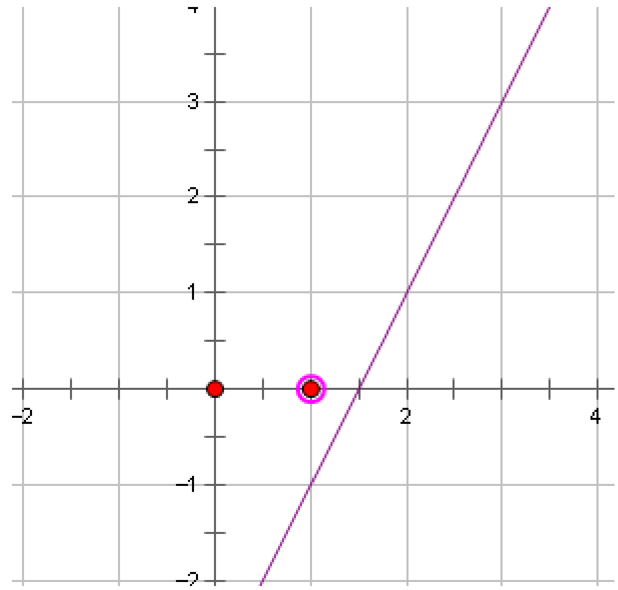
Lesson 2: The Limiting of a Function

L.G. : I can determine the limit of a function using appropriate techniques.

Definition of a Limit

What happens to the value of a function *f*, as *x* gets closer and closer to a particular value of *a*? Does *f*(*x*) tend to "home in" on some specific value, that is, does it have a **limit**?



Consider $f(x) = 2x - 3$

The limiting value of the function as $x \to 2$ would be

$$2x - 3 \to 2(2) - 3 \to 1$$

We say $\lim_{x \to 2} 2x - 3 = 1$ because …

- As the function approaches 2 from the right
  i.e. $\lim_{x \to 2^+} 2x - 3 = 1$ **Right-hand limit**
- and as the function approaches 2 from the left
  i.e. $\lim_{x \to 2^-} 2x - 3 = 1$ **Left-hand limit**

> The Limit of a function
>
> $$\lim_{x \to a} f(x) = L$$
>
> means that *f*(*x*) approaches the value *L*, as *x* approaches the value *a*
>
> If $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$ then $\lim_{x \to a} f(x)$ **does not exist**. If $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$ then $\lim_{x \to a} f(x) = L$

Limits of Polynomial Functions

For polynomial functions such as, linear, quadratic, and cubic functions, the functions have a value for every value of *x*. These functions are continuous for all *x*, $x \in R$ and hence have a limiting value for all *x*.

To determine the limit of these functions at a specific value, we simply substitute the value of *x* into the function.

> If $f(x)$ is a polynomial function and $a \in R$ then, $\lim_{x \to a} f(x) = f(a)$

Example 1: Determine the following limits.

a) $\lim_{x \to 0} 3$

b) $\lim_{x \to 3} 5x^3$

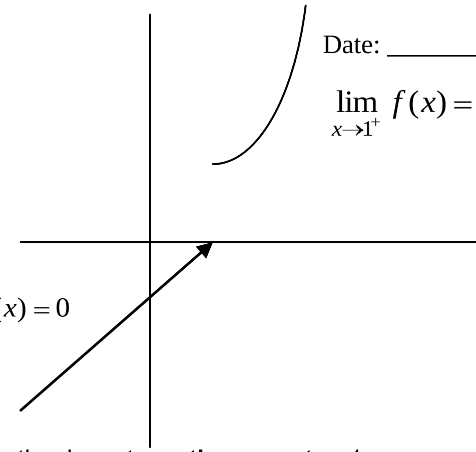
c) $\lim_{x \to -1} 3x^3 - 2x^2 + x - 11$

Homework: p37 #3, 4ace, **5, 6. 7b, 8,10df, 11ac, 12a,13 or 14 p193 #4** (End Behaviour Limits)

Note: Not all limits can be evaluated by substitution.

Example 2: Consider this piecewise function (step function)

$$f(x)=\begin{cases}(x-1)^2+3, & x>1\\ x-1, & x\le 1\end{cases}$$



$\lim\limits_{x\to 1^+} f(x)=3$

$\lim\limits_{x\to 1^-} f(x)=0$

we have $\lim\limits_{x\to 1^+} f(x)=3$ and $\lim\limits_{x\to 1^-} f(x)=0$

$\therefore \lim\limits_{x\to 1^+} f(x)\neq \lim\limits_{x\to 1^-} f(x)$ so **the limit does not exist** and the function is **not continuous** at *x*=1

Limits of Rational Functions

L.G. : I can determine the limit of a rational function using appropriate techniques.

Definition

Let $h(x)=\dfrac{f(x)}{g(x)}$ be a rational function. Let a be any real number that is in the domain of h. Then the

$$\lim_{x\to a} h(x)=\frac{f(a)}{g(a)} \text{ provided } g(a)\neq 0$$

Example 1: Determine the following limits.

a) $\lim\limits_{x\to 0}\dfrac{x^2+3x+6}{x+2}$

b) $\lim\limits_{x\to 1}\dfrac{x^2-3x}{x-2}$

IMPORTANT NOTE: Not all limits can be evaluated by substitution.

Consider $\lim\limits_{x\to 1}\dfrac{1}{x-1}$, substitution results in $\dfrac{1}{0}$ which is undefined.

The notion of **<u>infinity</u>** (∞) refers to cases where a quantity increases without bound or limits. The symbol ∞ does not represent a real number. It describes unbounded behavior and the limit does not exist

We consider the **right-hand limit** $\lim\limits_{x\to 1^+} f(x)$, and the **left- hand limit** $\lim\limits_{x\to 1^-} f(x)$

x	0.9	0.99	0.999	0.9999	1.0001	1.001	1.01	1.1
$\dfrac{1}{x-1}$								

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We notice that $\lim_{x \to 1^-} \frac{1}{x-1} =$ and $\lim_{x \to 1^+} \frac{1}{x-1} =$

The $\lim_{x \to 1} \frac{1}{x-1}$ does not exist but the left and right hand limits give us an understanding of the behaviour of the function at the **vertical asymptote** $x = 1$.

The Undefined Form $\left[\frac{k}{0}, k \varepsilon \Re\right]$

Substituting x=a to find the $\lim_{x \to a} h(x) = \frac{f(a)}{g(a)}$ may result in the indeterminate form $\left[\frac{k}{0}\right]$. In this case, the limit does not exist and there is a vertical asymptote at x =a.

The Indeterminate Form $\left[\frac{0}{0}\right]$

Substituting x=a to find the $\lim_{x \to a} h(x) = \frac{f(a)}{g(a)}$ may result in the indeterminate form $\left[\frac{0}{0}\right]$. To find the limit in this case, **first factor** the rational function.

Example 3: Determine the following limits.

a) $\lim_{x \to 0} \frac{x + 3x^2}{4x}$ b) $\lim_{x \to 2} \frac{x^2 - 4x + 4}{x^3 - 4x}$ c) $\lim_{x \to 2} \frac{1 - x^2}{1 - x}$ d) $\lim_{x \to 0} \frac{2 - \sqrt{4 + x}}{x}$

End Behaviour Limits

When you are finding a limit at infinity (∞), substituting can yield another indeterminate form $\frac{\infty}{\infty}$. To find the limit in this case, divide the functions in the numerator and denominator by the highest power of x in the denominator.

Example 4: Determine the following limits.

a) $\lim_{x \to \infty} \frac{1}{x}$ b) $\lim_{x \to \infty} \frac{x^2 - 4x + 1}{x - 4}$ c) $\lim_{x \to \infty} \frac{5x^2 - 3x + 4}{2x^2 + x - 7}$

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Homework: p37 #**3**, 4ace, **5, 6. 7b, 8,10df, 11ac, 12a,13 or 14 p193** #**4** (End Behaviour Limits)