MCV4U: Calculus & Vectors

Chain Rule of Derivatives

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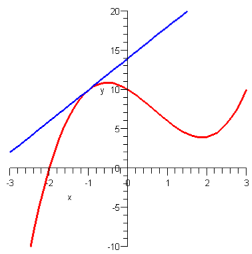

Product Rule

Recap

Determine the derivative of $f(x) = (x^4 - 3x^2 + 7x)\sqrt[3]{x}$.

Let $p(x) = x^4 - 3x^2 + 7x$ and $q(x) = x^{\frac{1}{3}}$.

Then $p'(x) = 4x^3 - 6x + 7$ and $q'(x) = \frac{1}{3}x^{-\frac{2}{3}}$.

$$f'(x) = (4x^3 - 6x + 7)\left(x^{\frac{1}{3}}\right) + \left(\tfrac{1}{3}x^{-\frac{2}{3}}\right)(x^4 - 3x^2 + 7x)$$
$$\text{or } \tfrac{13}{3}x^{\frac{10}{3}} - 7x^{\frac{4}{3}} + \tfrac{28}{3}x^{\frac{1}{3}}$$

Chain Rule

Sometimes we need to differentiate a *composite function*, $f(x) = g \circ h = g(h(x))$.

In this case, there is an "inner" function, $h(x)$, and an "outer" function, $g(x)$.

For instance, the function $f(x) = 2(x-3)^2$ can be thought of as being composed of the inner function, $x - 3$, and the outer function, $2x^2$.

To differentiate these types of functions, use the Chain Rule.

Chain Rule

If $f(x) = g(h(x))$, then $f'(x) = g'(h(x)) \cdot h'(x)$.

If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Chain Rule

A comprehensive proof of the chain rule is rather complicated, so consider an informal "argument" that explains the chain rule instead.

This average rate of change is defined by two quantities, Δi and Δd, which represent changes in the independent and dependent variables respectively.

If y is a function that depends on another function u, such that $y = f(u(x))$, then values of y will change according to changes in u, just as values of u will change according to changes in x.

If Δu, Δy and Δx represent changes in the variables, then the average rate of change in function y will be $\frac{\Delta y}{\Delta u}$, and the average rate of change in function u will be $\frac{\Delta u}{\Delta x}$.

Chain Rule

Since all of these changes are quantities, the relationship $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$ will hold.

Recall that the derivative is an expression that represents the instantaneous rate of change of a function at a specific point.

Thus, $\frac{dy}{dx} = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, where Δx andΔy are $(x + h) - x = h$ and $f(x+h) - f(x)$ respectively.

Substituting the above relationship, $\frac{dy}{dx} = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$.

Using limit properties, $\frac{dy}{dx} = \frac{\lim\limits_{\Delta x \to 0} \Delta y}{\lim\limits_{\Delta x \to 0} \Delta u} \cdot \frac{\lim\limits_{\Delta x \to 0} \Delta u}{\lim\limits_{\Delta x \to 0} \Delta x}$.

Chain Rule

As $\Delta x \to 0$, $\Delta u \to 0$.

Therefore, $\frac{dy}{dx} = \frac{\lim\limits_{\Delta u \to 0} \Delta y}{\lim\limits_{\Delta u \to 0} \Delta u} \cdot \frac{\lim\limits_{\Delta x \to 0} \Delta u}{\lim\limits_{\Delta x \to 0} \Delta x}$.

Note that $\frac{dy}{du} = \frac{\lim\limits_{\Delta u \to 0} \Delta y}{\lim\limits_{\Delta u \to 0} \Delta u}$ and $\frac{du}{dx} = \frac{\lim\limits_{\Delta x \to 0} \Delta u}{\lim\limits_{\Delta x \to 0} \Delta x}$.

Therefore, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ as required.

Chain Rule

Example

Determine the derivative of $f(x) = 3(4x^2 - 3x)^5$.

While the function could be expanded using the Binomial Theorem, it is faster to use the chain rule.

Let the inner function be $h(x) = 4x^2 - 3x$, with derivative $h'(x) = 8x - 3$.

The outer function $g(h) = 3h^5$, with derivative $g'(h) = 15h^4$.

$$\begin{aligned} f'(x) &= g'(h(x)) \cdot h'(x) \\ &= 15(4x^2 - 3x)^4(8x - 3) \end{aligned}$$

Chain Rule

Example

Determine the derivative of $f(x) = \dfrac{2}{(x^3 - 5x^2)^3}$.

Rewrite the function as $f(x) = 2(x^3 - 5x^2)^{-3}$.

The inner function is $h(x) = x^3 - 5x^2$, with derivative $h'(x) = 3x^2 - 10x$.

The outer function is $g(h) = 2h^{-3}$, with derivative $g'(h) = -6h^{-4}$.

$$\begin{aligned} f'(x) &= -6(x^3 - 5x^2)^{-4} \cdot (3x^2 - 10x) \\ \text{or } &\frac{60 - 18x}{(x-5)^4 x^7} \end{aligned}$$

Chain Rule

Example

Determine the value of $f'(-1)$ if $f(x) = (5x^3 - 1)(2x^2 + x)^4$.

In this example, we need to use both the product rule and the chain rules.

$$\begin{aligned} f'(x) =& 15x^2(2x^2 + x)^4 + (5x^3 - 1)(4)(2x^2 + x)^3(4x + 1) \\ f'(-1) =& 15(-1)^2(2(-1)^2 + (-1))^4 + \\ & (5(-1)^3 - 1)(4)(2(-1)^2 + (-1))^3(4(-1) + 1) \\ =& 87 \end{aligned}$$

The Chain Rule with Leibniz Notation

Using Leibniz notation, the chain rule is $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ where y is a function of u and u is a function of x.

Thus, u is the inner function, and y is the outer function.

In cases where functions are (or can be) broken down into such relationships, it may be easier to find the individual derivatives $\frac{dy}{du}$ and $\frac{du}{dx}$ separately, then multiply and substitute.

The Chain Rule with Leibniz Notation

Example

If $y = 2u^4 + 5u$ and $u = 5x^2 + 1$, determine $\frac{dy}{dx}$.

The derivative of u with respect to x is $\frac{du}{dx} = 10x$.

The derivative of y with respect to u is $\frac{dy}{du} = 8u^3 + 5$.

Therefore, the derivative of the composite function is

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= (8u^3 + 5)(10x) \\ &= (8(5x^2 + 1)^3 + 5)(10x) \end{aligned}$$

This can probably be simplified, but is in a workable state.

The Chain Rule with Leibniz Notation

Example

Determine the equation of the tangent to $y = \sqrt{u}$, if $u = 3x^2 + 4$, when $x = 2$.

The derivative of u with respect to x is $\frac{du}{dx} = 6x$.

The derivative of y with respect to u is $\frac{dy}{du} = \frac{1}{2\sqrt{u}}$.

Therefore, the derivative of the composite function is

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{1}{2\sqrt{u}} \cdot 6x \\ &= \frac{6x}{2\sqrt{3x^2 + 4}} \end{aligned}$$

The Chain Rule with Leibniz Notation

At $x = 2$, $y = \sqrt{3(2)^2 + 4} = 4$. Therefore, the point of tangency is $(2, 4)$.

Calculate the slope at $x = 2$.

$$\left.\frac{dy}{dx}\right|_{x=2} = \frac{6(2)}{2[4]} = \frac{3}{2}$$

Use the slope and point of tangency to find the equation of the tangent.

$$y = \tfrac{3}{2}(x-2) + 4 = \tfrac{3}{2}x + 1$$

The Chain Rule with Leibniz Notation

Even if a function is not explicitly broken up into two separate functions, this same technique can be used.

Example

If $y = \sqrt[3]{3x^2 - 7x} + 4$, determine $\frac{dy}{dx}$.

Let $u = 3x^2 - 7x$ and $y = \sqrt[3]{u} + 4 = u^{1/3} + 4$.

Then $\frac{du}{dx} = 6x - 7$ and $\frac{dy}{du} = \frac{1}{3}u^{-2/3}$.

Thus, $\frac{dy}{dx} = \frac{1}{3}u^{-2/3} \cdot (6x - 7) = \dfrac{(3x^2 - 7)^{-2/3}}{3} \cdot (6x - 7)$.

Questions?