practice: The Chain rule

The Chain Rule

If the derivatives $g'(x)$ and $f'(g(x))$ both exist and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then $F'(x)$ exists and is given by the product $F'(x) = f'(g(x))g'(x)$; that is

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) \tag{1}$$

The Chain Rule in Leibniz Notation

If $y = f(u)$, where $u = g(x)$, and f and g are differentiable, then y is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \tag{2}$$

B **1.** Find the derivatives of the following functions.

(a) $F(x) = (5 - 3x)^7$ (b) $F(x) = (2x^2 + 1)^{20}$

(c) $G(x) = (x^3 + x^2 - 2)^{\frac{3}{4}}$ (d) $G(x) = \sqrt{x^4 - x + 1}$

(e) $y = \sqrt[4]{x^2 + x}$ (f) $y = (1 + 3x + 4x^2)^{-3}$

(g) $y = \dfrac{1}{(x^3 + 2x^2 + 1)^2}$ (h) $y = \dfrac{4}{\sqrt{9 - x^2}}$

(i) $y = (1 + 2\sqrt{x})^6$ (j) $y = \sqrt{x + \sqrt{x}}$

(k) $y = x - \sqrt[5]{1 + x^5 - 6x^{10}}$ (l) $y = x^2 + (x^2 - 1)^5$

2. If $y = u^4 + 5u^2$, where $u = x^5 + 2x^2 + 1$, find $\dfrac{dy}{dx}$. Leave your answer in terms of u and x.

3. Find $\left.\dfrac{dy}{dx}\right]_{x=4}$ if $y = u^2 - 2u^5$ and $u = x - \sqrt{x}$.

4. Find $\left.\dfrac{dy}{dt}\right]_{t=1}$ if $y = \sqrt{1 + r^2}$ and $r = \dfrac{t + 1}{2t + 1}$.

5. Find $\left.\dfrac{ds}{dt}\right]_{t=4}$ if $s = v + \dfrac{50}{v}$ and $v = 3t - \sqrt{t}$.

6. Differentiate:

(a) $F(x) = x\sqrt{x^2 + 1}$ (b) $F(x) = (2x + 1)(4x - 1)^5$

(c) $G(x) = (x^2 - 1)^4(2 - 3x)$

(d) $G(x) = (x^4 - x + 1)^2(x^2 - 2)^3$

(e) $F(x) = \dfrac{x}{\sqrt{2x + 3}}$ (f) $f(t) = \dfrac{(1 + 2t)^5}{(3t^2 - 5)^2}$

(g) $g(x) = \left(\dfrac{x + 2}{x - 2}\right)^3$ (h) $h(t) = \left(\dfrac{t^2 + 1}{t + 1}\right)^{10}$

(i) $y = \sqrt{\dfrac{x^2 - 1}{x^2 + 1}}$ (j) $y = \dfrac{(2x + 3)^3}{\sqrt{4x - 7}}$

(k) $y = 3\sqrt{x}(2x + 1)^5 + \sqrt{4x - 3}$

(l) $y = \sqrt{1 + \sqrt[3]{x}}$

(m) $y = (t + \sqrt[3]{t + t^2})^{20}$ (n) $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

7. Find the equation of the tangent line to the curve $y = (x^2 - 3)^8$ at the point $(2, 1)$.

Answers:

1. (a) $F'(x) = -21(5 - 3x)^6$
   (b) $F'(x) = 80x(2x^2 + 1)^{19}$
   (c) $G'(x) = \dfrac{9x^2 + 6x}{4\sqrt[4]{x^3 + x^2 - 2}}$
   (d) $G'(x) = \dfrac{4x^3 - 1}{2\sqrt{x^4 - x + 1}}$
   (e) $y' = \dfrac{2x + 1}{4(x^2 + x)^{\frac{3}{4}}}$
   (f) $y' = -\dfrac{3(3 + 8x)}{(1 + 3x + 4x^2)^4}$
   (g) $y' = -\dfrac{2(3x^2 + 4x)}{(x^3 + 2x^2 + 1)^3}$
   (h) $y' = \dfrac{4x}{(9 - x^2)^{\frac{3}{2}}}$
   (i) $y' = \dfrac{6(1 + 2\sqrt{x})^5}{\sqrt{x}}$
   (j) $y' = \dfrac{2\sqrt{x} + 1}{4\sqrt{x}\sqrt{x + \sqrt{x}}}$
   (k) $y' = 1 - \dfrac{x^4 - 12x^9}{(1 + x^5 - 6x^{10})^{\frac{4}{5}}}$
   (l) $y' = 2x + 10x(x^2 - 1)^4$
2. $(4u^3 + 10u)(5x^4 + 4x)$ 3. -117
4. $-\dfrac{2}{9\sqrt{13}}$ 5. $\frac{11}{8}$
6. (a) $F'(x) = \dfrac{2x^2 + 1}{\sqrt{x^2 + 1}}$
   (b) $F'(x) = 6(8x + 3)(4x - 1)^4$
   (c) $G'(x) = (3 + 16x - 27x^2)(x^2 - 1)^3$
   (d) $G'(x) = 2(x^4 - x + 1)(x^2 - 2)^2 \times (7x^5 - 8x^3 - 4x^2 + 3x + 2)$
   (e) $F'(x) = \dfrac{x + 3}{(2x + 3)^{\frac{3}{2}}}$
   (f) $f'(t) = \dfrac{2(1 + 2t)^4(3t^2 - 6t - 25)}{(3t^2 - 5)^3}$
   (g) $g'(x) = \dfrac{-12(x + 2)^2}{(x - 2)^4}$
   (h) $h'(t) = \dfrac{10(t^2 + 1)^9(t^2 + 2t - 1)}{(t + 1)^{11}}$
   (i) $y' = \dfrac{2x}{(x^2 + 1)^{\frac{3}{2}}\sqrt{x^2 - 1}}$
   (j) $y' = \dfrac{4(2x + 3)^2(5x - 12)}{(4x - 7)^{\frac{3}{2}}}$
   (k) $y' = \dfrac{3(2x + 1)^4[22x + 1]}{2\sqrt{x}} + \dfrac{2}{\sqrt{4x - 3}}$
   (l) $y' = \dfrac{1}{6x^{\frac{2}{3}}\sqrt{1 + \sqrt[3]{x}}}$
   (m) $y' = 20(t + \sqrt[3]{t + t^2})^{19}\left(1 + \dfrac{1 + 2t}{3(t + t^2)^{\frac{2}{3}}}\right)$
   (n) $y' = \dfrac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}}\left(1 + \dfrac{1 + \frac{1}{2\sqrt{x}}}{2\sqrt{x + \sqrt{x}}}\right)$
7. $32x - y - 63 = 0$ 8. $4x - 2y - 7 = 0$

8. Find the equation of the tangent line to the curve $y = \dfrac{1}{\sqrt{20 - x^4}}$ at the point $\left(2, \frac{1}{2}\right)$.

9. If $F(x) = f(g(x))$, where $g(2) = 4$, $g'(2) = 3$, and $f'(4) = 5$, find $F'(2)$.

10. If $G(x) = h(p(x))$, where $h(5) = 1$, $h'(5) = 2$, $h'(1) = 3$, $p(1) = 5$, and $p'(1) = 7$, find $G'(1)$.

C 11. If f is a differentiable function, find expressions for the derivatives of the following functions.

(a) $F(x) = f(x^4)$ (b) $G(x) = [f(x)]^4$

(c) $H(x) = f(\sqrt{x})$ (d) $P(x) = \sqrt{f(x)}$

(e) $y = f(f(x))$ (f) $y = \sqrt{1 + [f(x)]^2}$

(g) $y = [f(x^2)]^2$ (h) $y = f([f(x)]^3)$

12. (a) Use the Chain Rule and the fact that $|x| = \sqrt{x^2}$ to show that

$$\frac{d}{dx}|x| = \frac{x}{|x|}$$

(b) Sketch the graphs of the function $f(x) = |x|$ and its derivative.

(c) Use the result of part (a) to differentiate the function $g(x) = x|x|$.

Answers:

9. 15 10. 14

11. (a) $F'(x) = 4x^3 f'(x^4)$ (b) $G'(x) = 4[f(x)]^3 f'(x)$

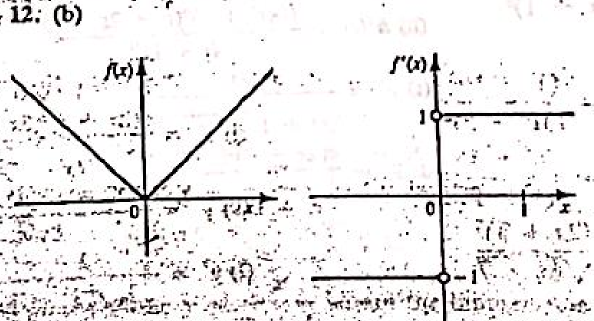
(c) $H'(x) = \dfrac{1}{2\sqrt{x}} f'(\sqrt{x})$

(d) $P'(x) = \dfrac{f'(x)}{2\sqrt{f(x)}}$ (e) $y' = f'(f(x))f'(x)$

(f) $y' = \dfrac{f(x)f'(x)}{\sqrt{1 + [f(x)]^2}}$ (g) $y' = 4xf(x^2)f'(x^2)$

(h) $y' = 3f'(x)[f(x)]^2 f'([f(x)]^3)$

12. (b)



(c) $g'(x) = 2|x|$

Practice: Derivatives of $y = e^x$ & $y = a^x$

#1: Differentiate each of the following:

a. $y = e^{3x}$

b. $s = e^{3t-5}$

c. $y = 2e^{10t}$

d. $y = e^{-3x}$

e. $y = e^{5-6x+x^2}$

f. $y = e^{\sqrt{x}}$

#2: Determine the derivative of each of the following:

a. $y = 2e^{x^3}$

b. $y = xe^{3x}$

c. $f(x) = \dfrac{e^{-x^3}}{x}$

d. $f(x) = \sqrt{x}e^{x}$

e. $h(t) = e^{t^2} + 3e^{-t}$

f. $g(t) = \dfrac{e^{2t}}{1 + e^{2t}}$

* #3: Determine the equation of the tangent to the curve defined by $y = \dfrac{2e^x}{1 + e^x}$ at the point $(0, 1)$.

* #4: a. If $f(x) = \frac{1}{3}(e^{3x} + e^{-3x})$, calculate $f'(1)$.

b. If $f(x) = e^{-\left(\frac{1}{x+1}\right)}$, calculate $f'(0)$.

c. If $h(z) = z^2(1 + e^{-z})$, calculate $h'(-1)$.

#5: If $y = \frac{5}{2}(e^{\frac{x}{5}} + e^{-\frac{x}{5}})$, prove that $y'' = \frac{y}{25}$.

#6: a. For the function $y = e^{-3x}$, determine $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, and $\frac{d^3y}{dx^3}$.

b. From the pattern in part a., state the value of $\frac{d^ny}{dx^n}$.

* #7: Determine the first and second derivatives of each function.

a. $y = -3e^x$

b. $y = xe^{2x}$

c. $y = e^x(4 - x)$

* #8: The distance s, in metres, fallen by a skydiver t seconds after jumping (and before the parachute opens) is $s = 160\left(\frac{1}{4}t - 1 + e^{-\frac{t}{4}}\right)$.

a. Determine the velocity, v, at time t.

b. Show that acceleration is given by $a = 10 - \frac{1}{4}v$.

c. Determine $v_T = \lim\limits_{t\to\infty} v$. This is the "terminal" velocity, the constant velocity attained when the air resistance balances the force of gravity.

d. At what time is the velocity 95% of the terminal velocity? How far has the skydiver fallen at that time?

#9: a. Use a table of values and successive approximation to evaluate each of the following:

i. $\lim\limits_{x\to\infty}\left(1 + \frac{1}{x}\right)^x$

ii. $\lim\limits_{x\to 0}(1 + x)^{\frac{1}{x}}$

b. Discuss your results.

#20: Differentiate each of the following functions:

a. $y = 2^{3x}$

b. $y = 3.1^x + x^3$

c. $s = 10^{3t-5}$

d. $w = 10^{(5-6n+n^2)}$

e. $y = 3^{x^2+2}$

f. $y = 400(2)^{x+3}$

#21: Determine the derivative of each function.

a. $y = x^5 \times (5)^x$

b. $y = x(3)^{x^2}$

c. $v = \frac{2^t}{t}$

d. $f(x) = \frac{\sqrt{3^x}}{x^2}$

#22: Determine the equation of the tangent to $y = 10^x$ at $(1, 10)$.

#23: A certain radioactive material decays exponentially. The percent, P, of the material left after t years is given by $P(t) = 100(1.2)^{-t}$.

a. Determine the half-life of the substance.

b. How fast is the substance decaying at the point where the half-life is reached?

#24: Historical data show that the amount of money sent out of Canada for interest and dividend payments during the period from 1967 to 1979 can be approximated by the model $P = (5 \times 10^8)e^{0.20015t}$, where t is measured in years ($t = 0$ in 1967) and P is the total payment in Canadian dollars.

Determine and compare the rates of increase for the years 1968 and 1978.

practice : Derivatives of $y=\log_a x$ & $y=\ln x$

part I :

1. Differentiate.
   (a) $f(x) = x^2 \ln x$ (b) $f(x) = \sqrt{\ln x}$
   (c) $g(x) = \ln(x^3+1)$ (d) $g(x) = \ln(5x)$
   (e) $y = \sin(\ln x)$ (f) $y = \ln(\sin x)$
   (g) $y = \dfrac{\ln x}{x^3}$ (h) $y = (x + \ln x)^3$
   (i) $y = \ln|2x+1|$ (j) $y = \ln\left(\dfrac{x+1}{x-1}\right)$
   (k) $y = \ln\sqrt{\dfrac{x}{2x+3}}$ (l) $y = \ln\dfrac{x}{\sqrt{x^2+1}}$
   (m) $y = \ln(\sec x + \tan x)$ (n) $y = \tan[\ln(1-3x)]$
2. (a) If $f(x) = \ln(\ln x)$, find $f'(x)$.
   (b) Find the domains of f and f'
3. Find the derivative of each function.
   (a) $f(x) = \log_2(x^2+1)$ (b) $g(x) = x\log_{10} x$
   (c) $F(x) = \log_5(3x-8)$ (d) $G(x) = \dfrac{1+\log_3 x}{x}$

5. Find the equation of the tangent line to each curve at the given point.
   (a) $y = \ln(x-1)$, $(2, 0)$ (b) $y = x^2\ln x$, $(1, 0)$
   (c) $y = 10^x$, $(1, 10)$ (d) $y = \log_{10} x$, $(100, 2)$

Answer key.

1. (a) $f'(x) = x(2\ln x + 1)$
   (b) $f'(x) = \dfrac{1}{2x\sqrt{\ln x}}$ (c) $g'(x) = \dfrac{3x^2}{x^3+1}$
   (d) $g'(x) = \dfrac{1}{x}$ (e) $y' = \dfrac{1}{x}\cos(\ln x)$
   (f) $y' = \cot x$ (g) $y' = \dfrac{1 - 3\ln x}{x^4}$
   (h) $y' = 3(x + \ln x)^2\left(1 + \dfrac{1}{x}\right)$
   (i) $y' = \dfrac{2}{2x+1}$
   (j) $y' = -\dfrac{2}{x^2-1}$ (k) $y' = \dfrac{3}{2x(2x+3)}$
   (l) $y' = \dfrac{1}{x(x^2+1)}$ (m) $y' = \sec x$
   (n) $y' = \dfrac{-3\sec^2[\ln(1-3x)]}{1-3x}$
2. (a) $f'(x) = \dfrac{1}{x\ln x}$ (b) $(1,\infty)$, $(1,\infty)$
3. (a) $f'(x) = \dfrac{2x}{(x^2+1)\ln 2}$
   (b) $g'(x) = \log_{10} x + \dfrac{1}{\ln 10}$
   (c) $F'(x) = \dfrac{3}{(3x-8)\ln 5}$
   (d) $G'(x) = \dfrac{1 - \ln x - \ln 3}{(\ln 3)x^2}$
5. (a) $x - y - 2 = 0$ (b) $x - y - 1 = 0$
   (c) $10(\ln 10)x - y - 10(\ln 10 - 1) = 0$
   (d) $x - 100(\ln 10)y + 100(2\ln 10 - 1) = 0$

Part II :

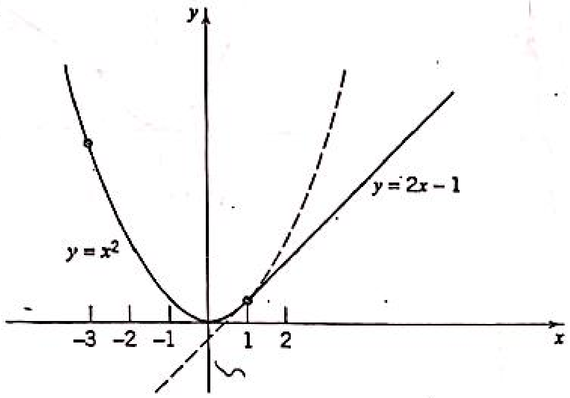
In Problems 573 to 597 differentiate the given functions.

573. $y = x^2\log_3 x$. 574. $y = \ln^2 x$.
575. $y = x\log_{10} x$. 576. $y = \sqrt{\ln x}$.
577. $y = \dfrac{x-1}{\log_2 x}$. 578. $y = x\sin x\ln x$.
579. $y = \dfrac{1}{\ln x}$. 580. $y = \dfrac{\ln x}{x^n}$.
581. $y = \dfrac{1-\ln x}{1+\ln x}$. 582. $y = \dfrac{\ln x}{1+x^2}$.
583. $y = x^n\ln x$. 584. $y = \sqrt{1+\ln^2 x}$.
585. $y = \ln(1-2x)$. 586. $y = \ln(x^2-4x)$.
[illegible] $y = \ln\sin x$. 588. $y = \log_3(x^2-1)$.
[illegible] $y = \ln\tan x$.
[illegible]1. $y = \ln^4\sin x$.
593. $y = (1+\ln\sin x)^n$. 594. $y = \log_2[\log_3(\log_5 x)]$.

Extrapractice: Differentiability & Derivatives

1: Find $f'(-3)$ and $f'(1)$ given that

$$f(x) = \begin{cases} x^2, & x \leq 1 \\ 2x - 1, & x > 1. \end{cases}$$



SOLUTION By definition

$$f'(-3) = \lim_{h\to 0} \frac{f(-3+h) - f(-3)}{h}.$$

However, for all x sufficiently close to -3, $f(x) = x^2$, we know that $f'(x) = 2x$. Thus, $f'(-3) = 2(-3) = -6$.

Now let's find

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}.$$

Since f is not defined by the same formula on both sides of 1, we will evaluate this limit by taking one-sided limits. Note that $f(1) = 1^2 = 1$.

To the left of 1, $f(x) = x^2$. Thus

$$\lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^-} \frac{(1+h)^2 - 1}{h}$$

$$= \lim_{h\to 0^-} \frac{(1 + 2h + h^2) - 1}{h} = \lim_{h\to 0^-} (2 + h) = 2.$$

To the right of 1, $f(x) = 2x - 1$. Thus

$$\lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^+} \frac{[2(1+h) - 1] - 1}{h} = \lim_{h\to 0^+} 2 = 2.$$

The limit of the difference quotient exists and is 2:

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = 2.$$

#2: You can see a similar change of direction in the graph of

$$f(x) = \begin{cases} x^2, & x \le 1 \\ \frac{1}{2}x + \frac{1}{2}, & x > 1 \end{cases}$$

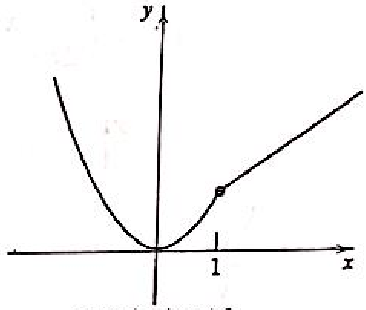
at the point (1, 1). Once again, f is everywhere continuous (verify this), but it is not differentiable at 1:

$$\lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^-} \frac{(1+h)^2 - 1}{h} = \lim_{h\to 0^-} \frac{h^2 + 2h}{h} = \lim_{h\to 0^-} (h+2) = 2,$$

$$\lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^+} \frac{\frac{1}{2}(1+h) + \frac{1}{2} - 1}{h} = \lim_{h\to 0^+} \left(\frac{1}{2}\right) = \frac{1}{2}.$$

Since these one-sided limits are different, the two-sided limit

$$\lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \quad \text{does not exist.}$$



no derivative at 1

Practice:

#1: In Exercises 27–32, draw the graph of each of the following functions and indicate where it is not differentiable.

27. $f(x) = |x + 1|$.

28. $f(x) = |2x - 5|$.

29. $f(x) = \sqrt{|x|}$.

30. $f(x) = |x^2 - 4|$.

31. $f(x) = \begin{cases} x^2, & x \le 1 \\ 2 - x, & x > 1. \end{cases}$

32. $f(x) = \begin{cases} x^2 - 1, & x \le 2 \\ 3, & x > 2. \end{cases}$

In Exercises 33–36, find $f'(c)$ if it exists.

33. $f(x) = \begin{cases} 4x, & x < 1 \\ 2x^2 + 2, & x \ge 1; \end{cases} \quad c = 1.$

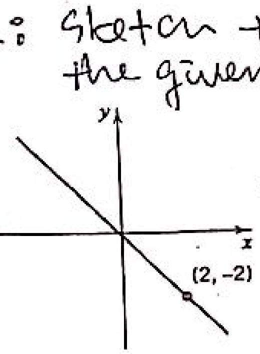
34. $f(x) = \begin{cases} 3x^2, & x \le 1 \\ 2x^3 + 1, & x > 1; \end{cases} \quad c = 1.$

35. $f(x) = \begin{cases} x + 1, & x \le -1 \\ (x+1)^2, & x > -1; \end{cases} \quad c = -1.$

36. $f(x) = \begin{cases} -\frac{1}{2}x^2, & x < 3 \\ -3x, & x \ge 3; \end{cases} \quad c = 3.$

#2: Sketch the graph of the derivative of the function with the given graph.

a)



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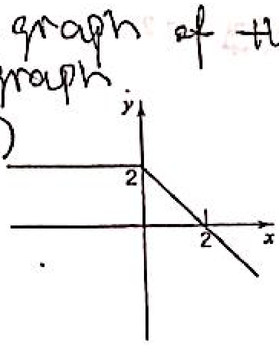
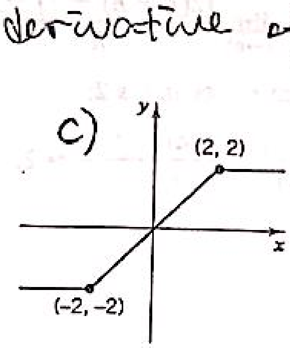


c)



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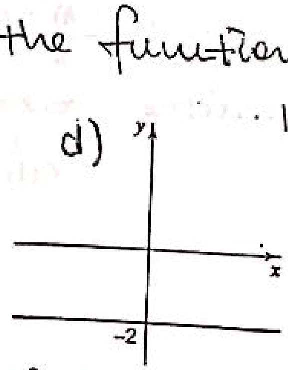

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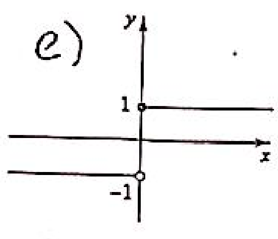
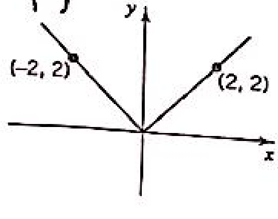


f)



#3: For the following three limits, each represents the derivative of a function f at a number c. Determine f and c in each case.

a) $\lim_{h\to 0} \frac{(1+h)^2-1}{h}$.

b) $\lim_{h\to 0} \frac{\sqrt{4+h}-2}{h}$.

c) $\lim_{h\to 0} \frac{\cos(\pi+h)+1}{h}$.

#4: Find A and B given that the function

$$f(x) = \begin{cases} x^3, & x \le 1 \\ Ax + B, & x > 1 \end{cases}$$

is differentiable at $x = 1$.

#5: Let $f(x) = \begin{cases} x^2 - x, & x \le 2 \\ 2x - 2, & x > 2. \end{cases}$

(a) Show that f is continuous at 2.
(b) Find $f'_-(2)$ and $f'_+(2)$.
(c) Is f differentiable at 2?

#6: Let $f(x) = x\sqrt{x}$, $x \ge 0$.

(a) Calculate $f'(x)$ for any $x > 0$.
(b) Calculate $f'_+(0)$.

#7: Let $f(x) = \sqrt{1-x}$ for $0 \le x \le 1$.

(a) Calculate $f'(x)$ for any $x \in (0, 1)$.
(b) Find $f'_+(0)$, if it exists.
(c) Find $f'_-(1)$, if it exists.

#8: Let $f(x) = \begin{cases} 1 - x^2, & x \le 0 \\ x^2, & x > 0. \end{cases}$

(a) Find $f'_-(0)$, if it exists.
(b) Find $f'_+(0)$, if it exists.
(c) Is f differentiable at 0?
of f.

#9: Find $f'(0)$ given that $h(0) = 3$ and $h'(0) = 2$.

a) $f(x) = xh(x)$.

b) $f(x) = 3x^2h(x) - 5x$.

c) $f(x) = h(x) - \frac{1}{h(x)}$.

d) $f(x) = h(x) + \frac{x}{h(x)}$.