MCR3U Unit 1: Introduction to functions
Lesson 1.3: Transformation of functions

Learning Goals:
I will be able to:

- **perform in the correct order, transformations on parent functions to obtain a new function**
- **recognize the transformations that have occurred to result in a given function**

Summary of Transformations

You will recall that we have the following basic ("parent") functions:

$f(x) = x$ → A (non-vertical) straight line
$f(x) = x^2$ → A parabola that opens upwards and has it vertex at the origin.
$f(x) = \sqrt{x}$ → The upper half of a sideways parabola with its vertex at the origin.
$f(x) = \frac{1}{x}$ → A hyperbola with asymptotes at x = 0 and y = 0. Only in quadrants 1 & 3.
$f(x) = |x|$ → A linear relationship with negative slope to the left of the origin and a positive slope to the right of the origin.

Transformations turn one of the parent graphs, $y = f(x)$ into a new graph whose equation can be written in the form

$$y = af[k(x - d)] + c$$

where $f(x)$ is one of the parent functions listed above. The transformations can be determined by examining the various parameters in this equation.

Parameter	Value	Effect on Graph
a	negative ($a < 0$)	Reflected over the x-axis
	$0 < \|a\| < 1$	Vertical compression
	$\|a\| > 1$	Vertical stretch
k (NEW!!!)	negative ($k < 0$)	Reflected over the y-axis
	$0 < \|k\| < 1$	Horizontal *stretch* by the factor $\frac{1}{\|k\|}$
	$\|k\| > 1$	Horizontal *compression* by the factor $\frac{1}{\|k\|}$
d (was h for a quadratic)	positive ($d > 0$)	Horizontal shift (translation) d units right
	negative ($d < 0$)	Horizontal shift d units left
c (was k for a quadratic)	positive ($c > 0$)	Vertical shift (translation) c unit up
	negative ($c < 0$)	Vertical shift c units down

What if more than one of the above transformations is to be applied? Does the order in which they are applied matter? Yes, it does. Note the following:

1. Horizontal and vertical transformations are *independent of each other*. The parameters k and d affect *only* the horizontal characteristics of the graph; the parameters a and c affect only the vertical characteristics of the graph (direction of opening, stretch/compression, and position). This means that you can apply the horizontal transformations (if any) and then the vertical ones, or vice-versa.

2. When working with either vertical or horizontal transformations, always apply the translations *last*. This means that you must apply k before d, and a before c.

For an example which applies all the transformations in one question, see Example 1 on page 61.

The function defined by $y = af[k(x-d)] + c$ describes the transformation of the graph of $f(x)$. How the equation for the transformed function looks depends on the parent function:

When $f(x) = x$ → transformed equation is $y = a(k(x-d)) + c$
When $f(x) = x^2$ → transformed equation is $y = a(k(x-d))^2 + c$
When $f(x) = \sqrt{x}$ → transformed equation is $y = a\sqrt{k(x-d)} + c$
When $f(x) = \frac{1}{x}$ → transformed equation is $y = \frac{a}{k(x-d)} + c$
When $f(x) = |x|$ → transformed equation is $y = a|k(x-d)| + c$

Important Notes:

- Before transformations can be applied, you *must* make the x coefficient equal to 1 (if it isn't already) by factoring the expression inside the $f()$ brackets to obtain the correct value for d:

  $$y = af[k(x-d)] + c$$

  ↖ k is factored out; x coefficient is equal to 1

  So, for example, if we had the equation

  $y = f(2x-6)$ ← the x coefficient is 2, not 1; the d value is *not* 6

  we would first factor the expression inside the brackets to get

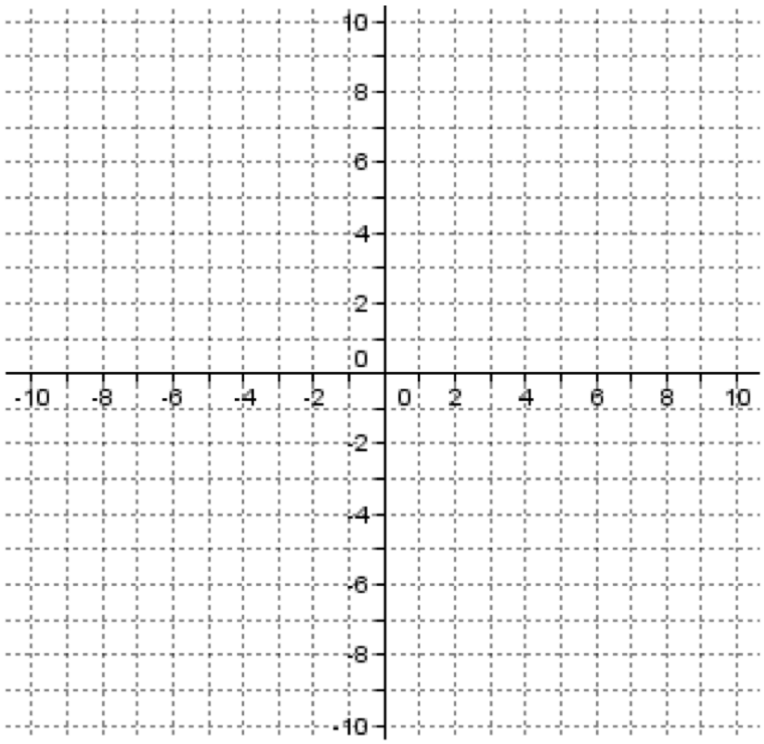
  $y = f[2(x-3)]$ ← the x coefficient is now equal to 1; $d = 3$

  before proceeding to do any transformations. Note that sometimes doing the factoring will cause d to be a fraction.

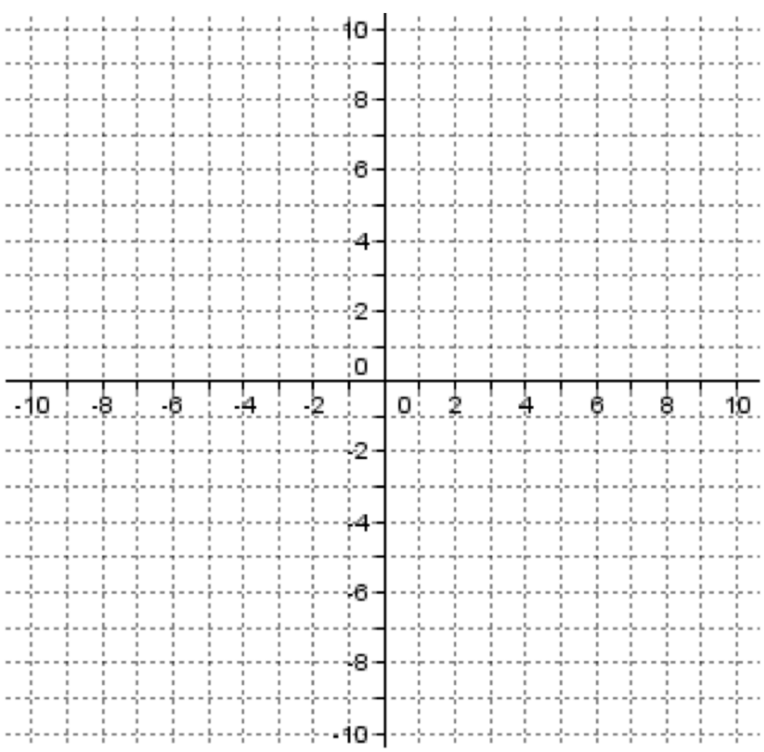
- When graphing by hand, a more accurate graph can be drawn by plotting:
  - the point(s) where the line or curve intersects the x or y axis and/or
  - points on the line or curve which have x and y coordinates which are integers or simple decimals which can be plotted accurately given the scale you are using (ex. (5, 8) is a point which is easy to plot accurately, while (5.8347, 7.4920) is not).

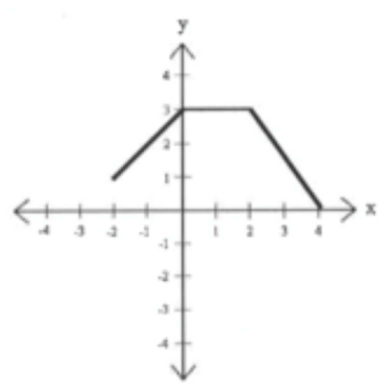
Let's try ourselves: Draw and list the steps of transformation required.

1) $f(x) = -2\left|\frac{1}{3}(x+2)\right| + 1$



2) $y = \frac{1}{2}f[-(x+1)] - 2$





Homework: Read textbook pg67. example 4 and summary on pg69. Work on pg70 – 73. #1, 2, 7a, 8bc, 9bc, 11, 12, 13, 15, 16, 17, 18, 19, 20,