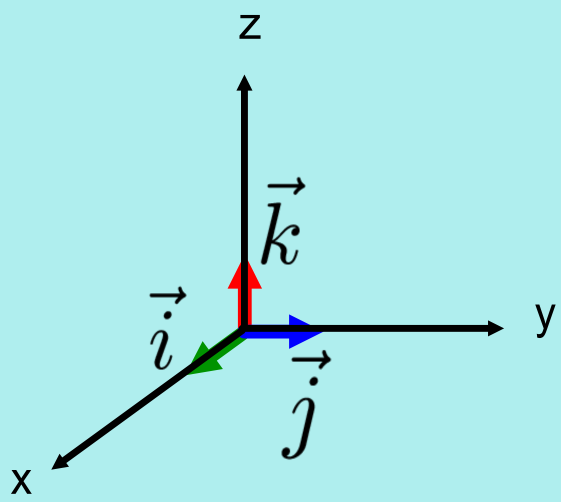
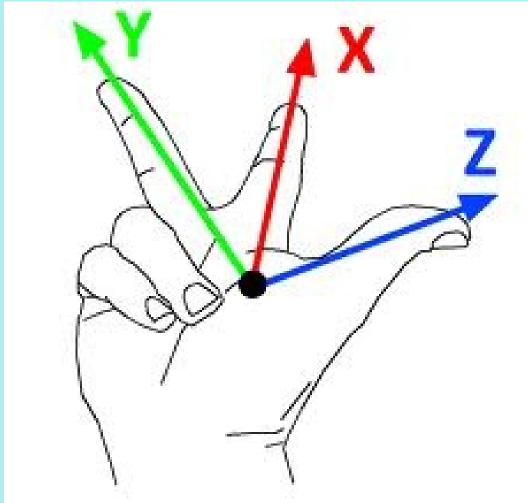
The Cross Product (Vector Product)

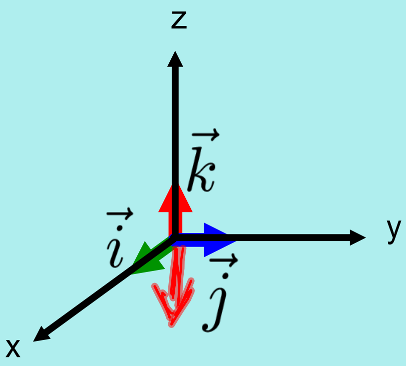
Recall: The right-hand rule





There is a vector operation, called the cross product,
which applies the right-hand rule in algebraic form.

The cross product can be used to algebraically define the relationship between the standard basis vectors.



$$\vec{i} \times \vec{j} = \vec{k} \qquad \vec{j} \times \vec{i} = -\vec{k}$$

$$\vec{j} \times \vec{k} = \vec{i} \qquad \vec{k} \times \vec{j} = -\vec{i}$$

$$\vec{k} \times \vec{i} = \vec{j} \qquad \vec{i} \times \vec{k} = -\vec{j}$$

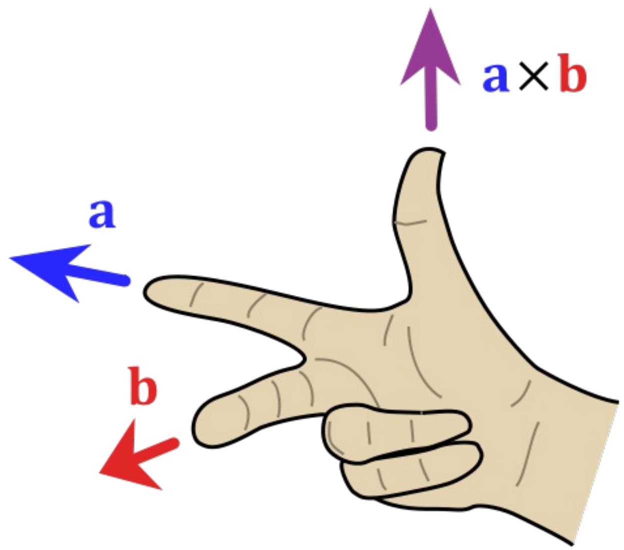
$\vec{i} \quad \vec{j} \quad \vec{k} \quad \vec{i} \quad \vec{j} \quad \vec{k}$

The Cross Product (Vector Product)

Given: $\vec{a} = (a_x, a_y, a_z)$ $\vec{b} = (b_x, b_y, b_z)$

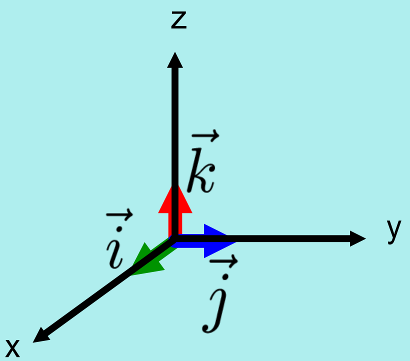
$$\vec{a} \times \vec{b} = (\underbrace{a_y b_z - a_z b_y}_{\vec{i}}, \underbrace{a_z b_x - a_x b_z}_{\vec{j}}, \underbrace{a_x b_y - a_y b_x}_{\vec{k}})$$

The vector $\vec{a} \times \vec{b}$ is perpendicular to each of the vectors $\vec{a}$ and $\vec{b}$, following the right-hand rule.



$$
\begin{aligned}
\vec{a} &= (a_x, a_y, a_z) & \vec{b} &= (b_x, b_y, b_z) \\
&= a_x\vec{i} + a_y\vec{j} + a_z\vec{k} & &= b_x\vec{i} + b_y\vec{j} + b_z\vec{k}
\end{aligned}
$$

$$
\begin{aligned}
\vec{a} \times \vec{b} &= (a_x\vec{i} + a_y\vec{j} + a_z\vec{k}) \times (b_x\vec{i} + b_y\vec{j} + b_z\vec{k}) \\
&\vdots \\
&= (a_yb_z - a_zb_y)\vec{i} + (a_zb_x - a_xb_z)\vec{j} + (a_xb_y - a_yb_x)\vec{k}
\end{aligned}
$$



	$\vec{i}$	$\vec{j}$	$\vec{k}$	$\vec{i}$
$\vec{a}$	a_x +	− a_y +	− a_z	a_x
$\vec{b}$	b_x	b_y	b_z	b_x

$\vec{k}$ $\vec{i}$ $\vec{j}$

$$\vec{a} \times \vec{b} = (a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x)$$

Ex.1 Find <u>both</u> possible cross products given:

$$\vec{u} = (2, 1, 4) \qquad \vec{v} = (1, 5, 6)$$

$$\begin{array}{c|cccc} & \vec{i} & \vec{j} & \vec{k} & \vec{i} \\ \vec{u} & 2 & 1 & 4 & 2 \\ \vec{v} & 1 & 5 & 6 & 1 \end{array}$$

$\vec{u} \times \vec{v} = (10-1)\vec{k} + (-14)\vec{i} + (-8)\vec{j}$

$= 9\vec{k} - 14\vec{i} - 8\vec{j}$

$= -14\vec{i} - 8\vec{j} + 9\vec{k}$

$= (-14, -8, 9)$

$$\vec{k} \qquad \vec{i} \qquad \vec{j}$$

$$\begin{array}{c|cccc} & \vec{i} & \vec{j} & \vec{k} & \vec{i} \\ \vec{v} & 1 & 5 & 6 & 1 \\ \vec{u} & 2 & 1 & 4 & 2 \end{array}$$

$\vec{v} \times \vec{u} = 14\vec{i} + 8\vec{j} - 9\vec{k}$

$= (14, 8, -9)$

$\vec{i} \times \vec{j} = \vec{k} \qquad \vec{j} \times \vec{i} = -\vec{k}$

Algebraic properties of cross product:

(1) It is NOT commutative. Changing the order will change the direction of the resulting vector (according to the right-hand rule).

$$\vec{a} \times \vec{b} = -\left(\vec{b} \times \vec{a}\right)$$

(2) Distributive for vector multiplication

$$\vec{a} \times \left(\vec{b} + \vec{c}\right) = \left(\vec{a} \times \vec{b}\right) + (\vec{a} \times \vec{c})$$

(3) Scalar multiplication

$$k\left(\vec{a} \times \vec{b}\right) = (k\vec{a}) \times \vec{b} = \vec{a} \times \left(k\vec{b}\right)$$

Ex.2 Find a vector perpendicular to both:

$$\vec{a} = 4\vec{i} - 3\vec{j} - 7\vec{k}$$

$$\vec{b} = 2\vec{i} - \vec{j} + 5\vec{k}$$

$\vec{a} \times \vec{b}$ or $\vec{b} \times \vec{a}$

is $\perp$ to $\vec{a}, \vec{b}$

$$\boxed{\vec{a} \times \vec{b} = -22\vec{i} - 34\vec{j} + 2\vec{k}}$$

$$\begin{array}{ccccc} & \vec{i} & \vec{j} & \vec{k} & \vec{i} \\ \vec{a} & 4 & -3 & -7 & 4 \\ \vec{b} & 2 & -1 & 5 & 2 \end{array}$$

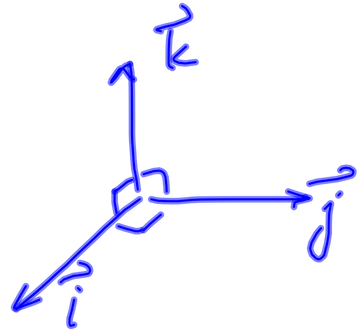
-4 $\quad -15$

$+6$ $\quad -7$

$\vec{a} \times \vec{b} = \vec{c}$

$\vec{c} \perp \vec{a}$

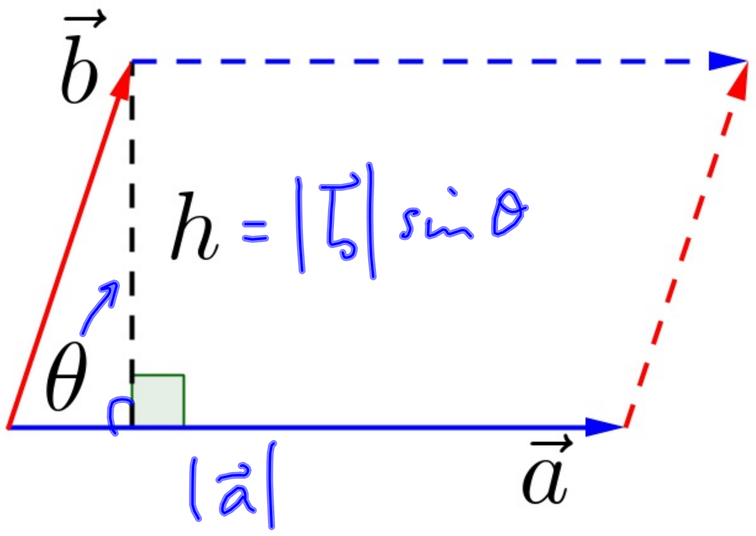
$\vec{c} \perp \vec{b}$



The magnitude of the cross product can also be related to the angle between the vectors (see p.411 for proof):

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|sin\theta$$

Example: Area of a Parallelogram



$$\sin\theta = \frac{h}{|\vec{b}|}$$

$$\begin{aligned} A &= \text{base} \times \text{height} \\ &= |\vec{a}||\vec{b}| \sin\theta \\ &= |\vec{a} \times \vec{b}| \end{aligned}$$

Assigned Work

p.407 # 1, 2, 3, 4abc, 5, 7, 9a, 13

p.415 # 5b, 6, 7