3.6 – Optimization Problems

Goal: To use derivatives to solve optimization problems.

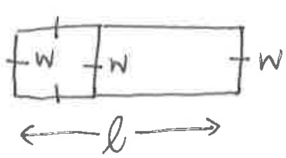
Calculus can be used to find critical points which are usually the maximum and minimum values of a function. This simple concept has countless applications. Consider a profit function, $P(x)$, which companies would want to MAXIMIZE, or a cost function, $C(x)$, which companies would want to MINIMIZE. Finding this "best", or optimal, value is called **optimization.**

Steps to solve Optimization Problems:

1. Identify what the question is asking
2. Define the variables and draw a diagram if it helps
3. Identify the quantity to be optimized and write an equation
4. Define the independent variable. Write all other variables in terms of the independent variable.
5. Define a function in terms of the independent variable
6. Differentiate the function
7. Determine the critical points (the max or min of the function)
8. Answer the question posed in the problem

Example 1. (P.201 #5) Ms Chau is trying to build two vegetable gardens that need to be fenced off from the animals. The two areas will share one common side and will be built with 60 m of fencing. One garden will be a square, and the other a rectangle. Find the dimensions that will maximize the total area.

1. Find dimensions of the total garden.

2. 

Let l be the length.
Let w be the width.

3. Optimize the area. $\boxed{A = l \cdot w}$.

4. Let w be the independent variable.

We know total fencing is 60m.

So,
$$3w + 2l = 60$$
$$2l = 60 - 3w$$
$$l = 30 - 1.5w.$$
isolate l, the variable.

5. $A = l \cdot w$
$$A = (30 - 1.5w)w$$
$$\boxed{A = 30w - 1.5w^2}$$

6. $A' = 30 - 3w$

7. $0 = 30 - 3w$
$$3w = 30$$
$$w = 10 \leftarrow \text{critical number}$$

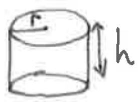
Find l, by subbing in $w = 10$.
$$l = 30 - 1.5(10)$$
$$= 30 - 15$$
$$= 15$$

8. Therefore, the dimensions of 10m by 15m will maximize the area of the garden.

<u>Check:</u> $A'' = -3 \leftarrow$ CD, so max.

Example 2. (p.201 #8a) Ms Chau is trying to create a 'can of worms' prank for April Fool's day. The cylindrical can must have a volume of 1 L to provide the most realistic experience. Find the height and radius of the can that will minimize the surface area.

1. Find h and r.

2. 

Let r = radius.
h = height.

3. Optimize SA.

$$\boxed{SA = 2\pi r h + 2\pi r^2}$$

4. Let r be the independent variable.

Given $V = 1L = 1000cm^3$.

$$V = \pi r^2 h$$

$$\left.\begin{aligned} 1000 &= \pi r^2 h \\ \boxed{h = \frac{1000}{\pi r^2}} \end{aligned}\right\} \text{isolate } h.$$

5. $SA = 2\pi r\left(\frac{1000}{\pi r^2}\right) + 2\pi r^2$

$$SA = \frac{2000}{r} + 2\pi r^2$$

$$SA = 2000r^{-1} + 2\pi r^2$$

6. $SA' = -2000r^{-2} + 4\pi r$

7. $0 = -2000r^{-2} + 4\pi r$

$$\frac{2000}{r^2} = 4\pi r$$

$$2000 = 4\pi r^3$$

$$r = \sqrt[3]{\frac{2000}{4\pi}}$$

$$r = \sqrt[3]{\frac{500}{\pi}}$$

$$\doteq 5.42 \text{ cm}$$

$$h = \frac{1000}{\pi(5.42)^2}$$

$$= 10.84 \text{cm}$$

8. A height of 10.84 cm + a radius of 5.42cm will minimize SA.

Check: $SA'' = 4000r^{-3} + 4\pi$

$$SA''(5.42) = 37.69$$

↑

CU, this is minimum value.

Example 3. (p.202 #13abc) For an outdoor concert, a ticket price of $30 typically attracts 5000 people. For each $1 increase in the ticket price, 100 fewer people will attend. The revenue, R, is the product of the number of people attending and the price per ticket. Find the ticket price that maximizes revenue.

1. Find ticket price.

2. Let x be the number of $1 increases.

3. Optimize revenue.

$R = (\#\ \text{ppl})(\text{price})$

$= (5000 - 100x)(30 + 1x)$

Starting # ppl

100 fewer people per $1 increase

starting price

$1 increases

4. Only have one variable which is x.

5. ↗

6. $R' = -100(30+x) + (5000-100x)(1)$

$0 = -3000 - 100x + 5000 - 100x$

$200x = 2000$

$\boxed{x = 10}$

To get max revenue, there will be ten $1 increases.

7. Ticket price $= 30 + 1(10)$

$= 40$

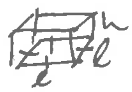
8. ∴ A $40 ticket price will maximize revenue.

Check:

$R'' = -200$ ← always concave down, which gives a max at CP.

Example 4. A cardboard box with a square base is to have a volume of 8 L. The cardboard for the box costs $0.1 cents/cm^2$, but the cardboard for the bottom is thicker, so its costs three times as much. Find the dimensions that will minimize the cost of the cardboard.

1. Find dimensions.

2. 

Let ℓ = length
h = height.

3. Optimize cost.

C = SA • cost

$= \underset{\text{bottom cost}}{0.3\ell^2} + \underset{\text{top cost}}{0.1\ell^2} + \underset{\text{cost of 4 sides}}{4(0.1)\ell h}$

$C = 0.4\ell^2 + 0.4\ell h$

4. Let ℓ be independent variable.

We know $V = 8L = 8000 cm^3$.

$V = \ell^2 h$

$8000 = \ell^2 h$

$\boxed{\frac{8000}{\ell^2} = h}$

5. $C = 0.4\ell^2 + 0.4\ell\left(\frac{8000}{\ell^2}\right)$

$C = 0.4\ell^2 + \frac{3200}{\ell}$

$\boxed{C = 0.4\ell^2 + 3200\ell^{-1}}$

6. $C' = 0.8\ell - 3200\ell^{-2}$

7. $0 = 0.8\ell - \frac{3200}{\ell^2}$

$0.8\ell = \frac{3200}{\ell^2}$

$0.8\ell^3 = 3200$

$\ell = \sqrt[3]{4000}$

$\ell \doteq 15.87 cm$

$h = \frac{8000}{15.87^2} = 31.76 cm$

8. ∴ With a length of 15.87cm and a height of 31.76cm, the cost of the cardboard will be minimized.

Check:

$C'' = 0.8 + \frac{6400}{\ell^3}$

$C''(15.87) = 2.40$ ← CU, so min. here!

IP p. 201 #4a, 9, 10b, 12 Note: Volume and surface area of typical shapes found on last page of textbook.