Course: MDM4U Teacher: Ms. Ella

Unit 2: Combination

Lesson 2.3: Binomial Theorem by using combination notations

Recall: We found patterns in Pascal's Triangle → the same patterns with combinations.

1	Row 0	$t_{0,0}$	${}_0C_0$
1 1	Row 1	$t_{1,0}$ $t_{1,1}$	${}_1C_0$ ${}_1C_1$
1 2 1	Row 2	$t_{2,0}$ $t_{2,1}$ $t_{2,2}$	${}_2C_0$ ${}_2C_1$ ${}_2C_2$
1 3 3 1	Row 3	$t_{3,0}$ $t_{3,1}$ $t_{3,2}$ $t_{3,3}$	${}_3C_0$ ${}_3C_1$ ${}_3C_2$ ${}_3C_3$
1 4 6 4 1	Row 4	$t_{4,0}$ $t_{4,1}$ $t_{4,2}$ $t_{4,3}$ $t_{4,4}$	${}_4C_0$ ${}_4C_1$ ${}_4C_2$ ${}_4C_3$ ${}_4C_4$
1 5 10 10 5 1	Row 5	$t_{5,0}$ $t_{5,1}$ $t_{5,2}$ $t_{5,3}$ $t_{5,4}$ $t_{5,5}$	${}_5C_0$ ${}_5C_1$ ${}_5C_2$ ${}_5C_3$ ${}_5C_4$ ${}_5C_5$

- Comparing these triangles we see ___________ ____
- Since $t_{n,r} = t_{n-1,r-1} + t_{n-1,r}$, we can see ____ ____

The Binomial Theorem: Recall that a binomial is a polynomial with just ______ terms, so it has the form ___________. Expanding $(a+b)^n$ is easy for small values of n, but becomes very laborious as n increases.

Look at the coefficients of each term. Do you notice a pattern?

For n = 0, $(a+b)^0 = 1$

For n = 1, $(a+b)^1 = a+b$

For n = 2,
$$(a+b)^2 = (a+b)(a+b) = a^2+ab+ab+b^2 = a^2+2ab+b^2$$

For n = 3,
$$(a+b)^3 = (a+b)(a+b)^2 = (a+b)(a^2+2ab+b^2) = a^3+2a^2b+ab^2+a^2b+2ab^2+b^3 = a^3+3a^2b+3ab^2+b^3$$

There is a relation between expanding binomials and ____________________________________.

Notice that in each successive term, the exponents on the "a" decrease by one, while the exponents on the "b" increase by one.

The coefficients of each term in the expansion of $(a+b)^n$ correspond to the terms in row n of Pascal's triangle. We can write these coefficients in combinational form using the Binomial Theorem.

$$(x+y)^n = \sum_{r=0}^{n} {}_nC_r x^{n-r}y^r = {}_nC_0x^ny^0 + {}_nC_1x^{n-1}y^1 + {}_nC_2x^{n-2}y^2 + \ldots + {}_nC_rx^{n-r}y^r + \ldots + {}_nC_{n-1}x^1y^{n-1} + {}_nC_nx^0y^n$$

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Example 1: Expand $(2a-3b)^4$

First term: ${}_4C_0(2a)^{4-0}(-3b)^0 = (2a)^4 = 16a^4$
Second Term: ${}_4C_1(2a)^{4-1}(-3b)^1 = 4(2a)^3(-3b) = 4(8a^3)(-3b) = -96a^3b$
Third Term: ${}_4C_2(2a)^{4-2}(-3b)^2 = 6(2a)^2(-3b)^2 = 6(4a^2)(9b^2) = 216a^2b^2$
Fourth Term: ${}_4C_3(2a)^{4-3}(-3b)^3 = 4(2a)(-3b)^3 = 4(2a)(-27b^3) = -216ab^3$
Fifth Term: ${}_4C_4(2a)^{4-4}(-3b)^4 = (-3b)^4 = 81b^4$
Full Expansion: $16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4$

Find the particular term: $\boldsymbol{t_{k+1} = {}_nC_k x^{n-k} y^k}$

Example 2: Given $(3x-4)^8$, determine the middle term of the expansion.

Example 3: Given $(5x-2y)^9$. Find the coefficient of the term containing x^5.

Example 4: Given $(2x^4-2y^2)^5$, find the coefficient of the term containing x^{12}.

Example 5: A term in the expansion of $(x+a)^7\ is\ \frac{21504x^5}{y^4}$. Find the value of a.

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Practice:

Given $(2x - 6)^{10}$, determine the middle term of the expansion.

Given $(5z + 9y)^6$ Find the coefficient of the term containing z^2

Given $(8x^6 - 7y^3)^9$ Find the coefficient of the term containing x^{36}

If a term in the expansion of $\left(2x^2 + \frac{m}{y}\right)^3$ is $\frac{54x^2}{y^2}$, the value of m is

A term in the expansion of $(mx - 4)^8$ is $1451520x^4$. The value of m is

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THEOREM 4: Let n be a positive integer. The number of positive integer solutions to $x_1 + x_2 + \cdots + x_r = n$ is $\binom{n-1}{r-1}$.

☆**Example 22.** Three friends have a total of 6 identical pencils, and each one has at least one pencil. In how many ways can this happen?
(A) 6 (B) 7 (C) 10 (D) 12 (E) 24

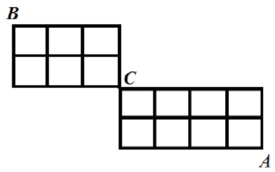
Example 23. In how many ways may we write the number 9 as the sum of three positive integer summands? Here order counts, so, for example, 1 + 7 + 1 is to be regarded different from 7 + 1 + 1.
(A) 16 (B) 27 (C) 30 (D) 14 (E) 28

THEOREM 5: Let n be a positive integer. The number of non-negative integer solutions to $y_1 + y_2 + \cdots + y_r = n$ is $\binom{n+r-1}{n}$ or $\binom{n+r-1}{r-1}$.

☆**Example 24.** Three friends have a total of 6 identical pencils. In how many ways can this happen?
(A) 10 (B) 12 (C) 28 (D) 24 (E) 16

THEOREM 6: The number of ways to walk from one corner to another corner of an m by n grid can be calculated by the following formula: $N = \binom{m+n}{n}$, where m is the number of rows and n the number of column.

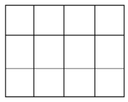
Example 25. How many ways are there to get from A to B if you can only go north or west?
(A) 120 (B) 130 (C) 140 (D) 150 (E) 160



<u>THEOREM 7: Counting How Many Rectangles</u>

For a rectangular grid with m vertical lines and n horizontal lines, the total number of rectangles that can be counted is $N = \binom{m}{2} \times \binom{n}{2}$.

Example 27. Consider the figure shown below. How many rectangles are there?



(A) 30 (B) 60 (C) 90 (D) 120 (E) 180

<u>THEOREM 8: Rising (Increasing) Number</u>

A rising number, such as 34689, is a positive integer where each digit is larger than the one to its left.

The number of integers with digits in increasing order can be calculated by $\binom{9}{n}$.

Example 28. How many 3-digit increasing numbers are there?
(A) 34 (B) 64 (C) 84 (D) 92 (E) 120

<u>THEOREM 9: Falling (Decreasing) Number</u>

A falling number is an integer whose decimal representation has the property that each digit except the units digit is larger than the one to its right. For example, 96521 is a falling number but 89642 is not. The number of integers with digits in decreasing order can be calculated: $\binom{10}{n}$.

Example 29. How many 3-digit falling numbers are there?
(A) 60 (B) 90 (C) 120 (D) 240 (E) 180

THEOREM 10: Palindrome Numbers

A **palindrome number** is a number that is the same when written forwards or backwards. The first few palindrome numbers are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33, 44, 55, 66, 77, 88, 99, 101, 111, and 121.

The number of palindromes with n digits can be found by $9\times10^{\left\lfloor\frac{n-1}{2}\right\rfloor}$ for $n > 1$.
$\lfloor x \rfloor$ is the floor function.

Example 30. How many 3-digit palindromes are there?
(A) 30 (B) 60 (C) 90 (D) 120 (E) 180

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Key Concepts

- Use the formula $(p + 1)(q + 1)(r + 1) \ldots - 1$ to find the total number of selections of at least one item that can be made from p items of one kind, q of a second kind, r of a third kind, and so on.
- A set with n distinct elements has 2^n subsets including the null set.
- For combinations with some identical elements, you often have to consider all possible cases individually.
- In a situation where you must choose *at least* one particular item, either consider the total number of choices available minus the number without the desired item or add all the cases in which it is possible to have the desired item.

Communicate Your Understanding

1. Give an example of a situation where you would use the formula $(p + 1)(q + 1)(r + 1) \ldots - 1$. Explain why this formula applies.
2. Give an example of a situation in which you would use the expression $2^n - 1$. Explain your reasoning.
3. Using examples, describe two different ways to solve a problem where *at least* one particular item must be chosen. Explain why both methods give the same answer.

Practise

A

1. How many different sums of money can you make with a penny, a dime, a one-dollar coin, and a two-dollar coin?
2. How many different sums of money can be made with one \$5 bill, two \$10 bills, and one \$50 bill?
3. How many subsets are there for a set with
   a) two distinct elements?
   b) four distinct elements?
   c) seven distinct elements?
4. In how many ways can a committee with eight members form a subcommittee with at least one person on it?

B

5. Determine whether the following questions involve permutations or combinations and list any formulas that would apply.
   a) How many committees of 3 students can be formed from 12 students?
   b) In how many ways can 12 runners finish first, second, and third in a race?
   c) How many outfits can you assemble from three pairs of pants, four shirts, and two pairs of shoes?
   d) How many two-letter arrangements can be formed from the word *star*?

Apply, Solve, Communicate

6. Seven managers and eight sales representatives volunteer to attend a trade show. Their company can afford to send five people. In how many ways could they be selected

a) without any restriction?

b) if there must be at least one manager and one sales representative chosen?

7. **Application** A cookie jar contains three chocolate-chip, two peanut-butter, one lemon, one almond, and five raisin cookies.

a) In how many ways can you reach into the jar and select some cookies?

b) In how many ways can you select some cookies, if you must include at least one chocolate-chip cookie?

8. A project team of 6 students is to be selected from a class of 30.

a) How many different teams can be selected?

b) Pierre, Gregory, and Miguel are students in this class. How many of the teams would include these 3 students?

c) How many teams would not include Pierre, Gregory, and Miguel?

9. The game of euchre uses only the 9s, 10s, jacks, queens, kings, and aces from a standard deck of cards. How many five-card hands have

a) all red cards?

b) at least two red cards?

c) at most two red cards?

10. If you are dealing from a standard deck of 52 cards,

a) how many different 4-card hands could have at least one card from each suit?

b) how many different 5-card hands could have at least one spade?

c) how many different 5-card hands could have at least two face cards (jacks, queens, or kings?

11. The number 5880 can be factored into prime divisors as $2 \times 2 \times 2 \times 3 \times 5 \times 7 \times 7$.

a) Determine the total number of divisors of 5880.

b) How many of the divisors are even?

c) How many of the divisors are odd?

12. **Application** A theme park has a variety of rides. There are seven roller coasters, four water rides, and nine story rides. If Stephanie wants to try one of each type of ride, how many different combinations of rides could she choose?

13. Shuwei finds 11 shirts in his size at a clearance sale. How many different purchases could Shuwei make?

14. **Communication** Using the summary on page 285, draw a flow chart for solving counting problems.

15. **a)** How many different teams of 4 students could be chosen from the 15 students in the grade-12 Mathematics League?

b) How many of the possible teams would include the youngest student in the league?

c) How many of the possible teams would exclude the youngest student?

16. **Inquiry/Problem Solving**

a) Use combinations to determine how many diagonals there are in

i) a pentagon **ii)** a hexagon

b) Draw sketches to verify your answers in part a).

17. A school is trying to decide on new school colours. The students can choose three colours from gold, black, green, blue, red, and white, but they know that another school has already chosen black, gold, and red. How many different combinations of three colours can the students choose?

18. Application The social convenor has 12 volunteers to work at a school dance. Each dance requires 2 volunteers at the door, 4 volunteers on the floor, and 6 floaters. Joe and Jim have not volunteered before, so the social convenor does not want to assign them to work together. In how many ways can the volunteers be assigned?

19. Jeffrey is a DJ at a local radio station. For the second hour of his shift, he must choose all his music from the top 100 songs for the week. Jeffery will play exactly 12 songs during this hour.

a) How many different stacks of discs could Jeffrey pull from the station's collection if he chooses at least 10 songs that are in positions 15 to 40 on the charts?

b) Jeffrey wants to start his second hour with a hard-rock song and finish with a pop classic. How many different play lists can Jeffrey prepare if he has chosen 4 hard rock songs, 5 soul pieces, and 3 pop classics?

c) Jeffrey has 8 favourite songs currently on the top 100 list. How many different subsets of these songs could he choose to play during his shift?

ACHIEVEMENT CHECK

Knowledge/Understanding	Thinking/Inquiry/Problem Solving	Communication	Application

20. There are 52 white keys on a piano. The lowest key is A. The keys are designated A, B, C, D, E, F, and G in succession, and then the sequence of letters repeats, ending with a C for the highest key.

a) If five notes are played simultaneously, in how many ways could the notes all be

i) As? **ii)** Gs?

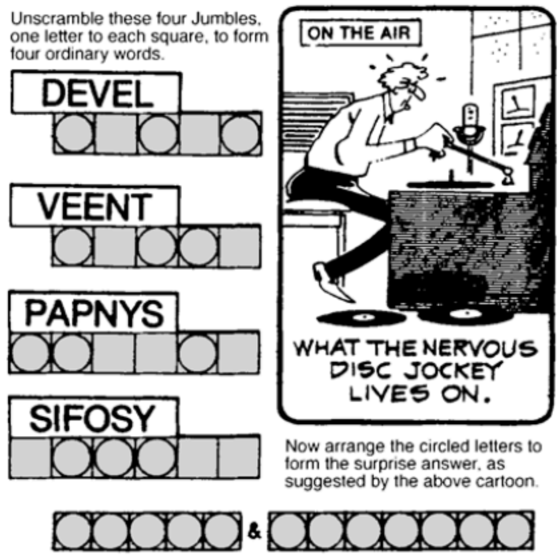
iii) the same letter? **iv)** different letters?

b) If the five keys are played in order, how would your answers in part a) change?

21. Communication

a) How many possible combinations are there for the letters in the three circles for each of the clue words in this puzzle?

b) Explain why you cannot answer part a) with a single ${}_nC_r$ calculation for each word.



22. Determine the number of ways of selecting four letters, without regard for order, from the word *parallelogram*.

C

23. Inquiry/Problem Solving Suppose the artist in Example 1 of this section had two apples, two oranges, and two pears in his refrigerator. How many combinations does he have to choose from if he wants to paint a still-life with

a) two pieces of fruit?

b) three pieces of fruit?

c) four pieces of fruit?

24. How many different sums of money can be formed from one \$2 bill, three \$5 bills, two \$10 bills, and one \$20 bill?

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Review of Key Concepts

5.1 Organized Counting With Venn Diagrams

Refer to the Key Concepts on page 270.

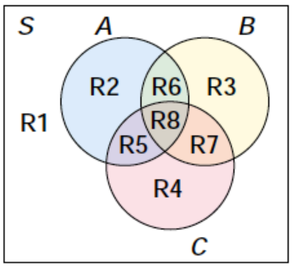
1. Which regions in the diagram below correspond to

a) the union of sets A and B?

b) the intersection of sets B and C?

c) $A \cap C$?

d) either B or S?



2. a) Write the equation for the number of elements contained in either of two sets.

b) Explain why the principle of inclusion and exclusion subtracts the last term in this equation.

c) Give a simple example to illustrate your explanation.

3. A survey of households in a major city found that

- 96% had colour televisions
- 65% had computers
- 51% had dishwashers
- 63% had colour televisions and computers
- 49% had colour televisions and dishwashers
- 31% had computers and dishwashers
- 30% had all three

a) List the categories of households not included in these survey results.

b) Use a Venn diagram to find the proportion of households in each of these categories.

5.2 Combinations

Refer to the Key Concepts on page 278.

4. Evaluate the following and indicate any calculations that could be done manually.

a) ${}_{41}C_8$ **b)** ${}_{33}C_{15}$

c) ${}_{25}C_{17}$ **d)** ${}_{50}C_{10}$

e) ${}_{10}C_8$ **f)** ${}_{15}C_{13}$

g) ${}_{5}C_4$ **h)** ${}_{25}C_{24}$

i) ${}_{15}C_{11}$ **j)** ${}_{25}C_{20}$

k) ${}_{16}C_8$ **l)** ${}_{30}C_{26}$

5. A track and field club has 12 members who are runners and 10 members who specialize in field events. The club has been invited to send a team of 3 runners and 2 field athletes to an out-of-town meet. How many different teams could the club send?

6. A bridge hand consists of 13 cards. How many bridge hands include 5 cards of one suit, 6 cards of a second, and 2 cards of a third?

7. Explain why combination locks should really be called permutation locks.

5.3 Problem Solving With Combinations

Refer to the Key Concepts on page 286.

8. At Subs Galore, you have a choice of lettuce, onions, tomatoes, green peppers, mushrooms, cheese, olives, cucumbers, and hot peppers on your submarine sandwich. How many ways can you "dress" your sandwich?

9. Ballots for municipal elections usually list candidates for several different positions. If a resident can vote for a mayor, two councillors, a school trustee, and a hydro commissioner, how many combinations of positions could the resident choose to mark on the ballot?

10. There are 12 questions on an examination, and each student must answer 8 questions including at least 4 of the first 5 questions. How many different combinations of questions could a student choose to answer?

11. Naomi invites eight friends to a party on short notice, so they may not all be able to come. How many combinations of guests could attend the party?

12. In how many ways could 15 different books be divided equally among 3 people?

13. The camera club has five members, and the mathematics club has eight. There is only one member common to both clubs. In how many ways could a committee of four people be formed with at least one member from each club?

5.4 The Binomial Theorem

Refer to the Key Concepts on page 293.

14. Without expanding $(x + y)^5$, determine

a) the number of terms in the expansion

b) the value of k in the term $10x^ky^2$

15. Use Pascal's triangle to expand

a) $(x + y)^8$

b) $(4x - y)^6$

c) $(2x + 5y)^4$

d) $(7x - 3)^5$

16. Use the binomial theorem to expand

a) $(x + y)^6$

b) $(6x - 5y)^4$

c) $(5x + 2y)^5$

d) $(3x - 2)^6$

17. Write the first three terms of the expansion of

a) $(2x + 5y)^7$

b) $(4x - y)^6$

18. Describe the steps in the binomial expansion of $(2x - 3y)^6$.

19. Find the last term in the binomial expansion of $\left(\frac{1}{x^2} + 2x\right)^5$.

20. Find the middle term in the binomial expansion of $\left(\sqrt{x} + \frac{5}{\sqrt{x}}\right)^8$.

21. In the expansion of $(a + x)^6$, the first three terms are $1 + 3 + 3.75$. Find the values of a and x.

22. Use the binomial theorem to expand and simplify $(y^2 - 2)^6(y^2 + 2)^6$.

23. Write $1024x^{10} - 3840x^8 + 5760x^6 - 4320x^4 + 1620x^2 - 243$ in the form $(a + b)^n$. Explain your steps.

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Chapter Test

ACHIEVEMENT CHART

Category	Knowledge/ Understanding	Thinking/Inquiry/ Problem Solving	Communication	Application
Questions	All	12	6, 12	5, 6, 7, 8, 9

1. Evaluate each of the following. List any calculations that require a calculator.

a) ${}_{25}C_{25}$

b) ${}_{52}C_1$

c) ${}_{12}C_3$

d) ${}_{40}C_{15}$

2. Rewrite each of the following as a single combination.

a) ${}_{10}C_7 + {}_{10}C_8$

b) ${}_{23}C_{15} - {}_{22}C_{14}$

3. Use Pascal's triangle to expand

a) $(3x - 4)^4$

b) $(2x + 3y)^7$

4. Use the binomial theorem to expand

a) $(8x - 3)^5$

b) $(2x - 5y)^6$

5. A student fundraising committee has 14 members, including 7 from grade 12. In how many ways can a 4-member subcommittee for commencement awards be formed if

a) there are no restrictions?

b) the subcommittee must be all grade-12 students?

c) the subcommittee must have 2 students from grade 12 and 2 from other grades?

d) the subcommittee must have no more than 3 grade-12 students?

6. A track club has 20 members.

a) In how many ways can the club choose 3 members to help officiate at a meet?

b) In how many ways can the club choose a starter, a marshal, and a timer?

c) Should your answers to parts a) and b) be the same? Explain why or why not.

7. Statistics on the grade-12 courses taken by students graduating from a secondary school showed that

- 85 of the graduates had taken a science course
- 75 of the graduates had taken a second language
- 41 of the graduates had taken mathematics
- 43 studied both science and a second language
- 32 studied both science and mathematics
- 27 had studied both a second language and mathematics
- 19 had studied all three subjects

a) Use a Venn diagram to determine the minimum number of students who could be in this graduating class.

b) How many students studied mathematics, but neither science nor a second language?

8. A field-hockey team played seven games and won four of them. There were no ties.

a) How many arrangements of the four wins and three losses are possible?

b) In how many of these arrangements would the team have at least two wins in a row?

9. A restaurant offers an all-you-can-eat Chinese buffet with the following items:

- egg roll, wonton soup
- chicken wings, chicken balls, beef, pork
- steamed rice, fried rice, chow mein
- chop suey, mixed vegetables, salad
- fruit salad, custard tart, almond cookie

a) How many different combinations of items could you have?

b) The restaurant also has a lunch special with your choice of one item from each group. How many choices do you have with this special?

10. In the expansion of $(1 + x)^n$, the first three terms are $1 - 0.9 + 0.36$. Find the values of x and n.

11. Use the binomial theorem to expand and simplify $(4x^2 - 12x + 9)^3$.

12. A small transit bus has 8 window seats and 12 aisle seats. Ten passengers board the bus and select seats at random. How many seating arrangements have all the window seats occupied if which passenger is in a seat

a) does not matter? **b)** matters?

ACHIEVEMENT CHECK

Knowledge/Understanding	Thinking/Inquiry/Problem Solving	Communication	Application

13. The students' council is having pizza at their next meeting. There are 20 council members, 6 of whom are vegetarian. A committee of 3 will order six pizzas from a pizza shop that has a special price for large pizzas with up to three toppings. The shop offers ten different toppings.

a) How many different pizza committees can the council choose if there must be at least one vegetarian and one non-vegetarian on the committee?

b) In how many ways could the committee choose *exactly* three toppings for a pizza?

c) In how many ways could the committee choose *up to* three toppings for a pizza?

d) The committee wants as much variety as possible in the toppings. They decide to order each topping exactly once and to have at least one topping on each pizza. Describe the different cases possible when distributing the toppings in this way.

e) For one of these cases, determine the number of ways of choosing and distributing the ten toppings.