\ \theta = \frac{opposite}{hypotenuse}$, $cosine\ of\ \theta = \frac{adjacent}{hypotenuse}$, $tangent\ of\ \theta = \frac{opposite}{adjacent}$

$cosecant\ of\ \theta =$ $secant\ of\ \theta =$ $cotangent\ of\ \theta =$

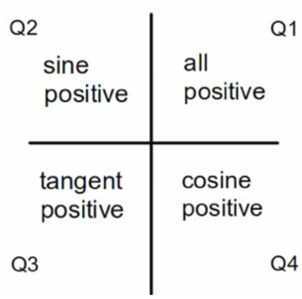
An angle is in standard position if the vertex is at the origin and the initial arm is along the positive x-axis. This angle can be described in terms of the point (x, y) at the end of the terminal arm.

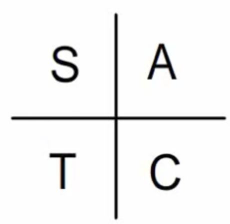


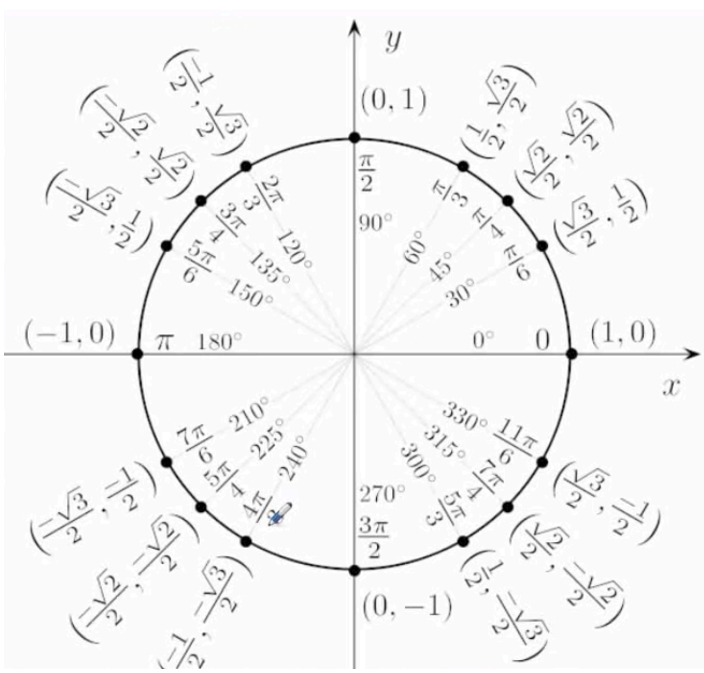
where: $r^2 = x^2 + y^2$

$sin\theta = \frac{y}{r}$ $\qquad cos\theta =$ $\qquad tan\theta =$

$csc\theta =$ $\qquad sec\theta =$ $\qquad cot\theta =$







The related acute angle (RAA) is the positive, acute angle between the nearest x-axis and the terminal arm.

Example 2: Evaluate using CAST rule, RA, and special triangles.

a) $sin\frac{5\pi}{3}$

b) $sec\,\frac{5\pi}{4}$

Example 3: Solve $tan\theta = -\frac{7}{24}$ for θ is between -2 π and 2 π.

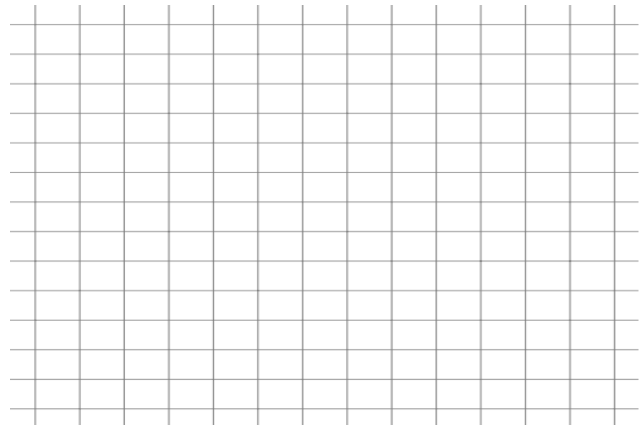
Unit 4 – Trigonometry

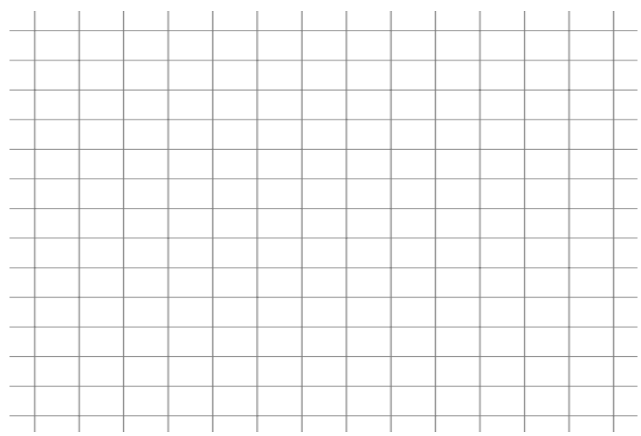
Lesson 2: Transformation of Trigonometric functions and sine/cosine/tangent graphs

Graphing from key points

For sine and cosine, use points from the x- and y-axes on the unit circle.

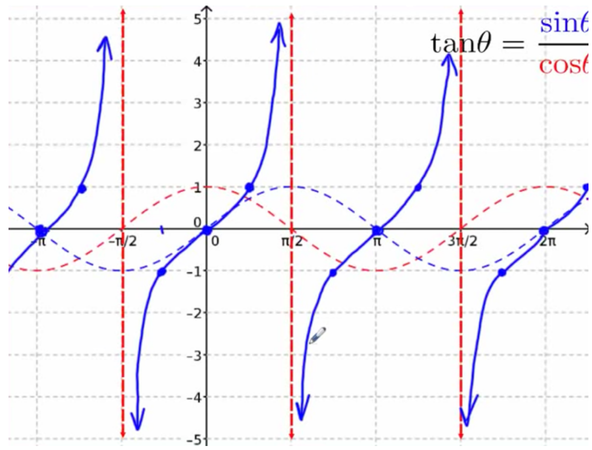
$\theta \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$





For tangent, use a cycle between two vertical asymptotes:

$$\theta \in \{-\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}\}$$

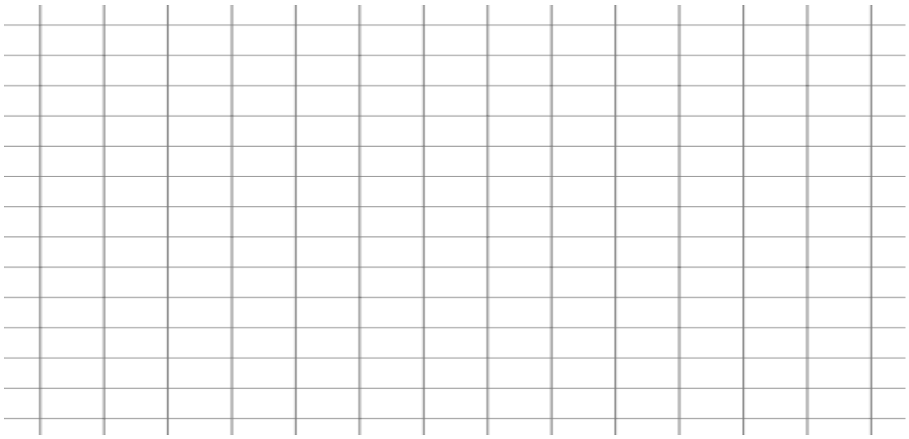


Summary:

Characteristics	Sine	Cosine	Tangent
Domain			
Range			
Zeros			
Asymptotes			
Period			

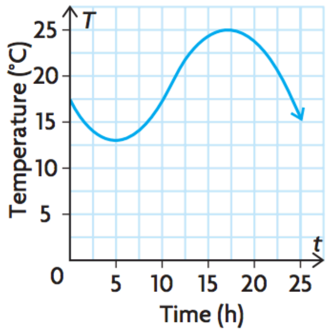
Example 1:

A mass on a spring is pulled toward the floor and released, causing it to move up and down. Its height, in centimetres, above the floor after t seconds is given by the function $h(t) = 10 \sin(2\pi t + 1.5\pi) + 15, 0 \leq t \leq 3$. Sketch a graph of height versus time. Then use your graph to predict when the mass will be 18 cm above the floor as it travels in an upward direction.



Example 2:

The following graph shows the temperature in Nellie's dorm room over a 24 h period.

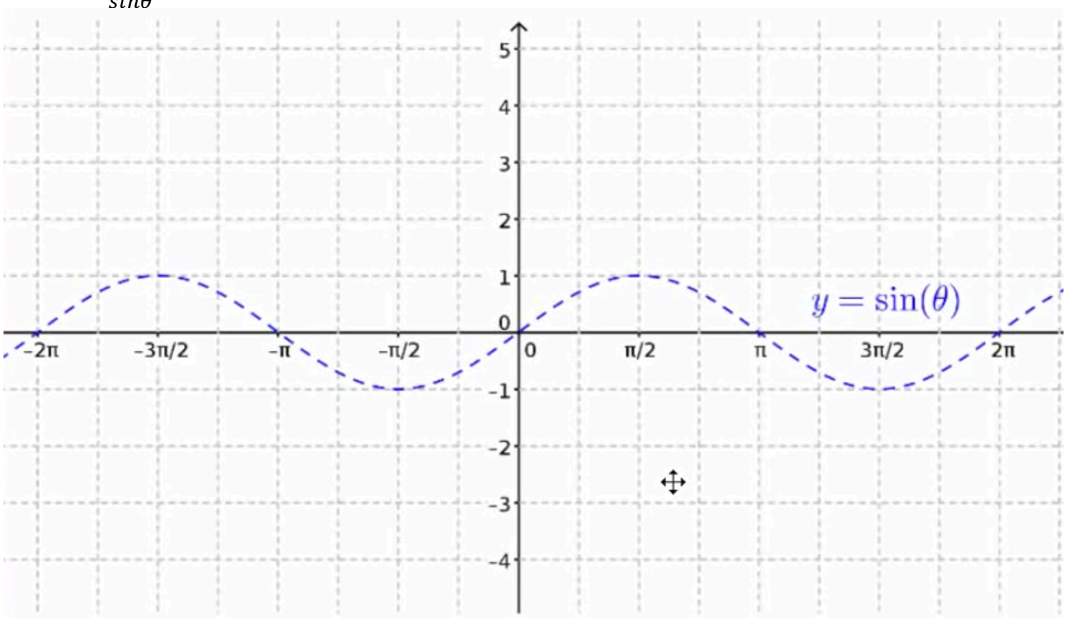


Determine the equation of this sinusoidal function.

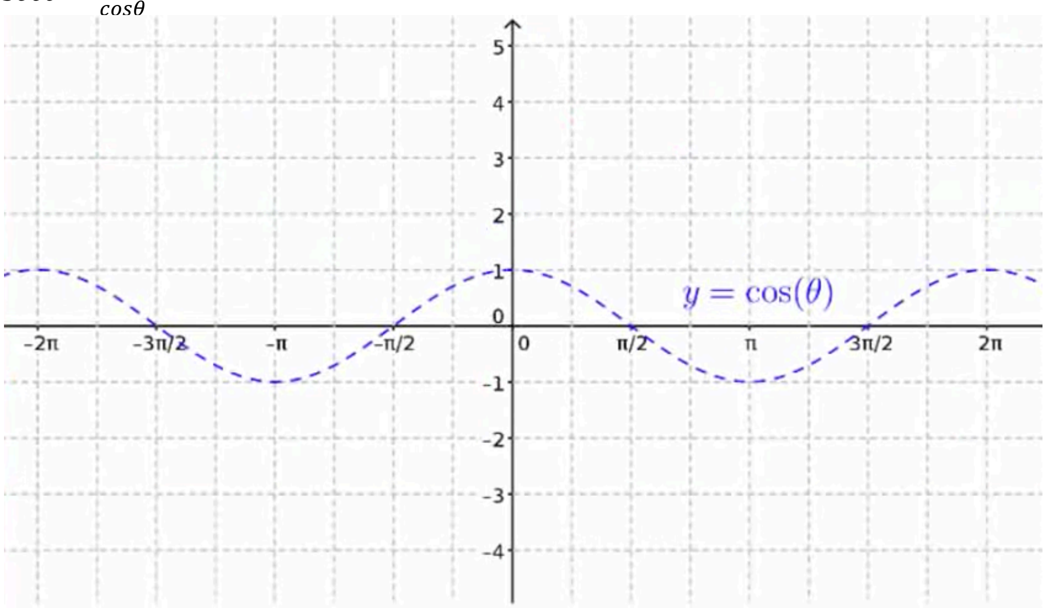
Unit 4 – Trigonometry

Lesson 3: Graphs of reciprocal Trig functions

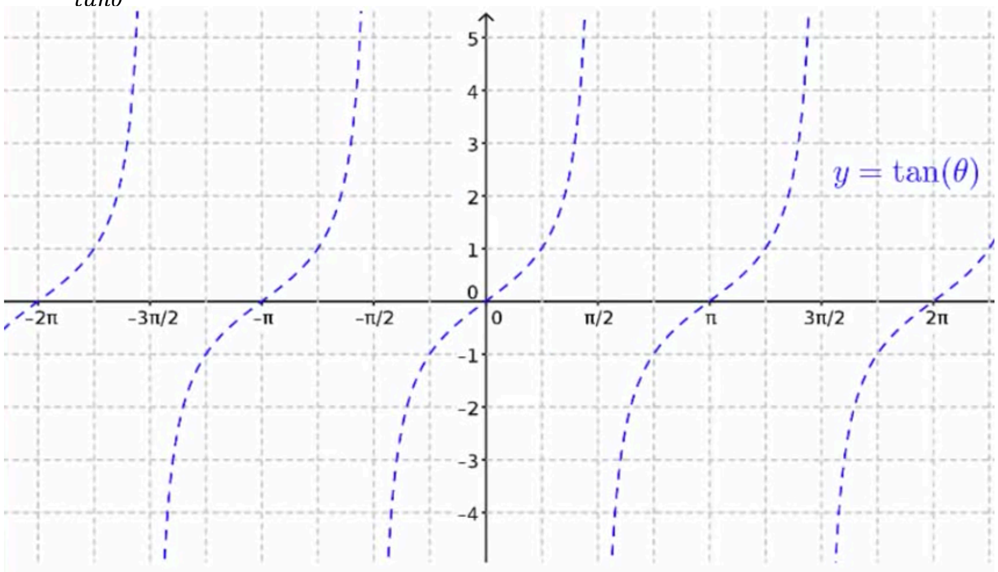
Graph: $csc\theta = \frac{1}{sin\theta}$



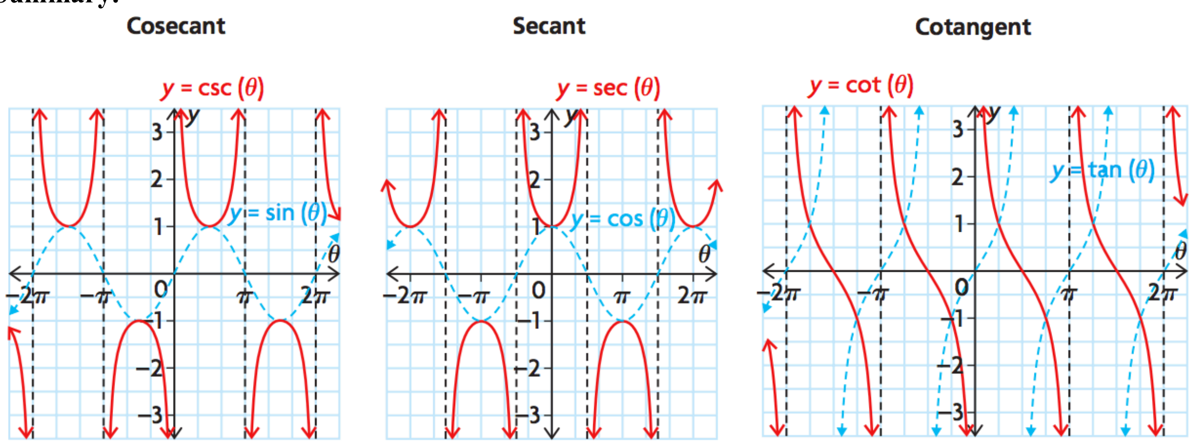
Graph: $sec\theta = \frac{1}{cos\theta}$



Graph: $cot\theta = \frac{1}{tan\theta}$



Summary:



Cosecant

- has vertical asymptotes at the points where $\sin\theta = 0$
- has the same period (2π) as $y = \sin\theta$
- has the domain $\{x \in \mathbf{R} \,|\, \theta \neq n\pi, n \in \mathbf{I}\}$
- has the range $\{y \in \mathbf{R} \,|\, |y| \geq 1\}$

Secant

- has vertical asymptotes at the points where $\cos\theta = 0$
- has the same period (2π) as $y = \cos\theta$
- has the domain $\{x \in \mathbf{R} \,|\, \theta \neq (2n - 1)\frac{\pi}{2}, n \in \mathbf{I}\}$
- has the range $\{y \in \mathbf{R} \,|\, |y| \geq 1\}$

Cotangent

- has vertical asymptotes at the points where $\tan\theta = 0$
- has zeros at the points where $y = \tan\theta$ has asymptotes
- has the same period (π) as $y = \tan\theta$
- has the domain $\{x \in \mathbf{R} \,|\, \theta \neq n\pi, n \in \mathbf{I}\}$
- has the range $\{y \in \mathbf{R}\}$

Unit 4 – Trigonometry
<u>Lesson 4: Modelling with Trigonometric Functions</u>

Example 1:

The tides at Cape Capstan, New Brunswick, change the depth of the water in the harbour. On one day in October, the tides have a high point of approximately 10 m at 2 p.m. and a low point of approximately 1.2 m at 8:15 p.m. A particular sailboat has a *draft* of 2 m. This means it can only move in water that is at least 2 m deep. The captain of the sailboat plans to exit the harbour at 6:30 p.m.

Can the captain exit the harbor safely in the sailboat at 6 p.m?

Example 2:

At one time, Maple Leaf Village (which no longer exists) had North America's largest Ferris wheel. The Ferris wheel had a diameter of 56 m, and one revolution took 2.5 min to complete. Riders could see Niagara Falls if they were higher than 50 m above the ground. Sketch three cycles of a graph that represents the height of a rider above the ground, as a function of time, if the rider gets on at a height of 0.5 m at $t = 0$ min. Then determine the time intervals when the rider could see Niagara Falls.

Unit 5 – Trigonometric Identities and Equations

Lesson 1: Equivalent Trigonometric Functions

Due to the periodic nature of trigonometric functions, there are multiple (infinite) ways to express equivalent functions.

1) Using the period:

   Both sine and cosine have a period of 2π, which means any phase shift by a multiple of the period will be equivalent.

   $sinx = sin(x + 2\pi) = sin\,(x - 2\pi)$

   $cosx = cos(x + 2\pi) = cos\,(x - 2\pi)$

   $cscx = csc(x + 2\pi) = csc\,(x - 2\pi)$

   $secx = sec(x + 2\pi) = sec\,(x - 2\pi)$

   similarly, for tangent and cotangent, $tanx = tan(x + \pi) = tan\,(x - \pi)$ and $cotx = cot(x + \pi) = cot\,(x - \pi)$

2) By symmetry:

   Recall, even functions: $f(x) = f(-x)$

   odd functions: $f(-x) = -f(x)$

   Cosine is even (reflective symmetry across the y-axis): $cosx = cos(-x)$

   Sine and tangent are odd (rotational symmetry): $sin(-x) = -sinx$

   $tan(-x) = -tanx$

3) Using C.A.S.T rule:

   Recall: you can identify equivalent trigonometric expressions by comparing principal angles drawn in standard position in quadrants II, III, and IV with their related acute angle, θ, in quadrant I.

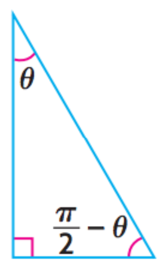
Principal Angle in Quadrant II	Principal Angle in Quadrant III	Principal Angle in Quadrant IV
$\sin(\pi - \theta) = \sin\theta$	$\sin(\pi + \theta) = -\sin\theta$	$\sin(2\pi - \theta) = -\sin\theta$
$\cos(\pi - \theta) = -\cos\theta$	$\cos(\pi + \theta) = -\cos\theta$	$\cos(2\pi - \theta) = \cos\theta$
$\tan(\pi - \theta) = -\tan\theta$	$\tan(\pi + \theta) = \tan\theta$	$\tan(2\pi - \theta) = -\tan\theta$

4) Using complimentary angles:

Recall: Complimentary angles add to $\frac{\pi}{2}$ (or 90°)

$sin\frac{\pi}{3} =$	$cos\frac{\pi}{6} =$	$csc\frac{\pi}{3} =$	$sec\frac{\pi}{6} =$
$cos\frac{\pi}{3} =$	$sin\frac{\pi}{6} =$	$sec\frac{\pi}{3} =$	$csc\frac{\pi}{6} =$
$tan\frac{\pi}{3} =$	$cot\frac{\pi}{6} =$	$cot\frac{\pi}{3} =$	$tan\frac{\pi}{6} =$

Hence: The above pattern is defined by <u>Cofunction Identities</u> by which describe trigonometric relationships between the complementary angles θ and $(\frac{\pi}{2} - \theta)$ in a right triangles.



$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

$$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$$

Example 1: Use R.A.A and cofunction identities to write an expression that is equivalent to each of the following expressions.

a) $\sec\frac{2\pi}{3} =$

b) $\tan\frac{7\pi}{6} =$

Practice:

Use the cofunction identities to write an expression that is equivalent to each of the following expressions.

a) $\sin \frac{\pi}{6}$

b) $\cos \frac{5\pi}{12}$

c) $\tan \frac{3\pi}{8}$

d) $\cos \frac{5\pi}{16}$

e) $\sin \frac{\pi}{8}$

f) $\tan \frac{\pi}{6}$

Write an expression that is equivalent to each of the following expressions, using the related acute angle.

a) $\sin \frac{7\pi}{8}$

b) $\cos \frac{13\pi}{12}$

c) $\tan \frac{5\pi}{4}$

d) $\cos \frac{11\pi}{6}$

e) $\sin \frac{13\pi}{8}$

f) $\tan \frac{5\pi}{3}$

Unit 5 – Trigonometric Identities and Equations

Lesson 2: Compound angle formulas & Double angle formula

Part I: Compound angle formulas

Key Idea

- The trigonometric ratios of compound angles are related to the trigonometric ratios of their component angles by the following compound angle formulas.

Addition Formulas

$\sin(a+b) = \sin a \cos b + \cos a \sin b$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$\tan(a+b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$

Subtraction Formulas

$\sin(a-b) = \sin a \cos b - \cos a \sin b$

$\cos(a-b) = \cos a \cos b + \sin a \sin b$

$\tan(a-b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$

Need to Know

- You can use compound angle formulas to obtain exact values for trigonometric ratios.
- You can use compound angle formulas to show that some trigonometric expressions are equivalent.

Example 1: Simplify each expression

a) $cos\frac{7\pi}{12}cos\frac{5\pi}{12} + sin\frac{7\pi}{12}sin\frac{5\pi}{12}$

b) $sin2xcosx - cos2xsinx$

Example 2: Determine the exact value of

a) $cos\ (15°)$

b) $tan\ (-\frac{5\pi}{12})$

Example 3: Evaluate $sin\ (a+b)$, where $sina = \frac{3}{5}$ and $sinb = \frac{5}{13}$, if a is in quadrant 1 and b is in quadrant 2.

Part II: Double angle formula

Key Idea

- The double angle formulas show how the trigonometric ratios for a double angle, 2θ, are related to the trigonometric ratios for the original angle, θ.

Double Angle Formula for Sine

$\sin 2\theta = 2 \sin \theta \cos \theta$

Double Angle Formulas for Cosine

$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$

$\cos 2\theta = 2 \cos^2 \theta - 1$

$\cos 2\theta = 1 - 2 \sin^2 \theta$

Double Angle Formula for Tangent

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

Need to Know

- The double angle formulas can be derived from the appropriate compound angle formulas.
- You can use the double angle formulas to simplify expressions and to calculate exact values.
- The double angle formulas can be used to develop other equivalent formulas.

Example 4: Simplify each of the following expressions and them evaluate

a) $2\sin\frac{\pi}{8}cos\frac{\pi}{8}$

b) $\frac{2tan\frac{\pi}{6}}{1-tan^2\frac{\pi}{6}}$

Example 5: If $cos\theta = -\frac{2}{3}\ and\ 0 \leq \theta \leq 2\pi$, determine the value of $cos2\theta$ and $sin2\theta$.

Practice: If $tan\theta = -\frac{3}{4}$, where $\frac{3\pi}{2} \leq \theta \leq 2\pi$, calculate the value of $cos2\theta$.

Example 6: Develop a formula for $sin\frac{x}{2}$, $\cos\frac{x}{2}$, $and\ tan\frac{x}{2}$. (Half angle formula)

Practice:

Jim needs to find the sine of $\frac{\pi}{8}$. If he knows that $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, how can he use this fact to find the sine of $\frac{\pi}{8}$? What is his answer?

Marion needs to find the cosine of $\frac{\pi}{12}$. If she knows that $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, how can she use this fact to find the cosine of $\frac{\pi}{12}$? What is her answer?

Unit 5 – Trigonometric Identities and Equations
Lesson 3: Proving Trigonometric Identities

Need to Know

- The following trigonometric identities are important for you to remember:

Identities Based on Definitions

Reciprocal Identities

$$\csc x = \frac{1}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{1}{\tan x}$$

Identities Derived from Relationships

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Double Angle Formulas

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= 2\cos^2 x - 1$$

$$= 1 - 2\sin^2 x$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

Addition and Subtraction Formulas

$$\sin (x + y) = \sin x \cos y + \cos x \sin y$$

$$\sin (x - y) = \sin x \cos y - \cos x \sin y$$

$$\cos (x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos (x - y) = \cos x \cos y + \sin x \sin y$$

$$\tan (x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan (x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Example 1: Prove that $\frac{\cos(x-y)}{\cos(x+y)} = \frac{1+tanxtany}{1-tanxtany}$

Example 2: Prove that $tan2x - 2tan2xsin^2x = sin2x$

Practice: Textbook pg417 – 418.

8. Prove that $\frac{1 + \tan x}{1 + \cot x} = \frac{1 - \tan x}{\cot x - 1}$.

9. Prove each identity.

a) $\frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta + \sin \theta \cos \theta} = 1 - \tan \theta$

b) $\tan^2 x - \sin^2 x = \sin^2 x \tan^2 x$

c) $\tan^2 x - \cos^2 x = \frac{1}{\cos^2 x} - 1 - \cos^2 x$

d) $\frac{1}{1 + \cos \theta} + \frac{1}{1 - \cos \theta} = \frac{2}{\sin^2 \theta}$

10. Prove each identity.

a) $\cos x \tan^3 x = \sin x \tan^2 x$

b) $\sin^2 \theta + \cos^4 \theta = \cos^2 \theta + \sin^4 \theta$

c) $(\sin x + \cos x)\left(\dfrac{\tan^2 x + 1}{\tan x}\right) = \dfrac{1}{\cos x} + \dfrac{1}{\sin x}$

d) $\tan^2 \beta + \cos^2 \beta + \sin^2 \beta = \dfrac{1}{\cos^2 \beta}$

e) $\sin\left(\dfrac{\pi}{4} + x\right) + \sin\left(\dfrac{\pi}{4} - x\right) = \sqrt{2} \cos x$

f) $\sin\left(\dfrac{\pi}{2} - x\right) \cot\left(\dfrac{\pi}{2} + x\right) = -\sin x$

11. Prove each identity.

T

a) $\dfrac{\cos 2x + 1}{\sin 2x} = \cot x$

b) $\dfrac{\sin 2x}{1 - \cos 2x} = \cot x$

c) $(\sin x + \cos x)^2 = 1 + \sin 2x$

d) $\cos^4 \theta - \sin^4 \theta = \cos 2\theta$

e) $\cot \theta - \tan \theta = 2 \cot 2\theta$

f) $\cot \theta + \tan \theta = 2 \csc 2\theta$

g) $\dfrac{1 + \tan x}{1 - \tan x} = \tan\left(x + \dfrac{\pi}{4}\right)$

h) $\csc 2x + \cot 2x = \cot x$

i) $\dfrac{2 \tan x}{1 + \tan^2 x} = \sin 2x$

j) $\sec 2t = \dfrac{\csc t}{\csc t - 2 \sin t}$

k) $\csc 2\theta = \dfrac{1}{2}(\sec \theta)(\csc \theta)$

l) $\sec t = \dfrac{\sin 2t}{\sin t} - \dfrac{\cos 2t}{\cos t}$

Unit 5 – Trigonometric Identities and Equations

Lesson 4: Solving Linear and quadratic trigonometric equations

Example: Solve each equation for $0 \leq x \leq 2\pi$.

a) $cos2x = 2sinxcosx$

b) $3sinx + 3cos2x = 2$