Name: **Date:**

2A – 1.4 Limit of a Function and 1.5 Properties of Limits

Lesson Goals:

- Be able to evaluate a limit using substitution
- Be able to evaluate a limit of indeterminant form using a variety of algebraic strategies

1) Limits

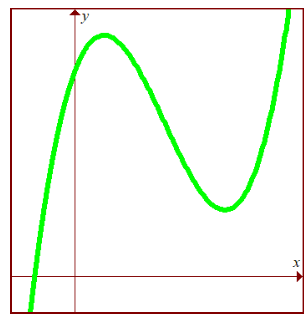
- Limits allow us to calculate the behaviour of an expression (or function) at points infinitely close to a value, called the limiting factor.

$$\lim_{x\to 2}(4x^2+9x-1)$$

- We will use limits to:
  - Find slope of tangents
  - Find the end behaviour of functions, as $x \to \pm\infty$, $y \to ?$
  - Find behaviour as function approaches a vertical asymptote

2) Limits of Polynomial Functions

- Polynomial functions, $P(x)$, are continuous at every number, so $\lim_{x\to a} P(x) = P(a)$.



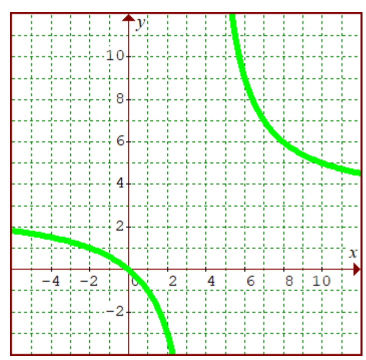
Example 1: Evaluate.

a) $\lim_{x\to 5}(x^2+2x-3)$

b) $\lim_{x\to 4}(x^3+7x)$

3) Limits of Rational Functions

- Rational functions, $\frac{P(x)}{Q(x)}$, are continuous at most numbers, so $\lim_{x\to a}\frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}$ only if $Q(a) \neq 0$.



Example 2: Evaluate.

a) $\lim_{x\to -2}\left(\frac{3x}{x-4}\right)$

b) $\lim_{x\to 4}\left(\frac{3x}{x-4}\right)$

c) $\lim_{x\to 10}\left(\frac{3x}{x-4}\right)$

4) Indeterminate Form

- Substitution may not give enough information to determine the value of a limit.
- The limit may be an indeterminate form.
- After substitution, indeterminate form may look like:

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0(\infty) \qquad 1^{\infty} \qquad \infty-\infty \qquad 0^0 \qquad \infty^0$$

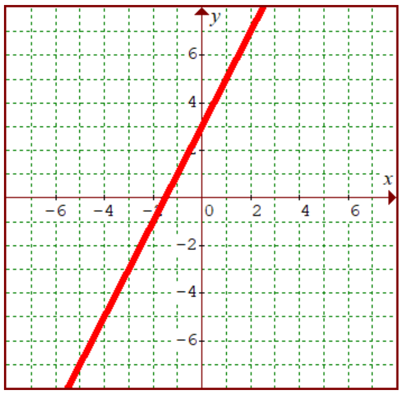
- If one of the above appears, we must start over:
  - Use algebra to simplify the limit expression
  - Then try substitution again

5) Indeterminant Form $\frac{0}{0}$

Example 3: Evaluate.

- Strategy #1: factor, divide out, substitute

a) $\lim_{x\to-1}\left(\frac{2x^2+5x+3}{x+1}\right)$



b) $\lim_{h\to 0}\left(\frac{(2+h)^2-4}{h}\right)$

Example 4: Evaluate.

- Strategy #2: multiply by common denominator, divide out, substitute

a) $\lim_{x\to-1}\left(\frac{4+\frac{4}{3x+2}}{x+1}\right)$

b) $\lim_{x\to 2}\left(\frac{\frac{1}{3x}-\frac{1}{6}}{x-2}\right)$

Example 5: Evaluate.

- Strategy #3: multiply by the conjugate, divide out, substitute

a) $\lim_{x \to 9} \left(\frac{\sqrt{x}-3}{x-9}\right)$

b) $\lim_{x \to 2} \left(\frac{x-2}{\sqrt{x+2}-\sqrt{2x}}\right)$

Example 6: Evaluate.

- Strategy #4: substitute a "nice" expression, divide out, substitute

a) $\lim_{x \to 9} \left(\frac{\sqrt{x}+3}{x-9}\right)$

b) $\lim_{x \to -8} \left(\frac{2+\sqrt[3]{x}}{8+x}\right)$

6) Summary of Limit Strategies

- Try substitution first!
  - If you get "undefined", $\frac{k}{0}$, $k \in R, k \neq 0$, then "no limit exists" (usually due to a vertical asymptote).
  - If you get a real number answer, the number is the answer to the limit question.
  - If you get "indeterminate" $\frac{0}{0}$, you need to try one (or more) of the technique below:
    - If the limit's denominator is a polynomial:
      - Fully factor, cancel and state restrictions, then try substituting again.
    - If the limit has a square root:
      - Rationalize the denominator or numerator by multiplying by the conjugate, cancel and state restrictions, then try substitution again. Note: for hard questions, you may need to rationalize twice!
    - If the limit is made up of compound or complicated looking fractions (a fraction in a fraction):
      - Fully simplify the fraction into simplest form, cancel and state restrictions, then try substitution again
    - If the limit has variable with cube or fourth roots, or rational exponent:
      - Change the variable so that the expression is easier to work with, factor, cancel and state restrictions, and then try substituting again

Homework: Worksheet (Optional: Page 45 #4, 6-9)

1.4 Limit of a Function and 1.5 Properties of Limits Worksheet

1. $\lim_{x\to 3}(2x-5)$
2. $\lim_{x\to -1}\frac{3x+4}{x-1}$
3. $\lim_{x\to 5}\frac{x-1}{x-5}$
4. $\lim_{x\to -5}\frac{x+5}{x^2-25}$
5. $\lim_{x\to 2}\frac{x^3-8}{x^2+x-6}$
6. $\lim_{x\to 0}\frac{4x^3-x^2}{x^2+10x}$
7. $\lim_{x\to -3}\frac{x^2-2x-15}{x^2+10x+21}$
8. $\lim_{x\to -2}\frac{6x+12}{x^2+3x+2}$
9. $\lim_{x\to 1}\frac{x^2-x-2}{x^2-2x+1}$
10. $\lim_{x\to -1}\frac{(x^2-1)(x+1)}{x^4-1}$
11. $\lim_{x\to 9}\frac{\sqrt{x}-1}{x-8}$
12. $\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$
13. $\lim_{x\to 2}\frac{\sqrt{2x}-2}{x-2}$
14. $\lim_{x\to 5}\frac{\sqrt{x+4}-3}{x-5}$
15. $\lim_{x\to 2}\frac{\frac{4}{x}-\frac{2}{5}}{x+1}$
16. $\lim_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{x-2}$
17. $\lim_{x\to -3}\frac{\frac{1}{3x}+\frac{1}{9}}{x+3}$
18. $\lim_{x\to 1}\frac{\frac{1}{2x-3}+1}{x-1}$

19. $\lim_{x\to 8}\frac{\sqrt[3]{x}-2}{x-8}$ → Let $\sqrt[3]{x}=a$, then $x=a^3$, and factor the denominator using difference of cubes.

20. $\lim_{x\to 16}\frac{\sqrt[4]{x}-2}{x-16}$ → Let $\sqrt[4]{x}=a$, then $x=a^4$, and FULLY factor the denominator using difference of squares.

21. $\lim_{x\to 4}\frac{\sqrt{x}-2}{\sqrt{x^3}-8}$
22. $\lim_{x\to 1}\frac{x^{1/6}-1}{x-1}$
23. $\lim_{x\to 0}\frac{(x+8)^{1/3}-2}{x}$

Unofficial Answers:

1. 1
2. $-\frac{1}{2}$
3. No limit
4. $-\frac{1}{10}$
5. $\frac{12}{5}$
6. 0
7. -2
8. -6
9. No limit
10. 0
11. 2
12. $\frac{1}{2}$
13. $\frac{1}{2}$
14. $\frac{1}{6}$
15. $\frac{8}{15}$
16. $-\frac{1}{4}$
17. $-\frac{1}{27}$
18. -2
19. $\frac{1}{12}$
20. $\frac{1}{32}$
21. $\frac{1}{12}$
22. $\frac{1}{6}$
23. $\frac{1}{12}$