Vertical & Horizontal Asymptotes

A. Vertical Asymptotes

The graph of $f(x)$ has a vertical asymptote, $x = c$ if one of the following is true:

$$\lim_{x \to c^-} f(x) = \pm\infty$$

(approach VA from left)

$$\lim_{x \to c^+} f(x) = \pm\infty$$

(approach VA from right)

Trivial Case:

A rational function of the form $f(x) = \frac{p(x)}{q(x)}$

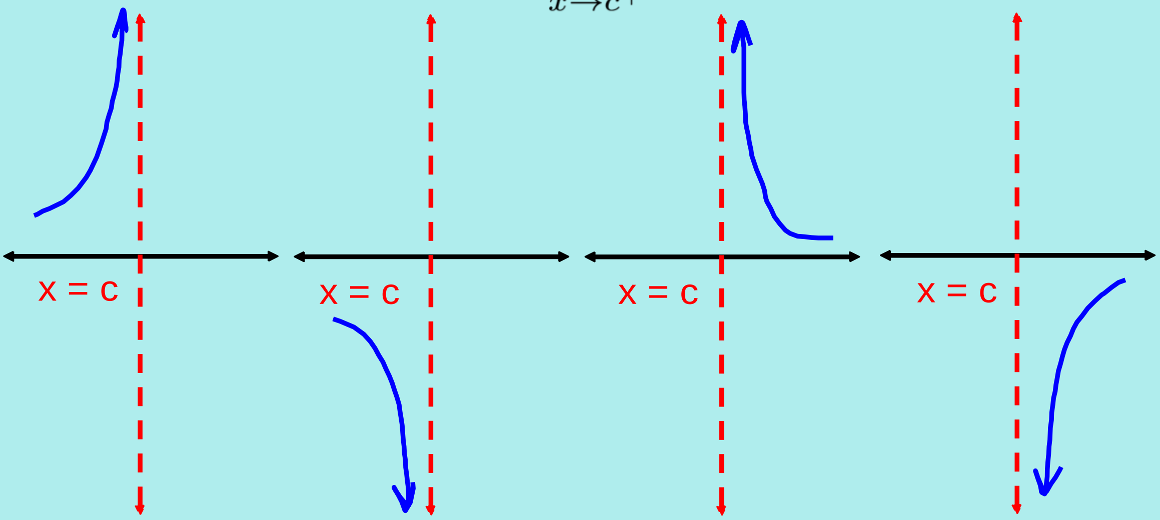
has a vertical asymptote at $x = c$ if

$q(c) = 0$ and $p(c) \neq 0$.

Behaviour of Graph at Vertical Asymptotes

$$\lim_{x \to c^-} f(x) = +\infty$$

$$\lim_{x \to c^+} f(x) = +\infty$$



$$\lim_{x \to c^-} f(x) = -\infty$$

$$\lim_{x \to c^+} f(x) = -\infty$$

Ex.1 Determine any vertical asymptotes (VA) of

$$f(x) = \frac{x}{x^2 + x - 2}$$

and describe the behaviour of the graph for values near the asymptotes.

$$f(x) = \frac{x}{(x+2)(x-1)} \qquad \underbrace{x \neq -2,\ x \neq 1}_{\text{VA's}}$$

VA: $x = -2$

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \frac{\overset{(-)\ -2}{x}}{\underset{(-)}{(x+2)}\underset{(-)\ -3}{(x-1)}} \qquad \lim_{x \to -2^+} \frac{\overset{(-2)}{x}}{\underset{(+\text{small})}{(x+2)}\underset{(-3)}{(x-1)}}$$

really small - number

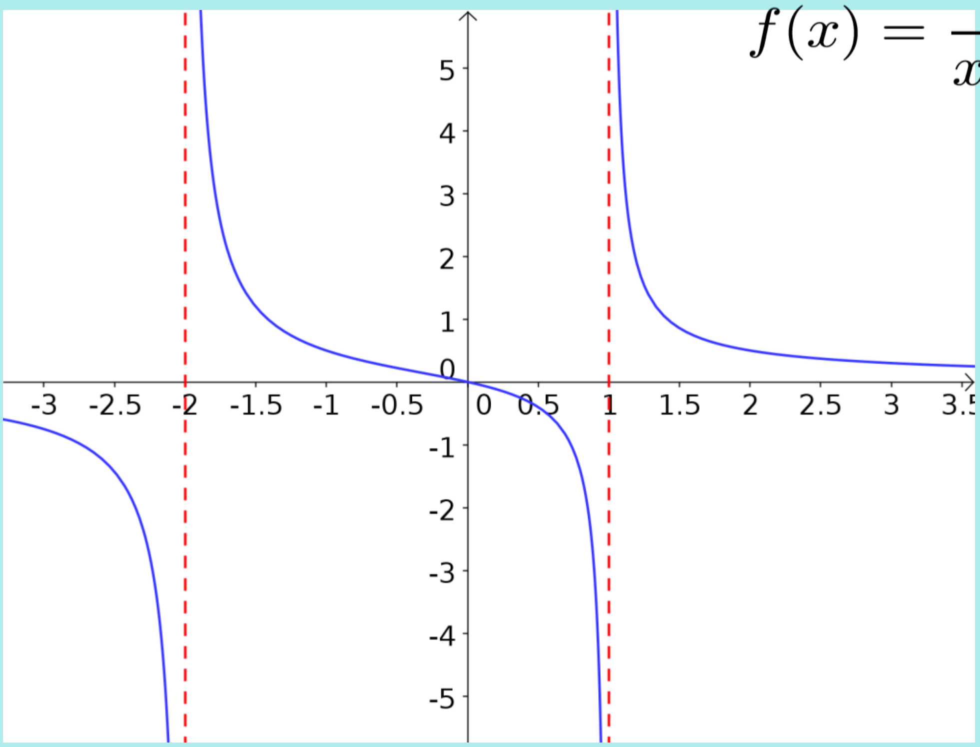
$$= +\infty$$

$$\lim_{x \to -2^-} f(x) = -\infty$$

VA: $x = 1$

$$\lim_{x \to 1^-} \frac{\overset{(+1)}{x}}{\underset{(+)}{(x+2)}\underset{(-\text{small})}{(x-1)}} = -\infty \qquad \lim_{x \to 1^+} \frac{x}{(x+2)(x-1)} = +\infty$$

Ex.1 Determine any vertical asymptotes (VA) of



$$f(x) = \frac{x}{x^2 + x - 2}$$

B. Horizontal Asymptotes

Consider the behaviour of the function as x tends to positive and negative infinity.

If $\lim_{x \to +\infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$,

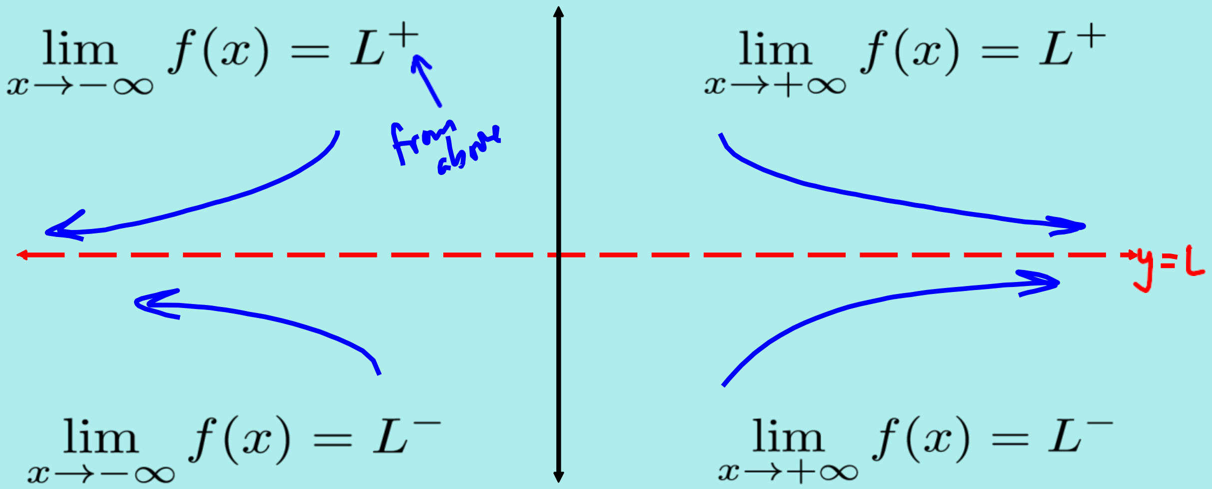
we say the line $y = L$ is a horizontal asymptote (HA).

Strategy for Rational Functions:

Factor the highest-order variable from the numerator and denominator (separately), and then apply the limit.

Note: It is important to consider whether the function approaches L from above or below.

Behaviour of Graph at Horizontal Asymptotes



$\lim_{x \to -\infty} f(x) = L^+$

$\lim_{x \to +\infty} f(x) = L^+$

$\lim_{x \to -\infty} f(x) = L^-$

$\lim_{x \to +\infty} f(x) = L^-$

Ex.2 Determine the equation of any horizontal asymptotes for

$$f(x) = \frac{3x^2}{x^2 - x - 6}$$

and behaviour.

factor x^2 out of both num & den.

$$f(x) = \frac{\cancel{x^2}(3)}{\cancel{x^2}\left(1 - \frac{x}{x^2} - \frac{6}{x^2}\right)}$$

$$= \frac{3}{1 - \frac{1}{x} - \frac{6}{x^2}}$$

HA: $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{3}{\underbrace{1 - \cancel{\frac{1}{x}}^{0} - \cancel{\frac{6}{x^2}}^{0}}_{1^-}}$

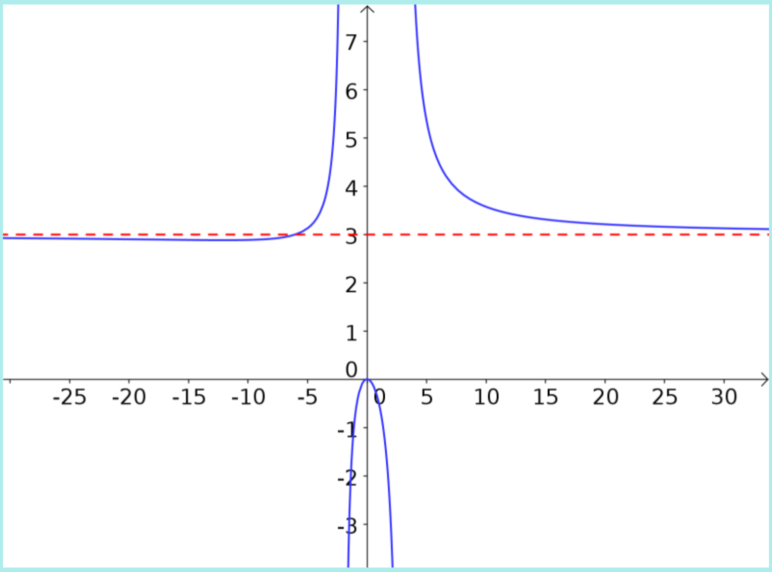
$= 3^+$

$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{3}{\underbrace{1 \underbrace{- \cancel{\frac{1}{x}}^{0}}_{+\text{small}} \underbrace{- \cancel{\frac{6}{x^2}}^{0}}_{-\text{smaller}}}_{+\text{smaller}}}$

$= 3^-$

Ex.2 Determine the equation of any horizontal asymptotes for

$$f(x) = \frac{3x^2}{x^2 - x - 6}$$



3^-

3^+

Assigned Work:

p.193 # (3, 4, 5)(skip d),
6abd, 9abc, 11, 12, 13, 15*