Asymptotes and Holes in Rational Functions

Vertical Asymptotes

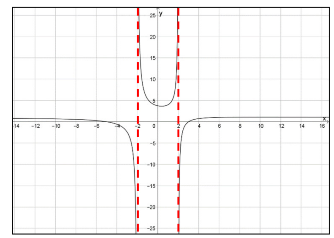
- Vertical asymptotes occur at x-values that make the denominator equal 0 (but not the numerator).

Examples

$$f(x)=\frac{x^2+2x-15}{x^2-4}$$

$$=\frac{(x+5)(x-3)}{(x+2)(x-2)}$$

- As x approaches -2 or 2, the denominator approaches 0.
- So, we'd be dividing by very small numbers, resulting in very large results (positive or negative).
- Vertical asymptotes are x = -2 and x = 2.

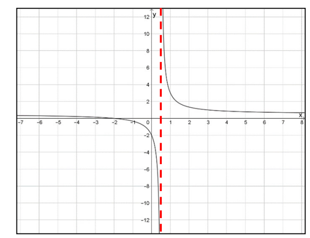


$$g(x)=\frac{x^2+5x+6}{2x^2+5x-3}$$

$$=\frac{(x+2)\cancel{(x+3)}}{(2x-1)\cancel{(x+3)}}$$

Vertical asymptote is $x=\frac{1}{2}$

NOTE: Something important does occur where x = -3. We'll discuss it later!



Horizontal Asymptotes

- Horizontal asymptotes are based on the end behaviour of a function.
- If the function approaches a specific value as x becomes very large (positively or negatively), a horizontal asymptote will occur.

Example

Consider the function $f(x)=\frac{x-5}{x^2-9}$.

As $x \to \infty$

x	$f(x)$
10	0.054945054
100	0.009508557
1 000	0.000995008
5 000	0.0001998
10 000	0.00009995
500 000	0.000001999
1 000 000	0.000000999

As $x \to -\infty$

x	$f(x)$
-10	-0.164835164
-100	-0.010509458
-1 000	-0.001005009
-5 000	-0.0002002
-10 000	-0.00010005
-500 000	-0.000002
-1 000 000	-0.000001

So, as $x \to \pm\infty$, $f(x) \to 0$, which creates a horizontal asymptote of y = 0.

NO!

? Do we always need to make tables in order to find horizontal asymptotes?

Finding Horizontal Asymptotes by Inspection

Consider the function shown below.

As $x \to \infty$, the denominator grows much more quickly than the numerator due to the x^2 term.

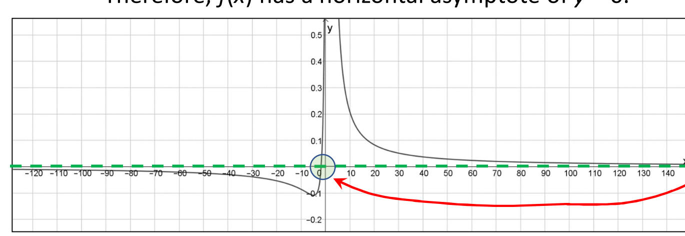
$$f(x)=\frac{4x+7}{3x^2-8x+5}$$

grows as $x \to \infty$

REALLY GROWS AS $x \to \infty$

Similarly, as $x \to -\infty$, the denominator becomes much larger (ignoring sign) than the numerator, causing $f(x)$ to approach a value of 0.

Therefore, $f(x)$ has a horizontal asymptote of y = 0.



NOTE: Since horizontal asymptotes simply describe the function's end behaviour, there is no issue with the graph passing through them.

If the numerator polynomial has a lower degree than the denominator polynomial, there will be a horizontal asymptote of y = 0.

What if the numerator polynomial has a higher degree than the denominator polynomial?

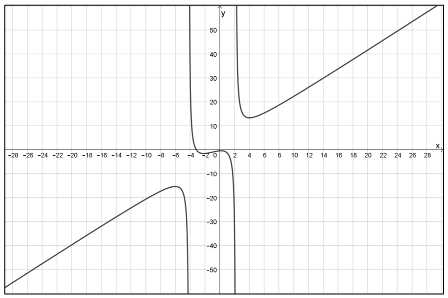
$$g(x) = \frac{2x^3 + 5x^2 - 3x + 4}{x^2 + 2x - 9}$$

REALLY GROWS AS $x \to \infty$ (numerator)

grows as $x \to \infty$ (denominator)

$g(x)$ does not approach any number as $x \to \pm\infty$

Therefore, $g(x)$ does not have a horizontal asymptote.



If the numerator polynomial has a higher degree than the denominator polynomial, there will not be a horizontal asymptote.

What if the numerator polynomial and the denominator polynomial have the same degree?

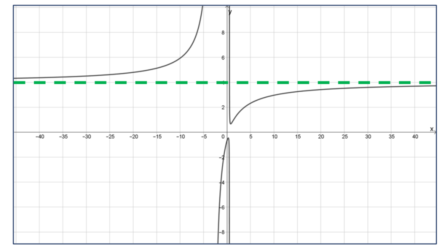
$$h(x) = \frac{8x^2 - 7x + 2}{2x^2 + 5x - 3} \xrightarrow{\text{as } x \to \pm\infty} \frac{8x^2}{2x^2} = 4$$

Therefore, $h(x)$ has a horizontal asymptote of $y = 4$.

Note

We can arrive at this conclusion by simply dividing the numerator polynomial's leading coefficient by the denominator polynomial's lading coefficient.

$$h(x) = \frac{8x^2 - 7x + 2}{2x^2 + 5x - 3}$$



If the numerator an denominator polynomials have the same degree, the horizontal asymptote can be found by dividing the leading coefficients.

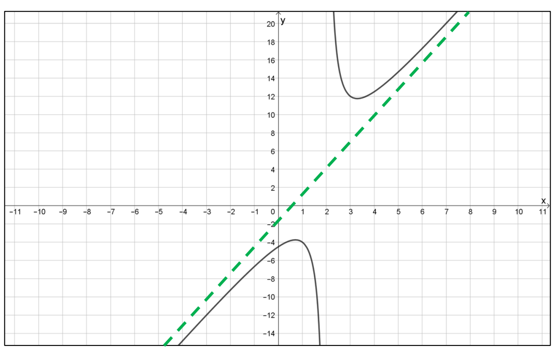
(Linear) Oblique Asymptotes

- Like horizontal asymptotes, oblique asymptotes are based on the end behaviour of a function.
- Oblique asymptotes occur when the degree of the numerator polynomial is one higher than the degree of the denominator polynomial.

$$f(x) = \frac{3x^2 - 8x + 9}{x - 2} = 3x - 2 + \frac{5}{x-2} \to 3x - 2 \text{ as } x \to \pm\infty$$

2	3	−8	9
	↓	6	−4
	3	−2	5

Therefore, $f(x)$ has an oblique asymptote of $y = 3x - 2$.



NOTE:
You might need to use long division instead of synthetic division.

NOTE:
Since oblique asymptotes simply describe the function's end behaviour, there is no issue with the graph passing through them.

Holes

- A hole occurs when the numerator polynomial and the denominator polynomial have a common binomial factor.

$$f(x) = \frac{3x^2 - 11x + 10}{x - 2}$$

$$= \frac{(3x-5)(x-2)}{x-2}$$

$$= 3x - 5, \quad x \neq 2$$

Therefore, $f(x)$ is simply the graph of $y = 3x - 5$ with a hole where $x = 2$.