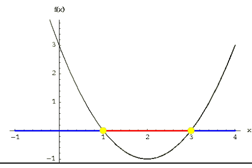
MHF4U: Advanced Functions

Solving Polynomial Inequalities

J. Garvin



Polynomial Inequalities

Sometimes, we wish to know the intervals on which a polynomial function is above or below a certain value.

For example, when is $5x^3 - 4x + 1 > 0$.

A *polynomial inequality* can be solved like a linear inequality, but may need to be analyzed using multiple cases.

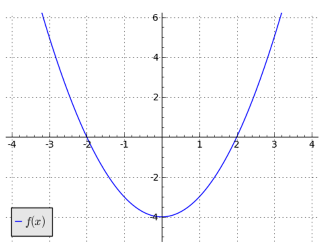
We can use the rules developed earlier to solve polynomial inequalities.

Solving Polynomial Inequalities Graphically

Example

Solve $x^2 - 4 < 0$.

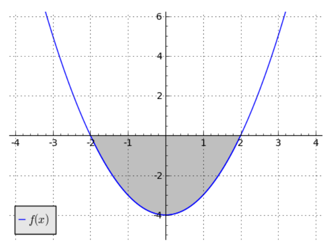
$f(x) = x^2 - 4$ is a quadratic function with its vertex at $(0, -4)$ and Q2-Q1 end behaviour, as show below.



Solving Polynomial Inequalities Graphically

$f(x) < 0$ when it is below the x-axis.

As shown on the graph, $f(x)$ is below the x-axis when it is between the two x-intercepts at -2 and 2.



Therefore, $x^2 - 4 < 0$ on the interval $(-2, 2)$.

Solving Polynomial Inequalities Algebraically

If a polynomial inequality, $P(x)$, has the form $P(x) \oplus 0$ (where $\oplus$ represents some inequality symbol) then the inequality can be found by determining the roots of the polynomial expression.

Sometimes this is easy to do using a graph, but in many cases we will need an algebraic solution instead.

This may be done using either *cases* or *intervals*.

Solving Polynomial Inequalities Using Cases

Example

Solve $2x^3 - 3x^2 - 11x + 6 \geq 0$.

Factor the polynomial expression to determine its x-intercepts.

$2(3)^3 - 3(3)^2 - 11(3) + 6 = 0$, so $x - 3$ is a factor.

3	2	-3	-11	6
+		6	9	-6
	2	3	-2	0

Therefore, $(x-3)(2x^2 + 3x - 2) \geq 0$.

Decomposing, $(x-3)(x+2)(2x-1) \geq 0$.

Solving Polynomial Inequalities Using Cases

Since there are three factors, the polynomial will be greater than or equal to zero when the product is non-negative.

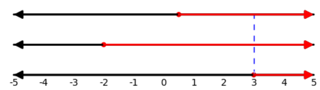
This will occur when exactly one of the factors is non-negative, or when all three factors are non-negative.

This gives us *four* cases to consider.

Solving Polynomial Inequalities Using Cases

Case 1: All factors are non-negative.

$$x-3 \geq 0 \qquad x+2 \geq 0 \qquad 2x-1 \geq 0$$
$$x \geq 3 \qquad x \geq -2 \qquad x \geq \frac{1}{2}$$

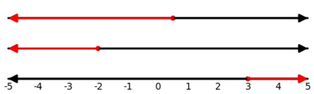


Since $[3, \infty)$ is common, it is a solution.

Solving Polynomial Inequalities Using Cases

Case 2: $x-3$ is non-negative.

$$x-3 \geq 0 \qquad x+2 < 0 \qquad 2x-1 < 0$$
$$x \geq 3 \qquad x < -2 \qquad x < \frac{1}{2}$$

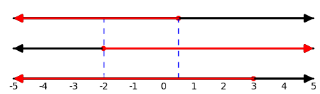


There is no region in common, so there is no solution for this case.

Solving Polynomial Inequalities Using Cases

Case 3: $x+2$ is non-negative.

$$x-3 < 0 \qquad x+2 \geq 0 \qquad 2x-1 < 0$$
$$x < 3 \qquad x \geq -2 \qquad x < \frac{1}{2}$$

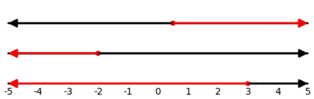


Since $\left[-2, \frac{1}{2}\right]$ is common, it is a solution.

Solving Polynomial Inequalities Using Cases

Case 4: $2x-1$ is non-negative.

$$x-3 < 0 \qquad x+2 < 0 \qquad 2x-1 \geq 0$$
$$x < 3 \qquad x < -2 \qquad x \geq \frac{1}{2}$$

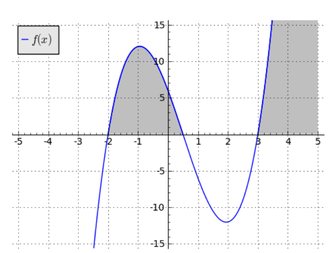


There is no region in common, so there is no solution for this case.

Solving Polynomial Inequalities Using Cases

Combining the previous cases gives a final solution of $2x^3 - 3x^2 - 11x + 6 \geq 0$ on $\left[-2, \frac{1}{2}\right] \cup [3, \infty)$.

Graphing $f(x) = 2x^3 - 3x^2 - 11x + 6$ shows that this is the case.



Solving Polynomial Inequalities Using Intervals

Using cases is an algebraic alternative to using a graph, but can be tedious.

Since a polynomial function changes sign when it crosses the x-axis, each x-intercept divides a polynomial function into intervals that are either positive or negative.

We can test values inside of each interval to determine whether the function is positive or negative within that interval.

Often, this is a much faster method than using cases.

Solving Polynomial Inequalities Using Intervals

Example

Solve $-2x^4 + 6x^3 + 6x^2 - 14x - 12 > 0$

First, factor the equation.

2	-2	6	6	-14	-12
+		-4	4	20	12
	-2	2	10	6	0

3	-2	2	10	-6
+		-6	-12	-6
	-2	-4	-2	0

Therefore, $-2(x-2)(x-3)(x^2+2x+1) > 0$.

Factoring the perfect square, $-2(x-2)(x-3)(x+1)^2 > 0$.

Note that the FT would identify $x+1$ as a factor, but would not indicate that it is order 2.

Solving Polynomial Inequalities Using Intervals

The polynomial has roots at 2, 3 and -1.

These values divide the polynomial into four intervals.

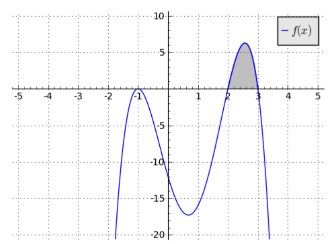
Test values within each interval to determine the sign.

Interval	$(-\infty, -1)$	$(-1, 2)$	$(2, 3)$	$(3, \infty)$
x	-2	0	$\frac{5}{2}$	4
Sign of $P(x)$	$-$	$-$	$+$	$-$

Solving Polynomial Inequalities Using Cases

By the table, $-2x^4 + 6x^3 + 6x^2 - 14x - 12 > 0$ on $(2, 3)$.

Graphing $f(x) = -2x^4 + 6x^3 + 6x^2 - 14x - 12$ shows that this is true.



Questions?