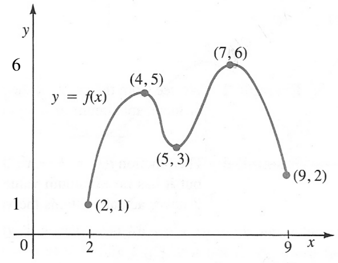
PROPERTIES OF FUNCTIONS

Up to this point, you have worked with several **parent functions**. A parent function simply refers to a base function before any transformations have been applied to it. The parent functions with which you should be familiar are as follows:



$f(x)=x,\ \ g(x)=x^2,\ \ h(x)=\sqrt{x},\ \ k(x)=\frac{1}{x},\ \ m(x)=b^x,\ \ n(x)=\sin x,\ \ p(x)=\cos x,\ \ q(x)=|x|$

In this lesson, we will consider several key properties that can be used to describe functions. To begin, consider the graph of a function *f* as shown above. (Note: This graph does not continue to the left and right.)

DOMAIN AND RANGE

Domain: $\{x\in\mathbb{R}\mid 2\le x\le 9\}$

Range: $\{y\in\mathbb{R}\mid 1\le y\le 6\}$

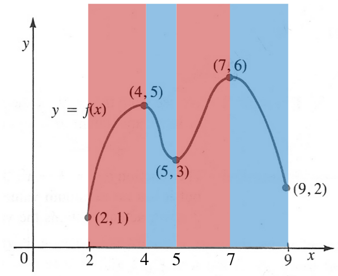
INTERVALS OF INCREASE AND DECREASE

Informally

A function is increasing if the graph rises from left to right and decreasing if the graph falls from left to right.

The function *f* on the right is **increasing** on the intervals $2<x<4$ and $5<x<7$

The function *f* is **decreasing** on the intervals $4<x<5$ and $7<x<9$



Note: The interval $7<x<9$ can also be written $(7, 9)$.

Note: We do not include the endpoints when stating intervals of increase/decrease.

Formally

A function *f* is **increasing** on an interval *I* if $f(x_1)<f(x_2)$ whenever $x_1<x_2$ and x_1 and x_2 are in *I*.
A function *f* is **decreasing** on an interval *I* if $f(x_1)>f(x_2)$ whenever $x_1<x_2$ and x_1 and x_2 are in *I*.

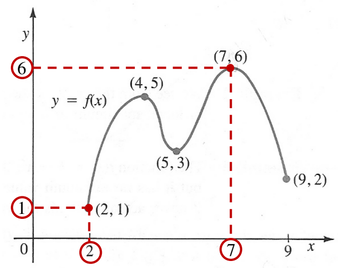
ABSOLUTE EXTREME VALUES

Informally

The **absolute maximum** of a function is its highest point.

The function *f* shown on the right has an **absolute maximum at 7.**

The **absolute maximum value is *f*(7)=6.**



The **absolute minimum** of a function is its lowest point.

The function *f* has an **absolute minimum at 2** and the **absolute minimum value is *f*(2)=1**.

Note: When we talk about where something occurs, we refer to an *x*-value. When we talk about a function's value, we refer to a *y*-value.

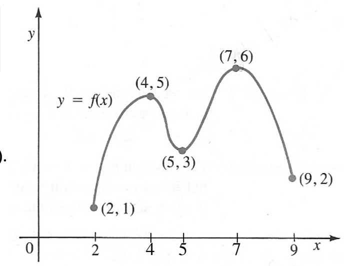
Formally

A function *f* has an **absolute maximum** at *a* if $f(a)\ge f(x)$ for all *x* in the domain of *f*. $f(a)$ is called the **absolute maximum value** of *f*.
A function *f* is has an **absolute minimum** at *b* if $f(b)\le f(x)$ for all *x* in the domain of *f*. $f(b)$ is called the **absolute minimum value** of *f*.

LOCAL EXTREME VALUES

Informally

Notice that if we restrict our attention to the interval $2\le x\le 5$, then the highest point is (4,5). For that reason, we say that *f* has a **local maximum value** of 5 where *x* = 4. That is, *f* has a local maximum value of *f*(4)=5.



Similarly, *f*(5)=3 is a **local minimum value** because it is the smallest value of *f* if we only consider values of *x* that are near 5.

f has another local maximum value of *f*(7)=6.

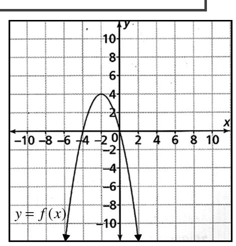
Note: In this course, maximum or minimum values that occur at endpoints are **not** considered to be local maximum or local minimum values.

Note: The interval $2\le x\le 5$ can also be written [2, 5].

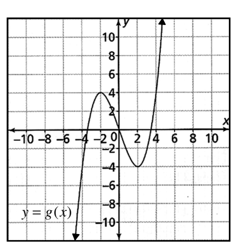
Formally

When a function *f* changes from increasing to decreasing at a point (*c*, *d*), the function has a **local maximum value** of $f(c)=d$.
When a function *f* changes from decreasing to increasing at a point (*c*, *d*), the function has a **local minimum value** of $f(c)=d$.

SOME EXAMPLES...



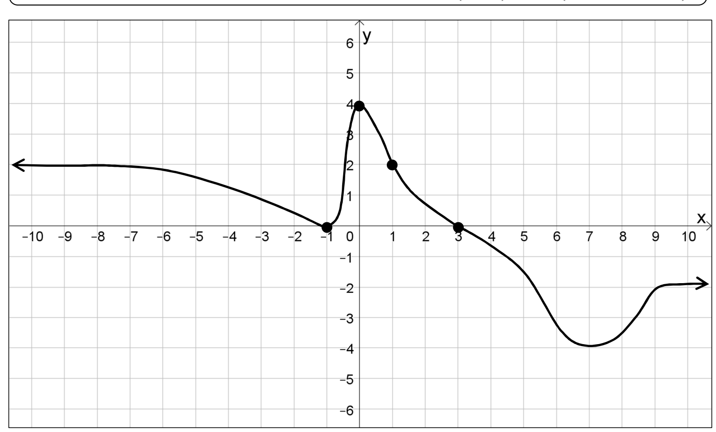
Domain: $\{x\in\mathbb{R}\}$
Range: $\{y\in\mathbb{R}\mid y\le 4\}$
Interval(s) of Increase: $x<-2$
Interval(s) of Decrease: $x>-2$
Absolute Maximum Value: $f(-2)=4$
Absolute Minimum Value: None
Local Maximum Value(s): $f(-2)=4$
Local Minimum Value(s): None



Domain: $\{x\in\mathbb{R}\}$
Range: $\{y\in\mathbb{R}\}$
Interval(s) of Increase: $x<-2,\ x>2$
Interval(s) of Decrease: $-2<x<2$
Absolute Maximum Value: None
Absolute Minimum Value: None
Local Maximum Value(s): $g(-2)=4$
Local Minimum Value(s): $g(2)=-4$

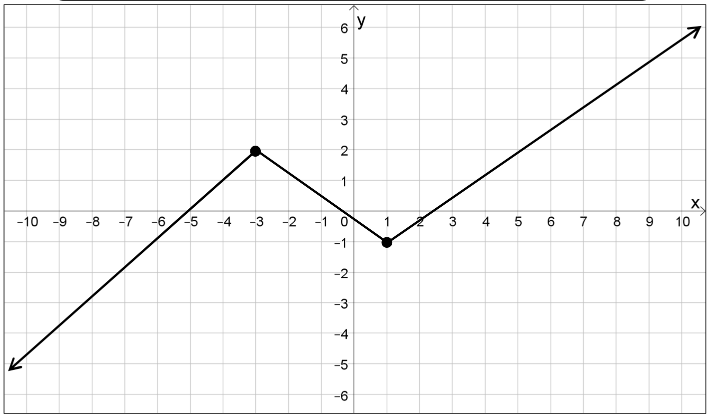
ANOTHER EXAMPLE...

Graph a function $y=f(x)$ that has all of the following properties: $f(1)=2,\ f(3)=f(-1)=0$, $f(0)=4,\ f(x)>0$ for $x<-1$, and $f(x)<0$ for $x>3$, domain $\{x\in\mathbb{R}\}$, range $\{y\in\mathbb{R}\mid -4\le y\le 4\}$.



YET ANOTHER EXAMPLE…

Graph a function $y = f(x)$ that has all of the following properties: $f(-3)=2$, $f(1)=-1$, increasing on the intervals $(-\infty,-3)$ and $(1,\infty)$, decreasing on the interval $-3<x<1$, domain $\{x\in\mathbb{R}\}$, range $\{y\in\mathbb{R}\}$.

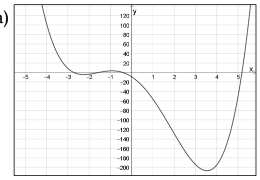


CONTINUITY

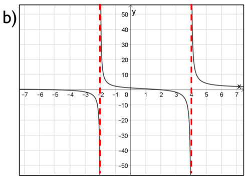
A function is **continuous** on an interval if it does not contain any holes or breaks in that interval.

A **discontinuity** occurs when there is a break in the function's graph. We refer to the location of a discontinuity by stating the *x*-value at which it occurs.

Examples

a) 

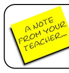
No discontinuity.
Continuous on the interval $(-\infty,\infty)$.

b) 

Discontinuous at $x = -2$ and $x = 4$ (vertical asymptotes)

c) $y=-3x^2+5x-4$
No discontinuity.
Continuous on the interval $(-\infty,\infty)$.

d) $y=\frac{1}{x}$
Discontinuous at $x = 0$.
(vertical asymptote)



The definition of a **continuous function** is slightly more complex than the definitions above and will be considered in future lessons!

SYMMETRY

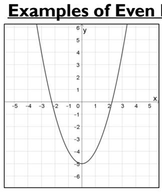
EVEN SYMMETRY
A function is an **even function** if $f(-x)=f(x)$ for every value of *x* in the function's domain.

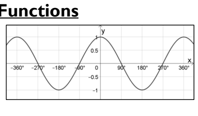
What does $f(-x)=f(x)$ mean?

- If you evaluate the function at any number and also the opposite of that number, you get the same result.
  - For example, $f(-2)$ would give the same result as $f(2)$.

- The graph of $f(x)$ is symmetrical about the *y*-axis.
  - The graph does the same on the left side of the *y*-axis (negative *x*-values) as it does on the right side of the *y*-axis (positive *x*-values).

Examples of Even Functions





x	y
-2	16
-1	8
0	0
1	8
2	16

$f(x)=3x^4-x^2+7$ SAME!

Proof:
$$f(-x)=3(-x)^4-(-x)^2+7$$
$$=3x^4-x^2+7$$
$$=f(x)$$
$$\therefore f(-x)=f(x)$$

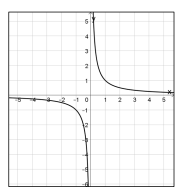
ODD SYMMETRY
A function is an **odd function** if $f(-x)=-f(x)$ for every value of *x* in the function's domain.

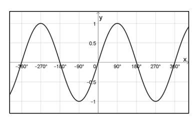
What does $f(-x)=-f(x)$ mean?

- If you evaluate the function at any number and also the opposite of that number, you get opposite results (opposite signs).
  - For example $f(-2)$ would give opposite result of $f(2)$.

- The graph of $f(x)$ has rotational symmetry about the origin.
  - The graph does the opposite on the left side of the *y*-axis (negative *x*-values) as it does on the right side of the *y*-axis (positive *x*-values).

Examples of Odd Functions





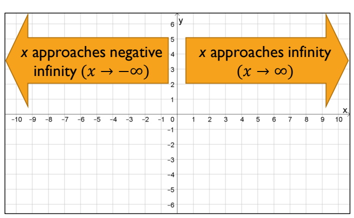
x	y
-2	16
-1	8
0	0
1	-8
2	-16

$f(x)=2x^3-7x$ OPPOSITE!

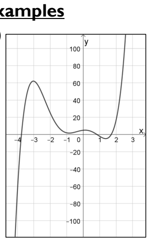
Proof:
$$f(-x)=2(-x)^3-7(-x)$$
$$=-2x^3+7x$$
$$=-f(x)$$
$$\therefore f(-x)=-f(x)$$

END BEHAVIOUR

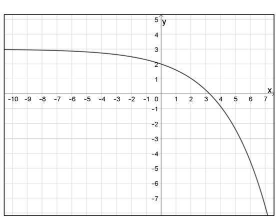
End behaviour describes what happens to a function's *y*-values as the *x*-values approach infinity in the positive and negative directions.



Examples

a) 

As $x\to\infty,\ y\to\infty$
As $x\to-\infty,\ y\to-\infty$

b) 

As $x\to\infty,\ y\to-\infty$
As $x\to-\infty,\ y\to3$

c) $f(x)=-3|x+6|+8$
As $x\to\infty,\ y\to-\infty$
As $x\to-\infty,\ y\to-\infty$

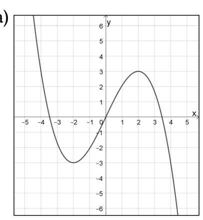
BRINGING IT ALL TOGETHER

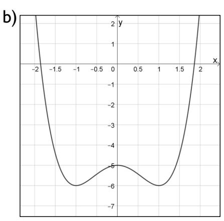
On a separate page, determine the following properties for each of the given functions.

PROPERTIES

- ❒Domain
- ❒Range
- ❒Intervals of Increase/Decrease
- ❒Absolute Maximum/Minimum Values
- ❒Local Maximum/Minimum Values
- ❒Continuity
- ❒Symmetry
- ❒End Behaviour

FUNCTIONS

a) 

b) 

c) $y=\frac{1}{x-4}+3$

d) $f(x)=-2x^2+12x+5$

e) $f(x)=-4|2x|+8$

f) $y=3^{-x}+2$