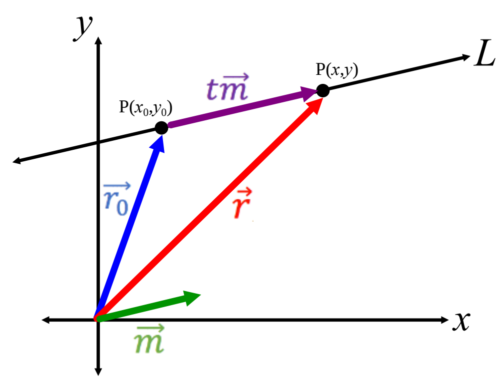
In two-space, a line can be defined many different types of equations:

| | | | |
|---|---|---|---|
| **i)** | Slope $y$-intercept form | → | $y = mx + b$ |
| **ii)** | Standard form (scalar equation) | → | $Ax + By + C = 0$ |
| **iii)** | Vector Equation | → | $\vec{r} = \vec{r_0} + t\vec{m}$ or $[x, y] = [x_0, y_0] + t[m_1, m_2]$ |
| **iv)** | Parametric Equation | → | $\begin{cases} x = x_0 + tm_1 \\ y = y_0 + tm_2 \end{cases}$ |

## Part 1: Vector Equation of a Line in Two-Space

In slope $y$-intercept form and standard form, we have equations that define all points $(x, y)$ on the line. A vector equation of a line is an equation that describes resultant vectors that start at the origin and end at a point on the line. In order to create these resultant vectors, we need a position vector that gives us a point on the line, $\vec{r_0} = [x_0, y_0]$, and a direction vector parallel to the line, $\vec{m} = [m_1, m_2]$. By adding the position and direction vectors together, we get a resultant vector, $\vec{r}$, that has its tip on a point on the line. By multiplying the direction vector by $t$ (to get $t\vec{m}$), the resultant vector, $\vec{r}$, can have its tip at any point on the line.




**Summary of Vector Equation of a Line:**

$\vec{r} = \vec{r_0} + t\vec{m}$ or $[x, y] = [x_0, y_0] + t[m_1, m_2]$

where

- $t \in \mathbb{R}$
- $\vec{r} = [x, y]$ is a position vector to any unknown point on the line
- $\vec{r_0} = [x_0, y_0]$ is a position vector to any known point on the line
- $\vec{m} = [m_1, m_2]$ is a direction vector parallel to the line

**Example 1:** For a line that goes through the points $A(1,4)$ and $B(3,1)$

**a)** Write a vector equation for the line

Find direction vector $\vec{m}$:

$\vec{m} = \overrightarrow{AB}$

$\vec{m} = [3-1, 1-4]$

$\vec{m} = [2, -3]$

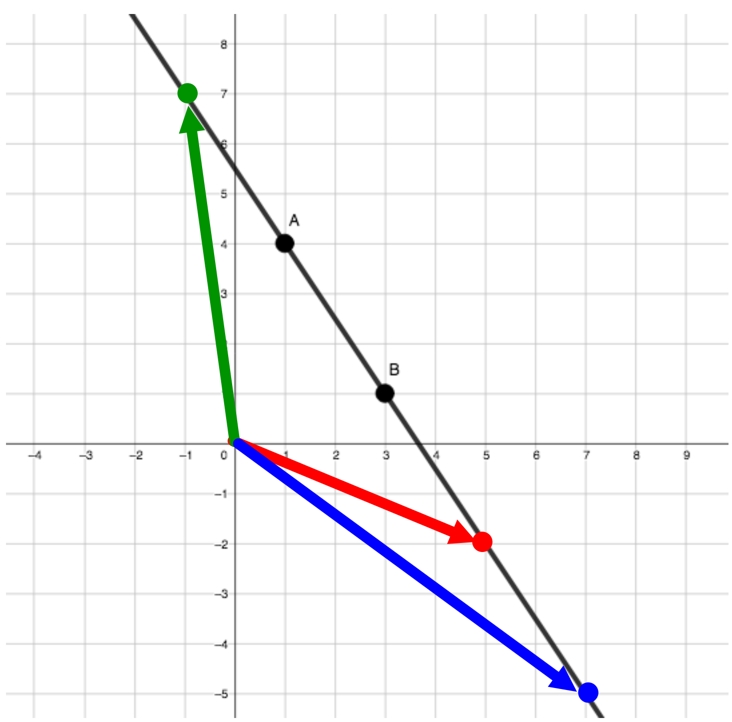
Use either point $A(1,4)$ or $B(3,1)$ for $\vec{r_0}$

A vector equation is: $[x, y] = [3,1] + t[2, -3]$

**b)** Determine three more position vectors to points on the line. Graph the line.

https://www.geogebra.org/graphing/j6ntqajv

| $t = 1$ | $t = 2$ | $t = -2$ |
|---|---|---|
| $[x, y] = [3,1] + 1[2, -3]$ | $[x, y] = [3,1] + 2[2, -3]$ | $[x, y] = [3,1] + (-2)[2, -3]$ |
| $[x, y] = [5, -2]$ | $[x, y] = [7, -5]$ | $[x, y] = [-1,7]$ |



**c)** Determine if the point (2,3) is on the line.

If (2,3) is on the line, there is a value of $t$ that makes the following equation true:

$[2,3] = [3,1] + t[2,-3]$

| Equate the $x$-coordinates: | Equate the $y$-coordinates: |
| --- | --- |
| $2 = 3 + 2t$ | $3 = 1 - 3t$ |
| $t = -\frac{1}{2}$ | $t = -\frac{2}{3}$ |

Since the $t$-values are not equal, the point (2,3) is NOT on the line.

**Part 2: Parametric Equation of a Line in Two-Space**

Remember that the Vector Equation of a line can be written as:

**i)** $\vec{r} = \vec{r_0} + t\vec{m}$ or $[x, y] = [x_0, y_0] + t[m_1, m_2]$

https://www.geogebra.org/graphing/f2h5x3wf

The vector equation of the line can be separated in to two parts, one for each variable.

The parametric equations of a line in two-space are:

$$x = x_0 + tm_1$$
$$y = y_0 + tm_2$$

The parametric equation of the line from example 1 would be:

$$\ell: \begin{cases} x = 3 + 2t \\ y = 1 - 3t \end{cases}$$

**Example 2:** Consider the line $\ell_1 : \begin{cases} x = 3 + 2t \\ y = -5 + 4t \end{cases}$

**a)** Find the coordinates of two points on the line.

For Point 1, let $t = 0$:

$x = 3 + 2(0)$ $\quad$ $y = -5 + 4(0)$

$x = 3$ $\quad$ $y = -5$

$P_1 = (3, -5)$

For Point 2, let $t = 1$:

$x = 3 + 2(1)$ $\quad$ $y = -5 + 4(1)$

$x = 5$ $\quad$ $y = -1$

$P_2 = (5, -1)$

**b)** Write a vector equation of the line.

$\vec{r_0} = [3, -5]$

$\vec{m} = [2,4] = 2[1,2]$ (use reduced version of direction vector if possible)

The vector equation is: $[x, y] = [3, -5] + t[1,2]$

**c)** Write the scalar equation of the line.

*To write the scalar equation, isolate t in both parametric equations, then set them equal to each other and rearrange in to standard form.*

$x = 3 + t$ $\quad$ $y = -5 + 2t$

$t = x - 3$ $\quad$ $t = \dfrac{y + 5}{2}$

$x - 3 = \dfrac{y + 5}{2}$

$2x - 6 = y + 5$

$2x - y - 11 = 0$

**d)** Determine if $\ell_1$ is parallel to $\ell_2$: $\begin{cases} x = 1 + 3t \\ y = 8 + 12t \end{cases}$

If they are parallel, the direction vectors will be scalar multiples of each other.

$[2,4] = k[3,12]$

$2 = 3k$

$k = \frac{2}{3}$

$4 = 12k$

$k = \frac{1}{3}$

Therefore, $\ell_1$ and $\ell_2$ are not parallel.

**Example 3:** Consider the line with scalar equation $4x + 5y + 20 = 0$

https://www.geogebra.org/graphing/tx4b9esu

**a)** Graph the line

$x$-int:

$4x + 5(0) + 20 = 0$

$x = -5$

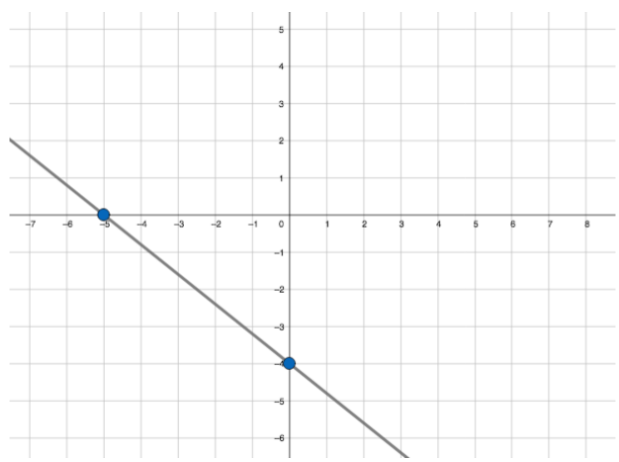
$(-5,0)$

$y$-int:

$4(0) + 5y + 20 = 0$

$y = -4$

$(0,-4)$



**b)** Determine a position vector that is perpendicular to the line.

Start by finding a direction vector for the line:

$\vec{m} = [0, -4] - [-5,0]$

$\vec{m} = [5, -4]$

To find a vector $\vec{n}$ that is perpendicular to $\vec{m}$, set $\vec{m} \cdot \vec{n} = 0$ and solve for values of $x$ and $y$ that make it true:

$[5, -4] \cdot [x, y] = 0$

$5x - 4y = 0$

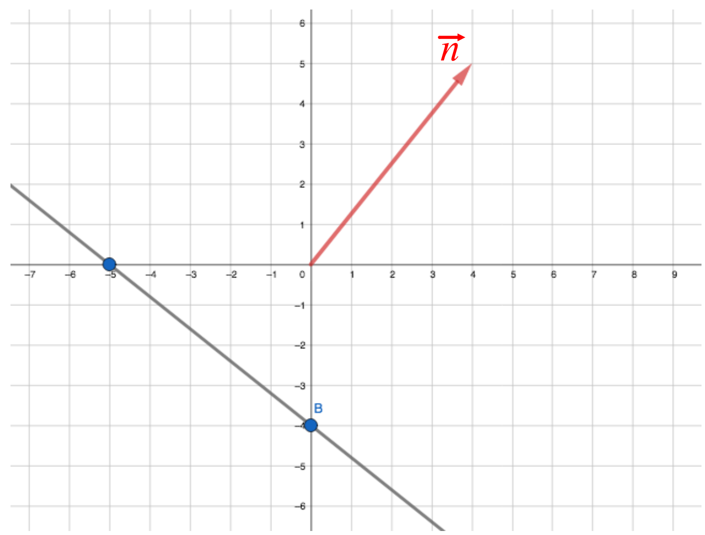
$x = \frac{4}{5}y$

Choose any value for $y$ and then solve for $x$

$x = \frac{4}{5}(5)$

$x = 4$

$\vec{n} = [4,5]$ is perpendicular to the line.




**c)** How does the position vector from part b) compare to the scalar equation?

The components of $\vec{n}$ correspond to the coefficients of $x$ and $y$ in the scalar equation.

**d)** Write a vector equation of the line

$\vec{r_0} = [0, -4]$

$\vec{m} = [5, -4]$

The vector equation is: $[x, y] = [0, -4] + t[5, -4]$

**Example 4:** Find a scalar equation for the line $[x, y] = [0,3] + t[2, -1]$

https://www.geogebra.org/graphing/ykexufu7

Perpendicular vector:

*Remember shortcut for finding a perpendicular vector is to swap coordinates and change 1 sign.*

$\vec{n} = [1,2]$

$x + 2y + C = 0$

$0 + 2(3) + C = 0$

$C = -6$

Scalar equation is $x + 2y - 6 = 0$