Chapter 6: Quadratic Equations:
Date Created: Friday, October 15, 2021
Date Revised: Wednesday, February 14, 2024

Three forms of Quadratic Relations:

1. $y = ax^2 + bx + c$ ← standard form
2. $y = a(x – h)^2 + k$ ← Vertex Form
3. $y = a(x – r)(x – s)$ ← x-intercept form or zero form.

Topic: 1: Determine the Minima and Maxima of a quadratic relation in the form of $y = ax^2 + bx + c$

Also called: Completing of Squares, $(x – h)^2$

Example 1: Convert $y = ax^2 + bx + c$ into $y = a(x – h)^2 + k$

$y = x^2 + 8x + 5$

Step 1: factor the x out from the first two terms: $x^2 + 8x$

$y = x^2 + 8x + 5$
$y = x(x + 8) + 5$

what is $(x + 8)^2 = (x + 8)(x + 8) = x^2 + 8x + 8x + 64)$ ← $(x + 8)^2$ is too big

What if $(x + 8/2)^2 = (x + 4)^2 = (x + 4)(x + 4) = x^2 + 4x + 4x + 16 = x^2 + 8x + 16$

To keep it balanced from the original equation we need to take away 16.
$= (x^2 + 8x)$
$= (x + 4)^2 – 16$

expand
$= (x + 4)^2 - 16$ ← brings us closer to completing the square and balancing the equation.
$= (x + 4)(x + 4) – 16$
$= x^2 + 8x + 16 – 16$
$= x^2 + 8x$
$= x(x + 8)$

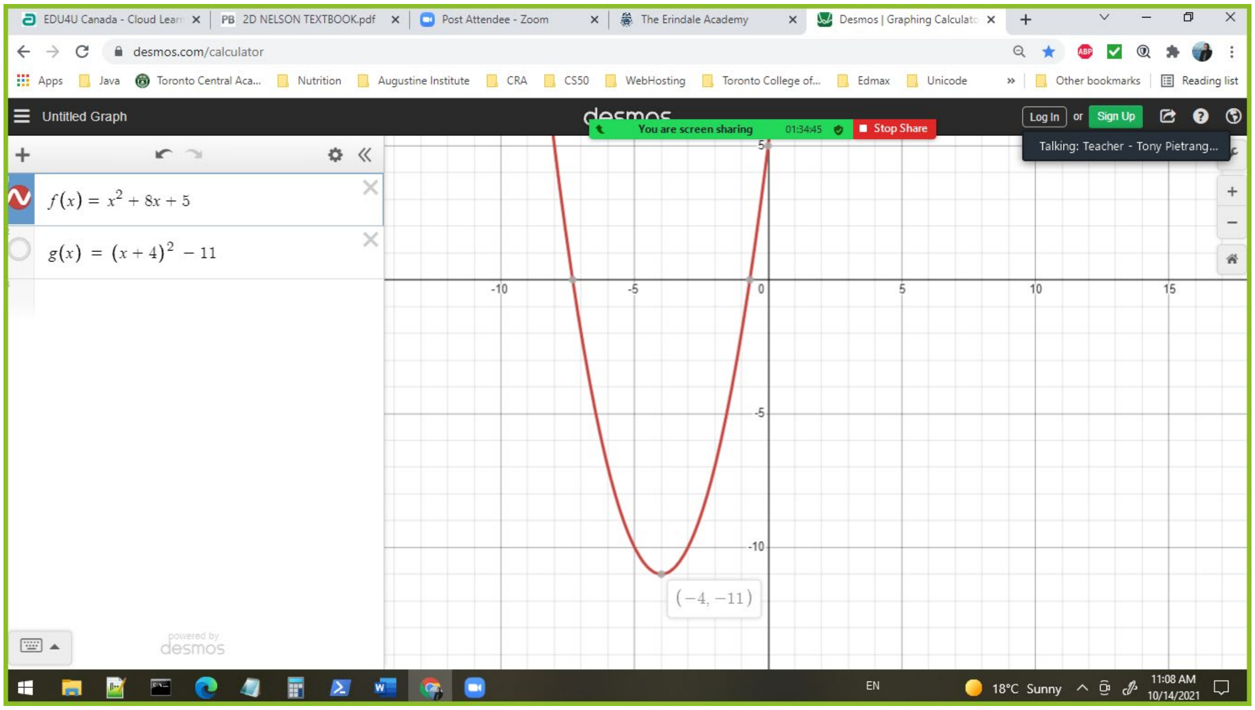
$y = x^2 + 8x + 5$
$y = x^2 + 8x + 5$
$y = x^2 + 8x + 4^2 – 4^2 + 5$
$y = (x^2 + 8x + 4^2) + 5 – 4^2$
$y = (x + 4)^2 + 5 – 16$

$y = (x + 4)^2 - 11$ 🡐 vertex $(x, y) = (h, k) = (-4, -11)$)

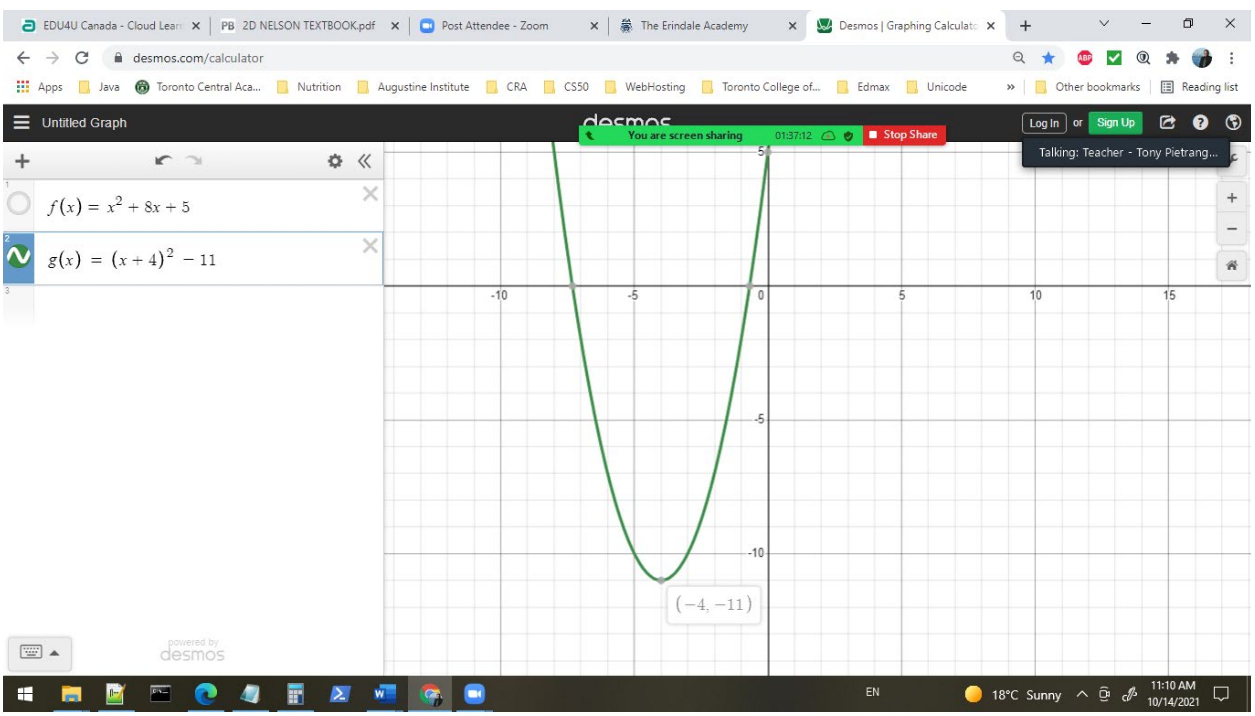
$y = (x - h)^2 - 11$ 🡐 h is -4 to make it $(x + 4)^2$

Desmos Graphing Software to plot both equations.

y = x^2+ 8x + 5



y = $(x + 4)^2$ – 11 ← vertex is at (x, y) = (-4, -11)



Note: Both equations form the same graph.

Example 2: Finding the Minimum and Maximum, where $a \neq 1$

$y = 2x^2 + 12x + 11 \rightarrow y = a(x - h)^2 + k$

Step 1: Factor the coefficient from first 2 terms.

$y = 2(x^2 + 6x) + 11$

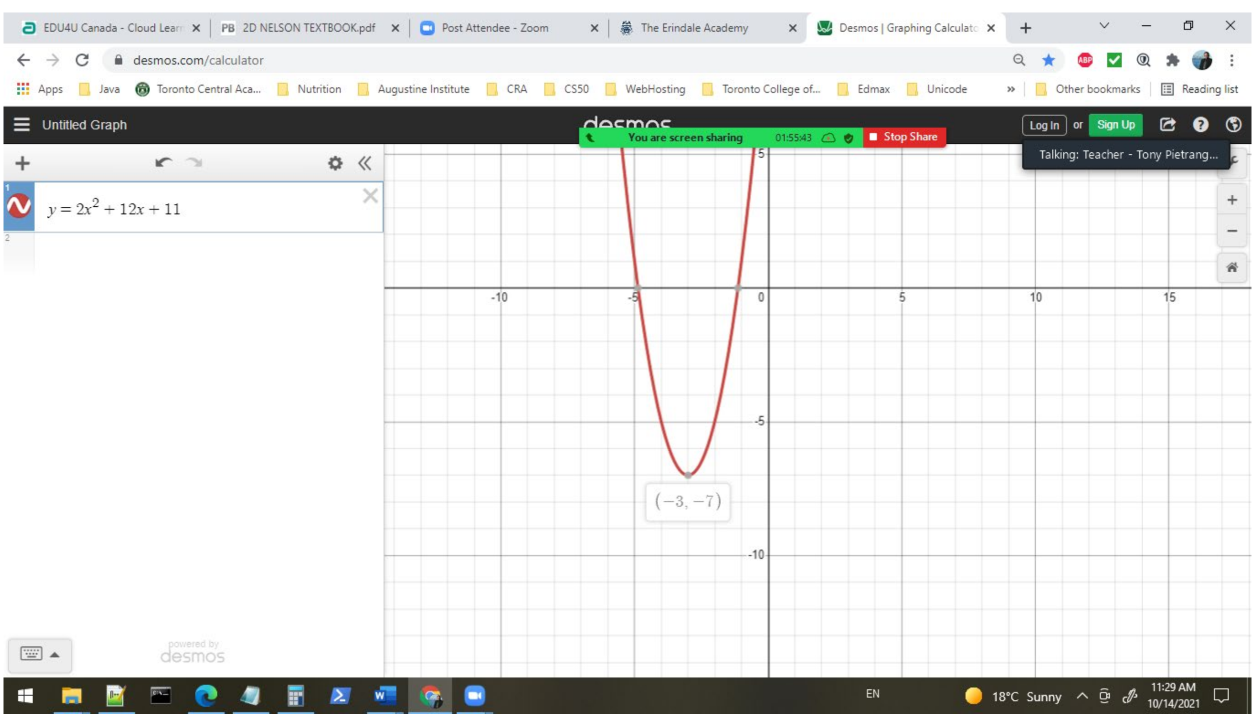
Step 2: Complete the square of the term inside the brackets.

$= x^2 + 6x$
$= x^2 + 6x + (6/2)^2 - (6/2)^2$
$= x^2 + 6x + (3)^2 - (3)^2$
$= (x^2 + 6x + 9) - 9$ ← $x^2 + 6x + 9 = (x + 3)^2$
$= (x + 3)^2 - 9$ ← substitute into the original equation

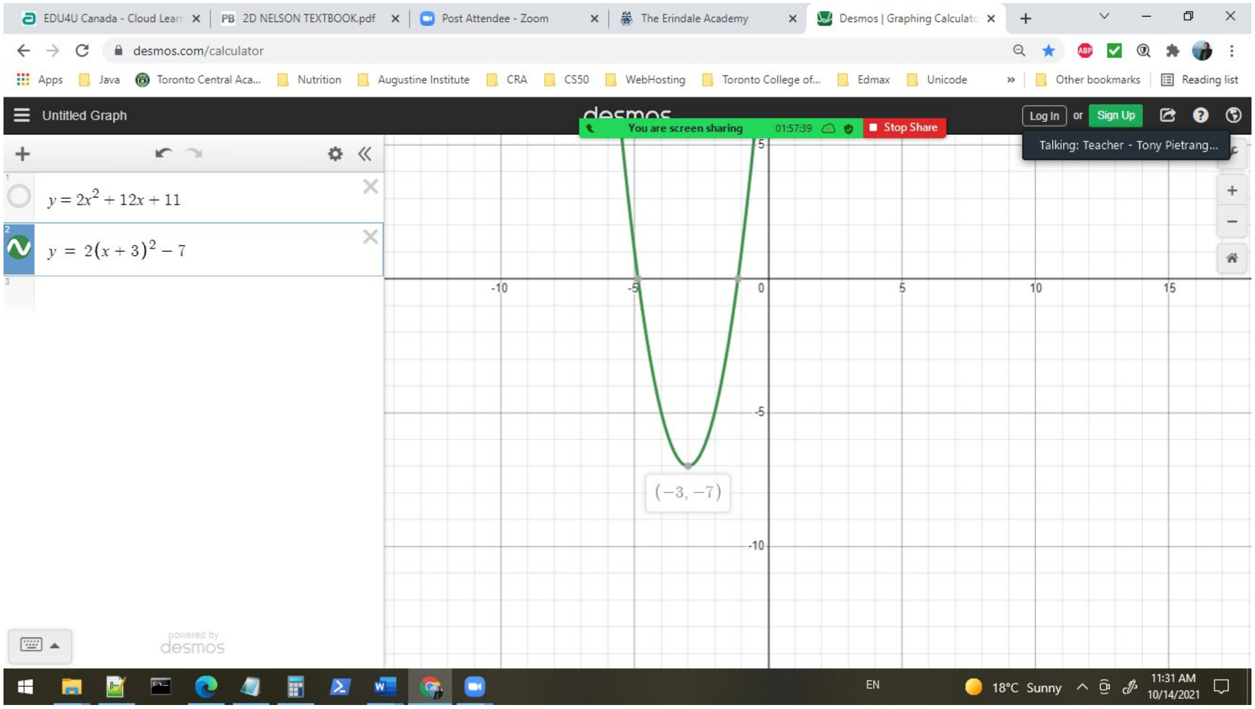
$y = 2(x^2 + 6x) + 11$
$y = 2[(x + 3)^2 - 9] + 11$
$y = 2[(x + 3)^2] + 11 + (-9)(2)$
$y = 2(x + 3)^2 + 11 - 18$
$y = 2(x + 3)^2 - 7$ ← vertex $(x, y) = (h, k) = (-3, -7)$

Plot using Desmos Graphing software.

$y = 2x^2 + 12x + 11$



$y = 2(x + 3)^2 - 7$



desmos.com/calculator
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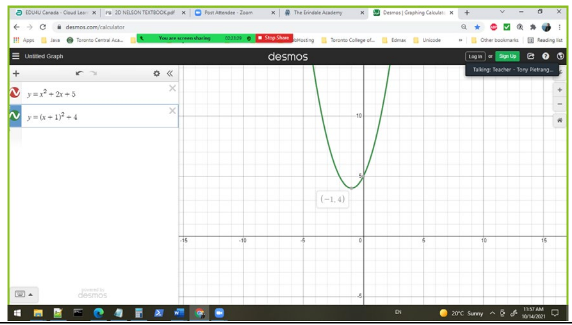
Talking: Teacher - Tony Pietrang...
$y = 2x^2 + 12x + 11$
$y = 2(x + 3)^2 - 7$
$(-3, -7)$

Activity 11: Completing the Squares, or Transforming Quadratic equation: $y = ax^2 + bx + c$ into vertex form $y = a(x – h)^2 + k$

<table>
<tr><td>Standard form
$y = ax^2 + bx + c$</td><td>Vertex Form
$y = a(x – h)^2 + k$</td></tr>
<tr><td>Question 1: Rewrite equations into the vertex form.</td><td></td></tr>
<tr><td>$y = x^2 +2x + 5$

Observation: a = 1
$y = x^2 +2x + 5$
$y = x^2 +2x + 1^2 – 1^2 + 5$
$y = (x^2 +2x + 1^2) – 1^2 + 5$
$y = (x + 1)^2 + 5 – 1$
$y = (x + 1)^2 + 4$</td><td>$y = (x + 1)^2 + 4$

vertex(x, y) = (h, k) = (-1, 4)


</td></tr>
<tr><td>$y = x^2 + 4x + 7$</td><td></td></tr>
<tr><td>$y = x^2 + 6x + 3$</td><td></td></tr>
<tr><td>Question 2: Determine the value of c, to complete the square.</td><td></td></tr>
<tr><td>$y = x^2 + 6x + c$</td><td></td></tr>
<tr><td>$y = x^2 + 14x + c$</td><td></td></tr>
<tr><td>$y = x^2 - 12x + c$</td><td></td></tr>
<tr><td>$y = x^2 - 10x + c$</td><td></td></tr>
<tr><td>$y = x^2 + 2x + c$</td><td></td></tr>
<tr><td>$y = x^2 - 80x + c$</td><td></td></tr>
<tr><td>Question 3: Rewrite the equations in the form of: $y=a(x – h)^2 + k$</td><td></td></tr>
<tr><td>$y = x^2 + 6x - 1$</td><td></td></tr>
<tr><td>$y = x^2 + 2x + 7$</td><td></td></tr>
<tr><td>$y = x^2 + 10x + 20$</td><td></td></tr>
<tr><td>$y = x^2 + 2x - 1$</td><td></td></tr>
<tr><td></td><td></td></tr>
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Course: MPM2D – Principles of Mathematics.
Date Created: Friday, October 15, 2021
Modified Date: Friday, February 16th, 2024

Goal/Topic: The Quadratic Formula:

Definition: Roots of an equation.

1. **The value(s) of the variables that makes an equation true**
   Example:
   $y = x^2 - x - 12$

   set $y = 0$
   $0 = (x - 4)(x + 3)$ ← these are the zeros of the equation.
   ← They are also the roots of the equation.

   ➔ **Below are the values that make the equation true.**

$x - 4 = 0$	**$X + 3 = 0$**
$X = 4$	**$X = -3$**

History:

Quadratic equations that can be factored are simple to solve.

But, what about quadratics that can not be factored.

The Greek mathematicians: - had methods to solve quadratics.
Euclid (300 BCE, same as BC)
Pythagoras (500 BCE, same as BC)

Hindu: Mathematicians:
Brahmagupta (700 AD, same as ACE)
Bhaskara (1100 AD, same as ACE)

Helped to develop the Quadratic Formula below:

The roots or zeros	Line of Symmetry
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$x = \frac{-b}{2a}$

Example: Reason you see: $\pm\sqrt{b^2 - 4ac}$

$a^2 = (-a)(-a)$

$a^2 = (a)(a)$

$16 = (-4)(-4)$

$16 = (4)(4)$

Reason you see, ± in front of factor above.

Let's us do a walk through to understand and develop the formula.

page 293: McGraw-Hill Ryerson

Example:	**Quadratic Formula**
$2x^2 + 5x + 1 = 0$	$ax^2 + bx + c = 0$
$x^2 + \frac{5}{2}x + \frac{1}{2} = 0$ 🡐 formula divide by 2	$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ 🡐 divide formula by a
$x^2 + \frac{5}{2}x + (\frac{5}{4})^2 - (\frac{5}{4})^2 + \frac{1}{2} = 0$ 🡐 complete the square	$x^2 + \frac{b}{2a}x + (\frac{b}{2a})^2 - (\frac{b}{2a})^2 + \frac{c}{a} = 0$ 🡐 complete the sq.
$(x^2 + \frac{5}{2}x + (\frac{5}{4})^2) - (\frac{5}{4})^2 + \frac{1}{2} = 0$	$(x^2 + \frac{b}{2a}x + (\frac{b}{2a})^2) - (\frac{b}{2a})^2 + \frac{c}{a} = 0$
$(x + (\frac{5}{4}))^2 - (\frac{5}{4})^2 + \frac{1}{2} = 0$ 🡐 using $(a + b)^2$	$(x + (\frac{b}{2a}))^2 - (\frac{b}{2a})^2 + \frac{c}{a} = 0$
$(x + (\frac{5}{4}))^2 - (\frac{25}{16}) + \frac{8}{16} = 0$ 🡐 add like terms	$(x + (\frac{b}{2a}))^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$ $(x + (\frac{b}{2a}))^2 - \frac{b^2}{4a^2} + \frac{c}{a}(\frac{4a}{4a}) = 0$
$(x + (\frac{5}{4}))^2 - \frac{17}{16} = 0$ 🡐 simplified	Simplified $(x + (\frac{b}{2a}))^2 - (\frac{b^2}{4a^2} - \frac{4ac}{4a^2}) = 0$ $(x + (\frac{b}{2a}))^2 - (\frac{b^2 - 4ac}{4a^2}) = 0$ 🡐 combine factors $(x + (\frac{b}{2a}))^2 = \frac{b^2 - 4ac}{4a^2}$
$(x + (\frac{5}{4}))^2 = \frac{17}{16}$ 🡐 square roots of by sides $x + \frac{5}{4} = \pm\sqrt{\frac{17}{16}} = \pm\frac{\sqrt{17}}{4}$ $x = \frac{-5}{4} \pm\frac{\sqrt{17}}{4}$	Square root both sides: $x + \frac{b}{2a} = \frac{\pm\sqrt{b^2-4ac}}{\sqrt{4a^2}}$ $x = -\frac{b}{2a} + \frac{\pm\sqrt{b^2-4ac}}{2a}$ $x = \frac{-b \pm\sqrt{b^2-4ac}}{2a}$ 🡐 solved; proof.

x = $\frac{-5 \pm \sqrt{17}}{4}$	

Example 1: Find the real roots of an equation.

a) $2x^2 + 9x + 6 = 0$

a = 2
b = 9
c = 6

Substitute into formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-9 \pm \sqrt{9^2 - 4(2)(6)}}{2(2)}$$

1. $2x^2 + 9x + 6 = 0$ a = 2 b = 9 c = 6 Solve for x:
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$x = \frac{-9 \pm \sqrt{9^2 - 4(2)(6)}}{2(2)}$ $x = \frac{-9 \pm \sqrt{81 - 48}}{4}$ $x = \frac{-9 \pm \sqrt{33}}{4}$ The exact roots are: root 1: $x_1 = \frac{-9 + \sqrt{33}}{4} \cong -0.81$ root 2: $x_2 = \frac{-9 - \sqrt{33}}{4} \cong -3.69$

$X_s = \frac{-0.81 + -3.69}{2} = \frac{-4.50}{2} = -2.25$

Formula for axis of symmetry:

$Xs = \frac{-b}{2a} = \frac{-b}{2a} = \frac{-9}{4} = -2.25$

Vertex

$y = 2x^2 + 9x + 6$

$y = f(x) = f(\frac{-9}{4}) = 2x^2 + 9x + 6$

$f(\frac{-9}{4}) = f(\frac{-9}{4}) = 2(\frac{-9}{4})^2 + 9(\frac{-9}{4}) + 6$

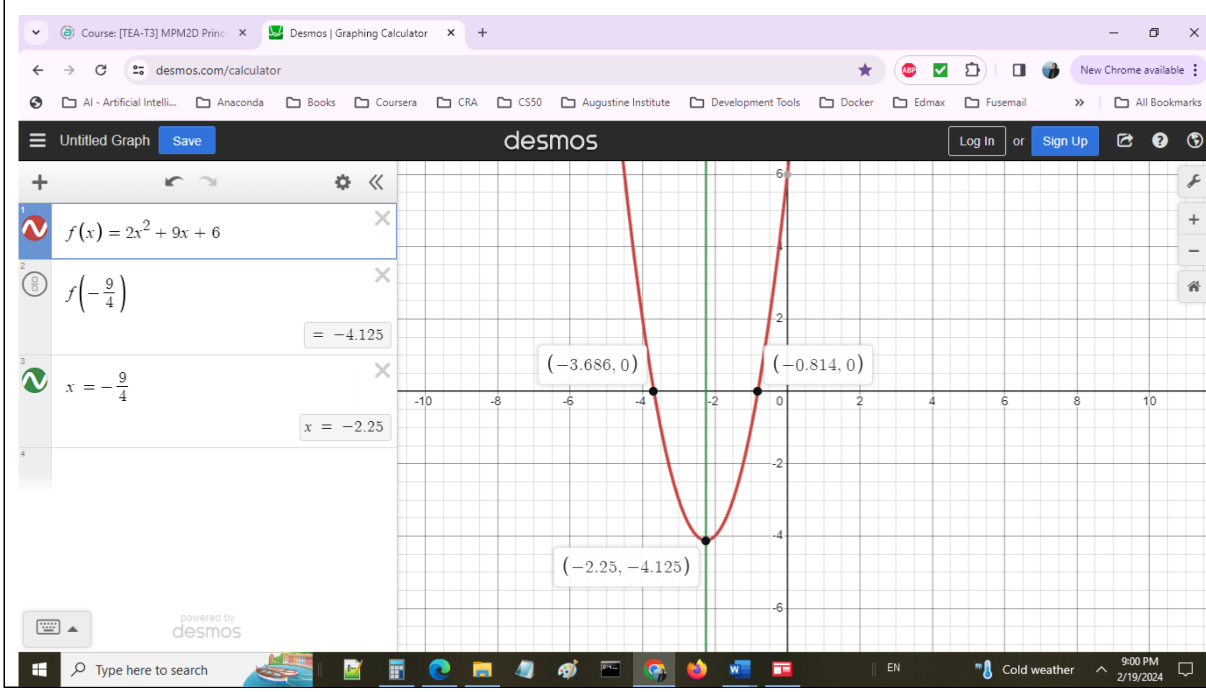
$f(\frac{-9}{4}) = 2(\frac{81}{16}) + (\frac{-81}{4}) + 6$

$f(\frac{-9}{4}) = (\frac{81}{8}) + (\frac{-162}{8}) + 6$

$f(\frac{-9}{4}) = (\frac{81}{8}) + (\frac{-162}{8}) + 6$

$f(\frac{-9}{4}) = (\frac{-81}{8}) + 6 = (\frac{-81}{8}) + (\frac{48}{8})$

$f(\frac{-9}{4}) = (\frac{-81}{8}) + (\frac{48}{8}) = (\frac{-33}{8}) = -4.125$



2. $4x^2 - 12x = -9$

rewrite into the form: $ax^2 + bx + c = 0$

$4x^2 - 12x + 9 = -9 + 9$
$4x^2 - 12x + 9 = 0$

a = 4
b = -12
c = 9

Solve for x:

$$x = \frac{-b \pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-12 \pm\sqrt{(-12)^2 - 4(4)(9)}}{2(4)}$$

$$x = \frac{-12 \pm\sqrt{144 - 144}}{2(4)} = \frac{-(-12)}{2(4)} = \frac{3}{2} = 1.50$$

$$D = \pm\sqrt{b^2 - 4ac} = \pm\sqrt{(-12)^2 - 4(4)(9)} = \sqrt{(-12)^2 - 4(4)(9)}$$

$$D = \pm\sqrt{(-12)^2 - 4(4)(9)} = \pm\sqrt{144 - 144} = 0$$

The discriminant, D, is zero. This means there is double root, which are the same.

∴ the graphic only touches the x-axis at only one point.

Axis of Symmetry:

$$Xs = \frac{-b}{2a} = \frac{-(-12)}{2(4)} = \frac{3}{2} = 1.50$$

Vertex

$y = 4x^2 - 12x + 9$

$$y = f(\frac{3}{2}) = f(\frac{3}{2}) = 4(\frac{3}{2})^2 - 12(\frac{3}{2}) + 9$$

$$y = f(\frac{3}{2}) = 4(\frac{3}{2})^2 - 12(\frac{3}{2}) + 9$$

$$y = f\left(\frac{3}{2}\right) = 4\left(\frac{3}{2}\right)^2 - 12\left(\frac{3}{2}\right) + 9$$

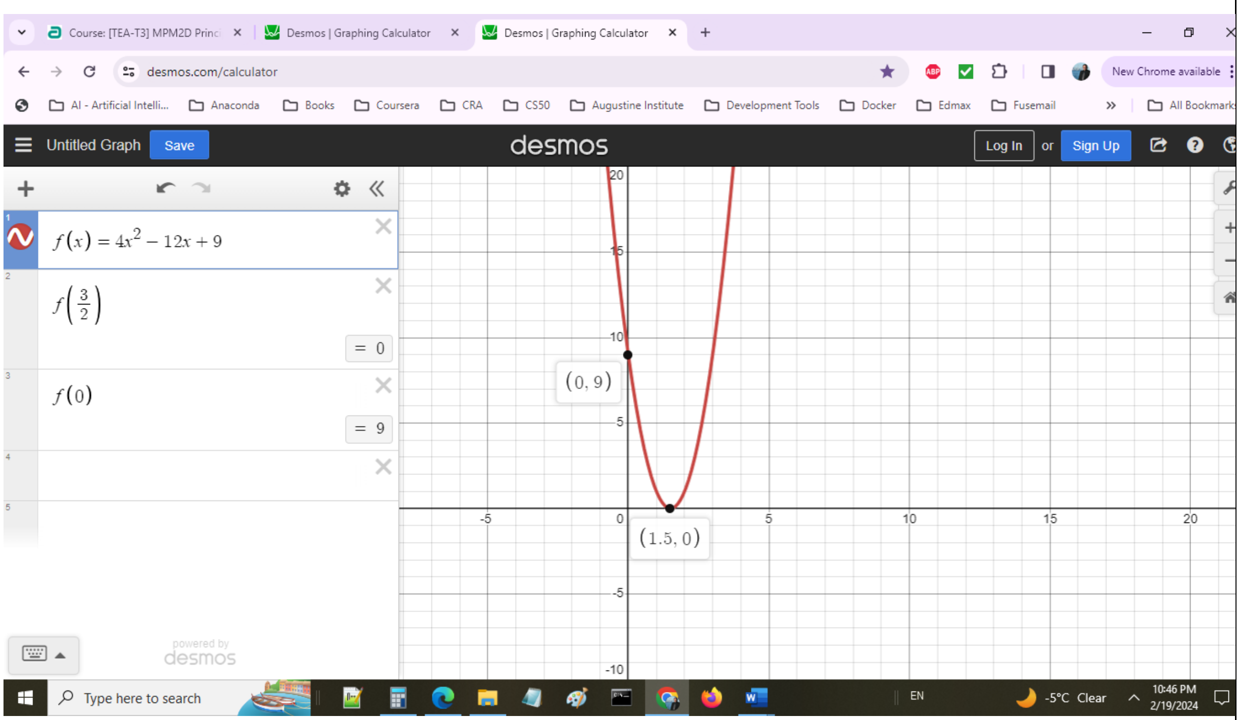
$$y = f\left(\frac{3}{2}\right) = 4\left(\frac{9}{4}\right) - 12\left(\frac{3}{2}\right) + 9$$

$$y = f\left(\frac{3}{2}\right) = 4\left(\frac{9}{4}\right) - 12\left(\frac{3}{2}\right) + 9$$

$$y = f\left(\frac{3}{2}\right) = \left(\frac{18}{2}\right) - \left(\frac{36}{2}\right) + 9$$

$$y = f\left(\frac{3}{2}\right) = 9 - 18 + 9 = 0$$

$$\text{vertex}(x,y) = \left(\left(\frac{3}{2}\right), 0\right)$$



Topic: Interpreting Quadratic Equation Roots:

Goal: Determine the number of roots of a quadratic equation, and relate these roots to the corresponding relation.

Quadratic relations may have two, one, or no x-intercepts illustrated by the following examples:

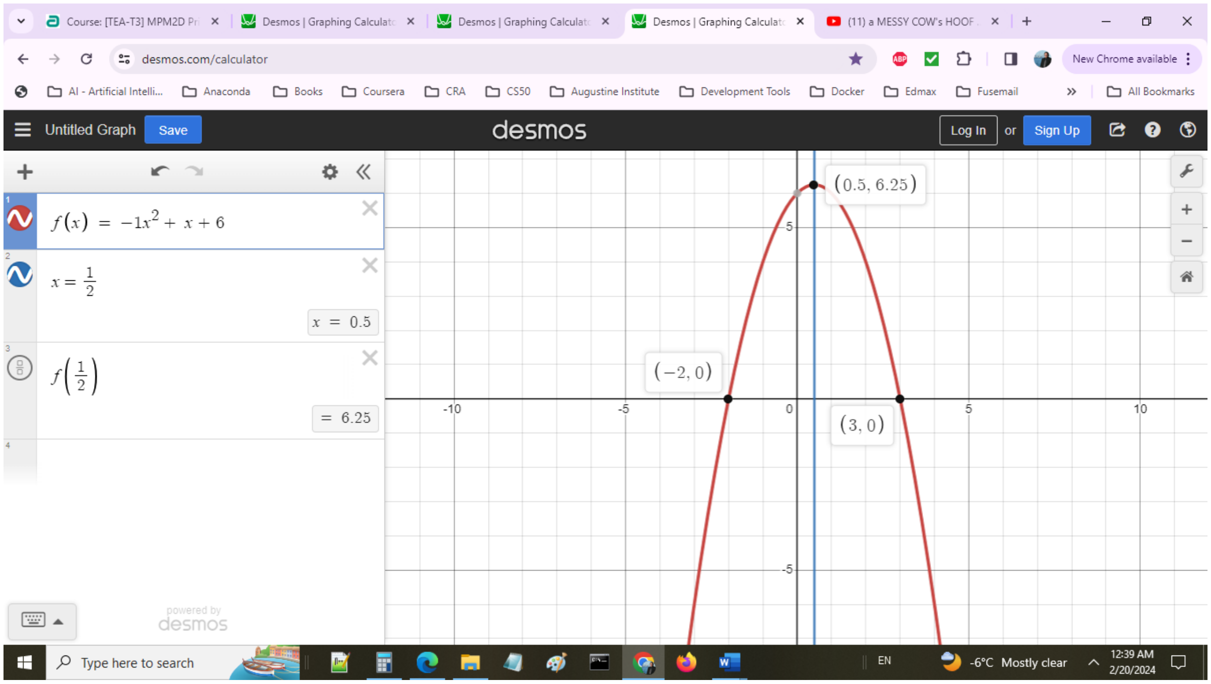
1. $-x^2 + x + 6 = 0$, has two roots
2. $x^2 - 6x + 9 = 0$, has one root
3. $2x^2 - 4x + 5 = 0$, has no roots.

Please see page 344 of the Nelson text book.

Case (1)

1. $-x^2 + x + 6 = 0$ $a = -1$ $b = 1$ $c = 6$ $x = \frac{-b \pm\sqrt{b^2-4ac}}{2a}$ Solve for x:
$x = \frac{-1 \pm\sqrt{1^2-4(-1)(6)}}{2(-1)}$ $x = \frac{-1 \pm\sqrt{1+24}}{2(-1)}$ $x = \frac{-1 \pm\sqrt{25}}{2(-1)}$ The exact roots are: root 1: $x_1 = \frac{-1 \pm\sqrt{25}}{-2} = \frac{-1+5}{-2} = \frac{-1+5}{-2} = \frac{4}{-2} = -2.0$ root 2: $x_2 = \frac{-1 -\sqrt{25}}{-2} = \frac{-1-5}{-2} = \frac{-6}{-2} = \frac{-6}{-2} = 3.0$ $X_S = \frac{-2.0 + 3.0}{2} = \frac{1}{2}$ Formula for axis of symmetry:

$Xs = \frac{-b}{2a} = \frac{-b}{2a} = \frac{-1}{2(-1)} = \frac{1}{2}$ Vertex $y = -x^2 + x + 6$ $y=f(x) = f(\frac{1}{2}) = -x^2 + x + 6$
$f(\frac{1}{2}) = f(\frac{1}{2}) = -x^2 + x + 6$ $f(\frac{1}{2}) = -1(\frac{1}{2})^2 + (\frac{1}{2}) + 6$ $f(\frac{1}{2}) = -(\frac{1}{4}) + (\frac{1}{2}) + 6$ $f(\frac{1}{2}) = -(\frac{1}{4}) + (\frac{2}{4}) + 6$ $f(\frac{1}{2}) = (\frac{25}{4}) = 6.25$ Vertex(x, y) = $((\frac{1}{2}), ((\frac{25}{4})) = (0.50, 6.25)$
1. **$-x^2 + x + 6 = 0$, has two roots,** that means that the graph will have two x-intercepts, P_1 **= (-2, 0),** P_2 **= (3, 0), with vertex (x, y) =** $((\frac{1}{2}), ((\frac{25}{4})) = (0.50, 6.25)$



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$f(x) = -1x^2 + x + 6$
$x = \frac{1}{2}$
$x = 0.5$
$f\left(\frac{1}{2}\right)$
$= 6.25$
(0.5, 6.25)
(−2, 0)
(3, 0)

Case (2)

2. $x^2 - 6x + 9 = 0$

a = 1
b = -6
c = 9

$$x = \frac{-b \pm\sqrt{b^2-4ac}}{2a}$$

Solve for x:

$$x = \frac{-(-6) \pm\sqrt{(-6)^2-4(1)(9)}}{2(1)}$$

$$x = \frac{6 \pm\sqrt{36-36}}{2(1)}$$

$$x = \frac{6 \pm\sqrt{0}}{2(1)} = \frac{6}{2} = 3$$

$$D = \pm\sqrt{b^2 - 4ac} = \pm\sqrt{(-6)^2 - 4(1)(9)} = \sqrt{(-6)^2 - 4(1)(9)}$$

$$D = \pm\sqrt{36 - 36} = \sqrt{0} = 0$$

The discriminant, D, is zero. This means there is double root, which are the same.

∴ the graphic only touches the x-axis at only one point.

Formula for axis of symmetry:

$$Xs = \frac{-b}{2a} = \frac{-b}{2a} = \frac{-(-6)}{2(1)} = \frac{6}{2} = \frac{6}{2} = 3$$

Vertex

$y = x^2 - 6x + 9$

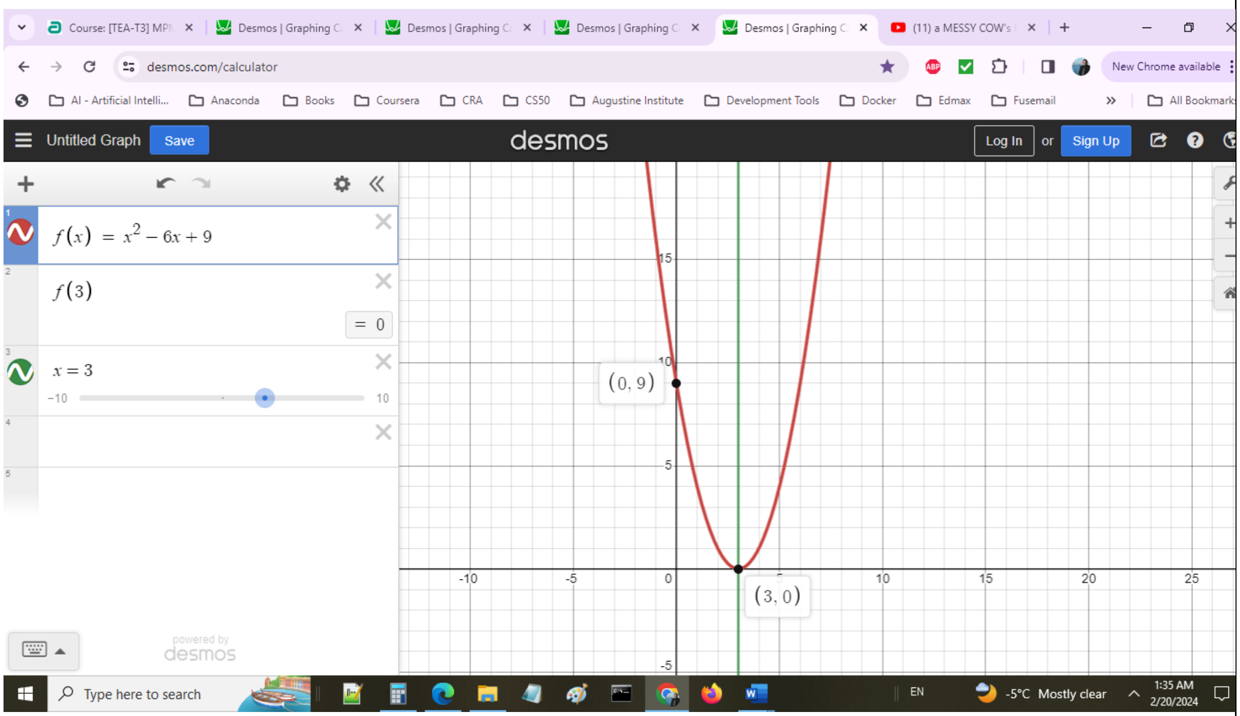
$y=f(x) = f(3) = (3)^2 - 6(3) + 9$

$f(3) = (3)^2 - 6(3) + 9$

$f(3) = 9 - 18 + 9 = 0$

Vertex(x, y) = (3.0, 0)

2. **x^2 - 6x + 9 = 0, has one root,** that means that the graph will have one x-intercepts, P_1 **= (3, 0), with vertex (x, y)** = (3.0, 0)



Case (3)

3. $2x^2 - 4x + 5 = 0$

a = 2
b = -4
c = 5

x = $\frac{-b \pm\sqrt{b^2-4ac}}{2a}$

Solve for x:

x = $\frac{-(-4) \pm\sqrt{(-4)^2-4(2)(5)}}{2(2)}$

x = $\frac{4 \pm\sqrt{16-40}}{2(1)}$

x = $\frac{4 \pm\sqrt{-24}}{2(1)}$

D = $\pm\sqrt{16-40}$

D = $\pm\sqrt{-24}$

The discriminant, D, has a negative square root, $\sqrt{-24}$. This means there is zero roots, for the quadratic equation-4, that is, no solutions.

∴ the graphic never crosses the x-axis.

Formula for axis of symmetry:

Xs = $\frac{-b}{2a} = \frac{-(-4)}{2(2)} = \frac{-(-4)}{2(2)} = \frac{4}{4} = 1$

Vertex

y = $2x^2 - 4x + 5$

y=f(x) = f(1) = $2x^2 - 4x + 5$

f(1) = $2(1)^2 - 4(1) + 5$

$f(1) = 2(1)^2 – 4(1) + 5$
$f(1) = 2 – 4 + 5$
$f(1) = 7 – 4 = 3$

Vertex (x, y) = (1, 3)

Y – intercept, when x = 0
$f(0) = 2x^2 - 4x + 5$
$f(0) = 2(0)^2 – 4(0) + 5$
$f(0) = 5$

Y – intercept, when x = 0, occurs at y = 5.

3. $2x^2 - 4x + 5 = 0$, **has no roots,** that means that the graph will have no x-intercepts, but **with vertex (x, y)** = (1, 3), the y – intercept is at (0, 5). See graph below for details.