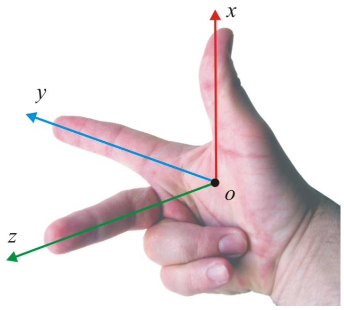
7.6 Cross Product

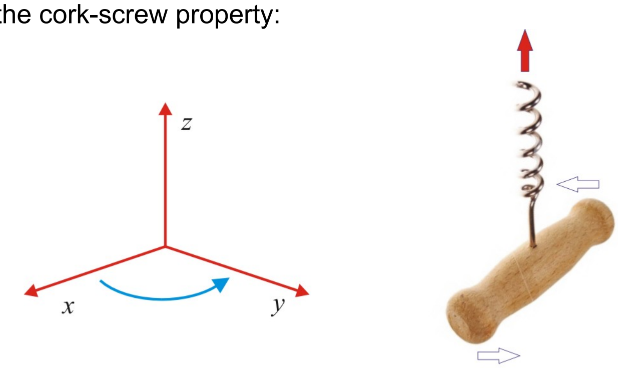
A Right Hand System

The *Right Hand System* is based on the position of first three fingers of the right hand as illustrated on the following figure:



B Cork-Screw Rule

The *cork-screw rule* describes a *right hand system* based on the cork-screw property:



If you rotate the x-axis towards the y-axis using the shortest path, the screw goes in the positive direction of the z-axis.

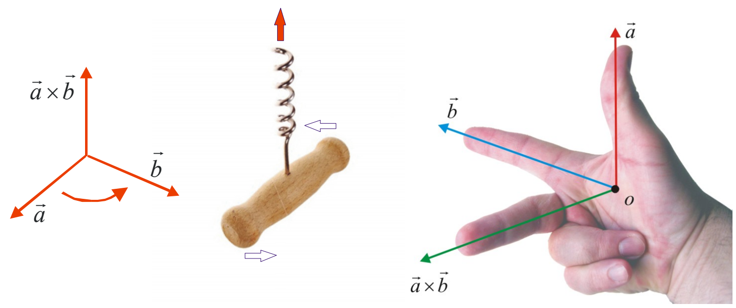
C Cross Product

The *cross product* between two vectors $\vec{a}$ and $\vec{b}$ is a *vector* quantity denoted by $\vec{a}\times\vec{b}$ having the following properties:

a) $\|\vec{a}\times\vec{b}\|=\|\vec{a}\|\|\vec{b}\|\sin\alpha$ where $\alpha=\angle(\vec{a},\vec{b})$

b) $\vec{a}\times\vec{b}$ is *perpendicular to both* $\vec{a}$ and $\vec{b}$ (is perpendicular to the plane determined by $\vec{a}$ and $\vec{b}$)

c) the vectors $\vec{a}$, $\vec{b}$, and $\vec{a}\times\vec{b}$ form a *right-handed system*



D Specific Cases

1. If $\vec{a}\,\|\,\vec{b}$ ($\alpha=0$ or $\alpha=\pi=180°$), then $\vec{a}\times\vec{b}=\vec{0}$.
2. If $\vec{a}\perp\vec{b}$ ($\alpha=\pi/2=90°$), then $\|\vec{a}\times\vec{b}\|=\|\vec{a}\|\|\vec{b}\|=maximum$
3. If $\vec{a}\equiv\vec{b}$ then $\vec{a}\times\vec{a}=\vec{0}$.

Ex 1. The magnitudes of two vectors $\vec{a}$ and $\vec{b}$ are $\|\vec{a}\|=2$ and $\|\vec{b}\|=3$ respectively, and the angle between them is $\alpha=60°$. Find the magnitude of the cross product of these vectors.

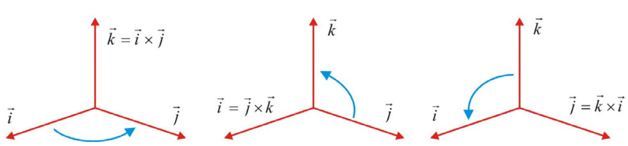
$$\|\vec{a}\times\vec{b}\|=\|\vec{a}\|\|\vec{b}\|\sin\alpha=(2)(3)\sin 60°=3\sqrt{3}$$

E Cross Product of Unit Vectors

The cross product of the *standard unit vectors* is given by:

$$\vec{i}\times\vec{i}=\vec{0} \qquad \vec{j}\times\vec{j}=\vec{0} \qquad \vec{k}\times\vec{k}=\vec{0}$$

$$\vec{i}\times\vec{j}=\vec{k} \qquad \vec{j}\times\vec{k}=\vec{i} \qquad \vec{k}\times\vec{i}=\vec{j}$$



D Cross Product of two Algebraic Vectors

The cross product of two algebraic vectors $\vec{a}=(a_x,a_y,a_z)=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}$ and $\vec{b}=(b_x,b_y,b_z)=b_x\vec{i}+b_y\vec{j}+b_z\vec{k}$ is given by:

$$\vec{a}\times\vec{b}=\vec{i}(a_yb_z-a_zb_y)+\vec{j}(a_zb_x-a_xb_z)+\vec{k}(a_xb_y-a_yb_x)$$

$$=\vec{i}\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix}+\vec{j}\begin{vmatrix}a_z & a_x\\ b_z & b_x\end{vmatrix}+\vec{k}\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}$$

$$\vec{a}\times\vec{b}=\begin{matrix}\vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j}\\ a_x & a_y & a_z & a_x & a_y\\ b_x & b_y & b_z & b_x & b_y\end{matrix}$$

Ex 2. For each case, find the cross product of the vectors $\vec{a}$ and $\vec{b}$.

a) $\vec{a}=(1,-2,0)$, $\vec{b}=(0,-1,2)$

$$\begin{array}{rrrrrrl} & \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} & \\ \vec{a}\times\vec{b}= & 1 & -2 & 0 & 1 & -2 & =\vec{i}(-4-0)+\vec{j}(0-2)+\vec{k}(-1-0) \\ & 0 & -1 & 2 & 0 & -1 & \end{array}$$

$\therefore \vec{a}\times\vec{b}=(-4,-2,-1)$

b) $\vec{a}=-\vec{i}+2\vec{j}$, $\vec{b}=\vec{i}-2\vec{j}-\vec{k}$

$$\begin{array}{rrrrrrl} & \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} & \\ \vec{a}\times\vec{b}= & -1 & 2 & 0 & -1 & 2 & =\vec{i}(-2-0)+\vec{j}(0-1)+\vec{k}(2-2) \\ & 1 & -2 & -1 & 1 & -2 & \end{array}$$

$\therefore \vec{a}\times\vec{b}=(-2,-1,0)$

c) $\vec{a}=(-1,1,-2)$, $\vec{b}=-2\vec{i}-\vec{j}+3\vec{k}$

$$\begin{array}{rrrrrrl} & \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} & \\ \vec{a}\times\vec{b}= & -1 & 1 & -2 & -1 & 1 & =\vec{i}(3-2)+\vec{j}(4+3)+\vec{k}(1+2) \\ & -2 & -1 & 3 & -2 & -1 & \end{array}$$

$\therefore \vec{a}\times\vec{b}=(1,7,3)$

E Properties of Cross Product

The following properties apply for the cross product:

1. $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$ (anti-commutative property)
2. $\lambda(\vec{a}\times\vec{b})=(\lambda\vec{a})\times\vec{b}=\vec{a}\times(\lambda\vec{b})$
3. $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ (distributive property)
4. $\vec{a}\times\vec{b}=0 \Leftrightarrow \vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$ or $\vec{a}\parallel\vec{b}$
5. $\vec{a}\times\vec{0}=\vec{0}$
6. $\vec{a}\times\vec{a}=\vec{0}$

Note: The dot and cross products have a higher priority in comparison to addition and subtraction operations.

d) $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{c}\cdot\vec{a})\vec{b}-(\vec{b}\cdot\vec{a})\vec{c}$ (triple cross product)

$[\vec{a}\times(\vec{b}\times\vec{c})]_x=a_y(\vec{b}\times\vec{c})_z-a_z(\vec{b}\times\vec{c})_y$

$=a_y(b_xc_y-b_yc_x)-a_z(b_zc_x-b_xc_z)$

$=(c_ya_y+c_za_z)b_x-(b_ya_y+b_za_z)c_x+a_xc_xb_x-a_xc_xb_x$

$=(\vec{c}\cdot\vec{a})b_x-(\vec{b}\cdot\vec{a})c_x=RS$

Ex 3. Use the cross product properties to prove the following relations:

a) $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})=2(\vec{a}\times\vec{b})$

$LS=(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})=\vec{a}\times\vec{a}+\vec{a}\times\vec{b}-\vec{b}\times\vec{a}-\vec{b}\times\vec{b}$

$=\vec{0}+\vec{a}\times\vec{b}-\vec{b}\times\vec{a}-\vec{0}=\vec{a}\times\vec{b}+\vec{a}\times\vec{b}=2(\vec{a}\times\vec{b})=RS$

b) $(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})+(\vec{a}\cdot\vec{b})(\vec{a}\cdot\vec{b})=(\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b})$

$LS=(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})+(\vec{a}\cdot\vec{b})(\vec{a}\cdot\vec{b})$

$=\|\vec{a}\times\vec{b}\|^2+(\vec{a}\cdot\vec{b})^2=\|\vec{a}\|^2\|\vec{b}\|^2\cos^2\theta+\|\vec{a}\|^2\|\vec{b}\|^2\sin^2\theta$

$=\|\vec{a}\|^2\|\vec{b}\|^2=(\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b})=RS$

c) $\vec{a}\cdot(\vec{b}\times\vec{c})=\vec{b}\cdot(\vec{c}\times\vec{a})=\vec{c}\cdot(\vec{a}\times\vec{b})$ (mixed product)

$\vec{a}\cdot(\vec{b}\times\vec{c})=a_x(b_yc_z-b_zc_y)+a_y(b_zc_x-b_xc_z)+a_z(b_xc_y-b_yc_x)$

$=b_x(c_ya_z-c_za_y)+b_y(c_za_x-c_xa_z)+b_z(c_xa_y-c_ya_x)$

$=\vec{b}\cdot(\vec{c}\times\vec{a})=RS$

Ex 4. Find an unit vector perpendicular to both $\vec{a}=(0,1,1)$ and $\vec{b}=(1,1,0)$.

The vector $\vec{u}=\dfrac{\vec{a}\times\vec{b}}{\|\vec{a}\times\vec{b}\|}$ is an unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. So:

$$\begin{array}{rrrrrrl} & \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} & \\ \vec{a}\times\vec{b}= & 0 & 1 & 1 & 0 & 1 & =\vec{i}(0-1)+\vec{j}(1-0)+\vec{k}(0-1) \\ & 1 & 1 & 0 & 1 & 1 & \end{array}$$

$=(-1,1,-1)$

$\vec{u}=\dfrac{(-1,1,-1)}{\sqrt{3}}=\left(\dfrac{-1}{\sqrt{3}},\dfrac{1}{\sqrt{3}},\dfrac{-1}{\sqrt{3}}\right)$

Ex 5. Classify as scalar, vector, or meaningless.

a) $\vec{a}+\vec{b}\times\vec{c}$ (vector)
b) $\vec{a}+\vec{b}\cdot\vec{c}$ (meaningless)
c) $\vec{a}\times\vec{b}-\vec{b}\times\vec{c}$ (vector)
d) $(\vec{b}\cdot\vec{c})\vec{a}$ (vector)
e) $(\vec{b}\cdot\vec{c})\times\vec{a}$ (meaningless)
f) $(\vec{b}\cdot\vec{c})(\vec{a}\times\vec{b})$ (vector)
g) $(\vec{b}-\vec{c})(\vec{a}\times\vec{b})$ (meaningless)
h) $(\vec{b}\cdot\vec{c})(\vec{a}\times\vec{b})-(\vec{b}+\vec{c})\times(\vec{c}\times\vec{b})$ (vector)
i) $(\vec{b}\cdot\vec{c})-\vec{a}\cdot(\vec{b}\times\vec{c})$ (scalar)

Reading: Nelson Textbook, Pages 401-407
Homework: Nelson Textbook: Page 407 #3, 4ab, 5, 8a, 11, 13