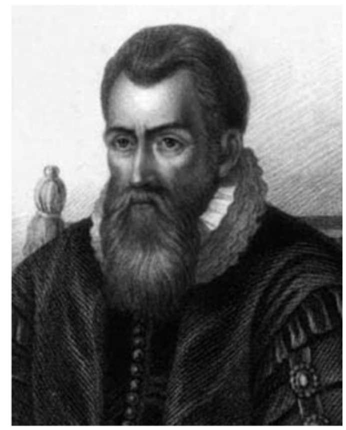
THE ERINDALE ACADEMY
1576 Dundas Street West, Mississauga ON L5C 1E5
www.teacademy.ca Tel: 905-232-1576
Email: info@teacademy.ca

Teacher: Ms. Ella

Unit 5 – Exponential and Logarithmic functions

Chapter 8.1 – 8.2: The logarithmic function and its graph



Many phenomena in the natural sciences (physics, chemistry, biology, astronomy) can be descried using exponential functions. To solve problems involving a function, it is often useful to use the inverse function.

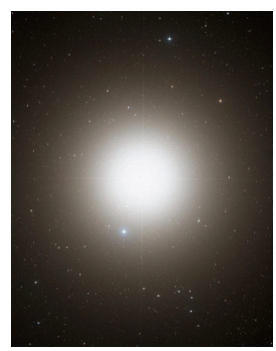
Invented by John Napier in the 17^{th} century, logarithmic functions (and the associated table of values generated using them) were the only effective numerical tools for dealing with exponential functions until the development of computers and calculator.

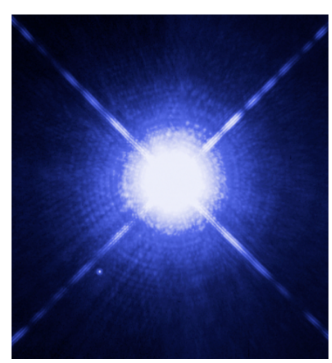
Some applications of logarithmic functions include:

- pH levels (acid/base) in chemistry
- Star brightness
- Sound intensity in physics/music
- Light intensity & absorption in physics/astronomy
- Richter scale for earthquakes in physics/geology

Concentration of hydrogen ions compared to distilled water		Examples of solutions at this pH
10 000 000	pH = 0	battery acid, strong hydrofluoric acid
1 000 000	pH = 1	hydrochloric acid secreted by stomach lining
100 000	pH = 2	lemon juice, gastric acid, vinegar
10 000	pH = 3	grapefruit, orange juice, soda
1000	pH = 4	tomato juice, acid rain
100	pH = 5	soft drinking water, black coffee
10	pH = 6	urine, saliva
1	pH = 7	“pure” water
$\frac{1}{10}$	pH = 8	seawater
$\frac{1}{100}$	pH = 9	baking soda
$\frac{1}{1000}$	pH = 10	Great Salt Lake, milk of magnesia
$\frac{1}{10\,000}$	pH = 11	ammonia solution
$\frac{1}{100\,000}$	pH = 12	soapy water
$\frac{1}{1\,000\,000}$	pH = 13	bleaches, oven cleaner
$\frac{1}{10\,000\,000}$	pH = 14	liquid drain cleaner

$$m_1 - m_{\text{ref}} = -2.5 \log_{10}\left(\frac{I_1}{I_{\text{ref}}}\right)$$





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Exponential relation: $y = a^x, a > 0, a \neq 1$; and Inverse relation is $x = a^y$, but there is no way to rearrange this algebraically, so a new representation is introduced – Logarithmic relation

$$y = log_a x, a > 0, a \neq 1, \text{ read as "log to the base a of x"}$$

The two most important logarithmic functions have bases of “10” and “e”, so a special notation is given:

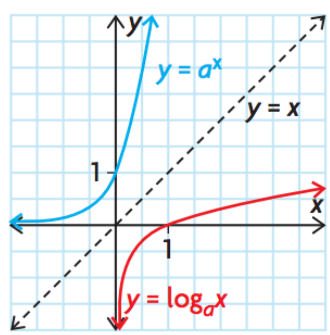
- $log_{10}x = logx$ is the “common log”
- $log_e x = lnx$ is the “natural log” where $e = 2.718$ is called “natural number”.

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{1 \times 2} + \frac{1}{1 \times 2 \times 3} + \frac{1}{1 \times 2 \times 3 \times 4} + \cdots$$

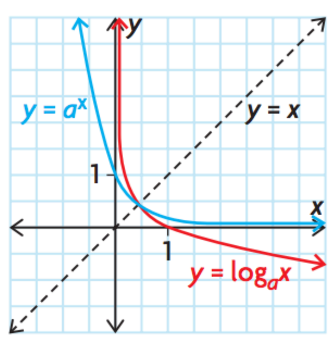
The graph of Logarithmic Function:

- The general shape of the graph of the logarithmic function depends on the value of the base.

When $a > 1$, the exponential function is an increasing function, and the logarithmic function is also an increasing function.



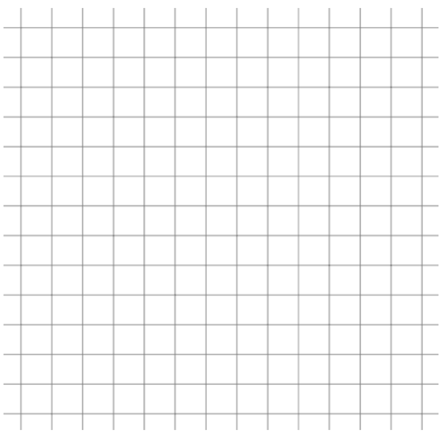
When $0 < a < 1$, the exponential function is a decreasing function and the logarithmic function is also a decreasing function.



- The y-axis is the vertical asymptote for the logarithmic function. The x-axis is the horizontal asymptote for the exponential function.
- The x-intercept of the logarithmic function is 1, while the y-intercept of the exponential function is 1.
- The domain of the logarithmic function is $\{x \in \mathbf{R} \mid x > 0\}$, since the range of the exponential function is $\{y \in \mathbf{R} \mid y > 0\}$.
- The range of the logarithmic function is $\{y \in \mathbf{R}\}$, since the domain of the exponential function is $\{x \in \mathbf{R}\}$.

Transformation of Logarithmic functions

Example 1: Use transformations to sketch the function $y = -2\,log\left[\frac{1}{2}(x-4)\right] + 1$



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Unit 5 – Exponential and Logarithmic functions

Chapter 8.3: Evaluating Logarithms

Some general rules to evaluate logarithmic terms:

Example 1: Solve

a) $log_3 3^2$

b) $log_4 4^7$

$$\boldsymbol{log_a a^x = x}$$

Example 2: Evaluate

a) $log_{10} 1$

b) $log_5 1$

$$\boldsymbol{log_a 1 = 0}$$

Example 3: Evaluate

a) $2^{log_2 x}$

b) $5^{log_5 x}$

$$\boldsymbol{a^{log_a x} = x}$$

Moreover, we can calculate the value of a logarithms or simplify by changing of the bases:

$$\boldsymbol{log_a x = \frac{log_{10} x}{log_{10} a}}$$

Evaluate.

a) $\log_6 \sqrt{6}$

b) $\log_5 125 - \log_5 25$

c) $\log_3 81 + \log_4 64$

d) $\log_2 \frac{1}{4} - \log_3 1$

e) $\log_5 \sqrt[3]{5}$

f) $\log_3 \sqrt{27}$

Evaluate.

a) $\log_3 3^5$

b) $5^{\log_5 25}$

c) $4^{\log_4 \frac{1}{16}}$

d) $\log_m m^n$

e) $a^{\log_a b}$

f) $\log_{\frac{1}{10}} 1$

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Unit 6 – Exponential and Logarithmic functions
Lesson 3: Logarithmic law

Recall: Exponent laws

$(a^x)(a^y) = a^{x+y}$
$(a^x) \div (a^y) = \frac{a^x}{a^y} = a^{x-y}, a \neq 0$
$a^{-x} = \frac{1}{a^x},\ a \neq 0$
$(a^x)^y = a^{xy}$
$a^0 = 1, a \neq 0$

Similarly, Log will also have a set of laws by which can make their calculation much easier.

Product rule: $log_a xy = log_a x + log_a y$

Quotient rule: $log_a\left(\frac{x}{y}\right) = log_a x - log_a y$

Power rule: $log_a x^r = r log_a x$

Example 1: simplify then evaluate.

a) $log_3 6 + log_3 4.5$

b) $log_2 48 - log_2 3$

c) $log_5 \sqrt[3]{25}$

Example 2: Rewrite as a single log to a common base: $log 12 + \frac{1}{2} log 7 - log 2$

Example 3: Use the properties of logarithms to express $log_a\sqrt{\frac{x^3y^2}{w}}$ in term of $log_a x, log_a y, log_a w$.

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Practice from Textbook: **pg475**

2. Express each of the following as a logarithm of a product or quotient.

a) $\log 5 + \log 7$
b) $\log_3 4 - \log_3 2$
c) $\log_m a + \log_m b$
d) $\log x - \log y$
e) $\log_6 7 + \log_6 8 + \log_6 9$
f) $\log_4 10 + \log_4 12 - \log_4 20$

3. Express each of the following in the form $r \log_a x$.

a) $\log 5^2$
b) $\log_3 7^{-1}$
c) $\log_m p^q$
d) $\log \sqrt[3]{45}$
e) $\log_7 (36)^{0.5}$
f) $\log_5 \sqrt[5]{125}$

4. K Use the laws of logarithms to simplify and then evaluate each expression.

a) $\log_3 135 - \log_3 5$
b) $\log_5 10 + \log_5 2.5$
c) $\log 50 + \log 2$
d) $\log_4 4^7$
e) $\log_2 224 - \log_2 7$
f) $\log \sqrt{10}$

5. Describe how the graphs of $y = \log_2(4x)$, $y = \log_2(8x)$, and $y = \log_2\left(\frac{x}{2}\right)$ are related to the graph of $y = \log_2 x$.

6. Evaluate the following logarithms.

a) $\log_{25} 5^3$
b) $\log_6 54 + \log_6 2 - \log_6 3$
c) $\log_6 6\sqrt{6}$
d) $\log_2 \sqrt{36} - \log_2 \sqrt{72}$
e) $\log_3 54 + \log_3 \left(\frac{3}{2}\right)$
f) $\log_8 2 + 3\log_8 2 + \frac{1}{2}\log_8 16$

7. Use the laws of logarithms to express each of the following in terms of $\log_b x$, $\log_b y$, and $\log_b z$.

a) $\log_b xyz$
b) $\log_b \left(\frac{z}{xy}\right)$
c) $\log_b x^2y^3$
d) $\log_b \sqrt{x^5yz^3}$

8. Explain why $\log_5 3 + \log_5 \frac{1}{3} = 0$.

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9. Write each expression as a single logarithm.

a) $3\log_5 2 + \log_5 7$

b) $2\log_3 8 - 5\log_3 2$

c) $2\log_2 3 + \log_2 5$

d) $\log_3 12 + \log_3 2 - \log_3 6$

e) $\log_4 3 + \frac{1}{2}\log_4 8 - \log_4 2$

f) $2\log 8 + \log 9 - \log 36$

10. **A** Use the laws of logarithms to express each side of the equation as a single logarithm. Then compare both sides of the equation to solve.

a) $\log_2 x = 2\log_2 7 + \log_2 5$

b) $\log x = 2\log 4 + 3\log 3$

c) $\log_4 x + \log_4 12 = \log_4 48$

d) $\log_7 x = 2\log_7 25 - 3\log_7 5$

e) $\log_3 x = 2\log_3 10 - \log_3 25$

f) $\log_5 x - \log_5 8 = \log_5 6 + 3\log_5 2$

11. Write each expression as a single logarithm. Assume that all the variables represent positive numbers.

a) $\log_2 x + \log_2 y + \log_2 z$

b) $\log_5 u - \log_5 v + \log_5 w$

c) $\log_6 a - (\log_6 b + \log_6 c)$

d) $\log_2 x^2 - \log_2 xy + \log_2 y^2$

e) $1 + \log_3 x^2$

f) $3\log_4 x + 2\log_4 x - \log_4 y$

12. Write $\frac{1}{2}\log_a x + \frac{1}{2}\log_a y - \frac{3}{4}\log_a z$ as a single logarithm. Assume that all the variables represent positive numbers.

13. Describe the transformations that take the graph of $f(x) = \log_2 x$ to the graph of $g(x) = \log_2(8x^3)$.

14. **T** Use different expressions to create two logarithmic functions that have the same graph. Demonstrate algebraically why these functions have the same graph.

15. **C** Explain how the laws of logarithms can help you evaluate $\log_3\left(\frac{\sqrt[5]{27}}{2187}\right)$.

Extending

16. Explain why $\log_x x^{m-1} + 1 = m$.

17. If $\log_b x = 0.3$, find the value of $\log_b x\sqrt{x}$.

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Unit 6 – Exponential and Logarithmic functions

Chapter 8.5 – 8.6 Solving Exponential and Logarithmic Equations

Example 1: Solve the following exponential equations.

1. $\frac{9^{x-3}}{3^{4x+1}} = 81^{x+2}$

2. $3 \cdot 3^{2x} - 28 \cdot 3^{x} + 9 = 0$

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3. $3^{x+2} = 4$

Practice: $5^{3x} = 30$

4. $2(6^{3x}) = 6(4^{2x-3})$

Practice: $27(2^{2x}) = 2(3^{x})$

Example 2: Solve the following log equations with unknown base.

a) $log_x 0.04 = -2$

b) $log_x 10 = 20$

Example 3: Solve the following log equations.

a) $log_2 x - log_2 3 = log_2 6$

b) $log(x+2) + log(x-1) = 1$

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11. The number of mold spores in a petri dish increases by a factor of 10 every week. If there are initially 40 spores in the dish, how long will it take for there to be 2000 spores?

12. **Half-life** is the time it takes for half of a sample of a radioactive element to decay. The function $M(t) = P\left(\frac{1}{2}\right)^{\frac{t}{h}}$ can be used to calculate the mass remaining if the half-life is h and the initial mass is P. The half-life of radium is 1620 years.

a) If a laboratory has 5 g of radium, how much will there be in 150 years?

b) How many years will it take until the laboratory has only 4 g of radium?

13. The function $s(d) = 0.159 + 0.118 \log d$ relates the slope, s, of a beach to the average diameter, d, in millimetres, of the sand particles on the beach. Which beach has a steeper slope: beach A, which has very fine sand with $d = 0.0625$, or beach B, which has very coarse sand with $d = 1$? Justify your decision.

Practice:

18. Solve the following system of equations algebraically.

$$y = \log_2(5x + 4)$$
$$y = 3 + \log_2(x - 1)$$

19. Solve each equation.

a) $\log_5(\log_3 x) = 0$ **b)** $\log_2(\log_4 x) = 1$

20. If $\left(\frac{1}{2}\right)^{x+y} = 16$ and $\log_{x-y} 8 = -3$, calculate the values of x and y.

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Extra Practice:

1. Bacteria are tripling. In 5 days, the population has quadrupled. What is the tripling time? Estimate the answer to 2 decimal places. Check your answer!
2. If $1 - 5^{2x+3} = 0$, find the value of x.
3. If $3^y = 5$, then determine the value of $3^{4y} - 11(3^{2y})$.
4. For the graph $f(x) = 2(2)^x$, you could say that it has vertically stretched $y = 2^x$. You could also get the same graph by shifting left 1 unit. Show how those two transformations result in the same graph, by altering the equation.
5. Write two different exponential functions that have an asymptote of $y = 5$.
6. For $f(x) = a(b)^x$,
7. What values should you have for a and b for it to be an increasing function?
8. Write two different equations of the form $f(x) = a(b)^x$ that are increasing functions.
9. Write two different equations of the form $g(x) = a(b)^x$ which are decreasing functions
10. For $y = b^{kx}$,
11. What values should you have for b and k to make it decreasing?
12. Write two different equations of the form $y = b^{kx}$ that are decreasing.
13. Write two different equations of the form $y = b^{kx}$ that are increasing.
14. Solve $4^{x^2+2} = 4^{3x}$
15. Solve $2^{x^2+5x} = \frac{1}{64}$
16. Solve $6^{x+1} + 6^{x+2} = 7$
17. Solve $3(3^{2x}) - 10(3^x) + 3 = 0$
18. Solve $2^{2x} - 2^{x+1} - 8 = 0$
19. Simplify $\sqrt{\sqrt[3]{\sqrt[4]{x}}}$
20. Simplify $\sqrt{x\sqrt{x\sqrt{x}}}$
21. Simplify $\sqrt[3]{x\sqrt{x\sqrt[4]{x}}}$
22. Solve the following system of equations:
$$2^{2x+y} = 32$$
$$2^{x-3y} = \frac{1}{2}$$
23. If $2^{3x} = 16^{y+1}$ and $2x = 5y - 17$, determine the value of $x + y$. Don't use a calculator.
24. Are there values of x that make the statement $x^{-2} > x^2$ true?
25. Let $x = m + n$, where m and n are positive integers satisfying the equation
$2^6 + m^n = 2^7$. Find the sum of all possible values of x.

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Unit 6 – Exponential and Logarithmic functions

Chapter 8.7: Solve Problems with Logarithmic Functions

pH Scale (hydrogen ion concentration):

$\mathbf{pH} = -log_{10}H^{+}$,

where pH is the scaled measurement (0 to 14), and H+ is the concentration of hydrogen ions (mol/L)

In chemistry, the pH (the measure of acidity or alkalinity of a substance) is based on a logarithmic scale. A logarithmic scale uses powers of 10 to compare numbers that vary greatly in size. For example, very small and very large concentrations of the hydrogen ion in a solution influence its classification as either a base or an acid.

Concentration of hydrogen ions compared to distilled water		Examples of solutions at this pH
10 000 000	pH = 0	battery acid, strong hydrofluoric acid
1 000 000	pH = 1	hydrochloric acid secreted by stomach lining
100 000	pH = 2	lemon juice, gastric acid, vinegar
10 000	pH = 3	grapefruit, orange juice, soda
1000	pH = 4	tomato juice, acid rain
100	pH = 5	soft drinking water, black coffee
10	pH = 6	urine, saliva
1	pH = 7	"pure" water
$\frac{1}{10}$	pH = 8	seawater
$\frac{1}{100}$	pH = 9	baking soda
$\frac{1}{1000}$	pH = 10	Great Salt Lake, milk of magnesia
$\frac{1}{10\,000}$	pH = 11	ammonia solution
$\frac{1}{100\,000}$	pH = 12	soapy water
$\frac{1}{1\,000\,000}$	pH = 13	bleaches, oven cleaner
$\frac{1}{10\,000\,000}$	pH = 14	liquid drain cleaner

A difference of one pH unit represents a tenfold (10 times) change in the concentration of hydrogen ions in the solution. For example, the acidity of a sample with a pH of 5 is 10 times greater than the acidity of a sample with a pH of 6. A difference of 2 units, from 6 to 4, would mean that the acidity is 100 times greater, and so on.

- A liquid with a pH less than 7 is considered *acidic*.
- A liquid with a pH greater than 7 is considered *alkaline*.
- A liquid with a pH of 7 is considered *neutral*. Pure distilled water has a pH value of 7.

Example:

The relationship between pH and hydrogen ion concentration is given by the formula $\text{pH} = -\log[H^+]$, where $[H^+]$ is the concentration of hydrogen ions in moles per litre (mol/L).

a) Calculate the pH if the concentration of hydrogen ions is 0.0001 mol/L.

b) The pH of lemon juice is 2. Calculate the hydrogen ion concentration.

c) If the hydrogen ion concentration is a measure of the strength of an acid, how much stronger is an acid with pH 1.6 than an acid with pH 2.5?

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Richter Scale (Earthquakes):

$\mathbf{M} = \boldsymbol{log_{10}A}$, where M is the magnitude (approximately 0 to 10), and A is the amplitude on the seismograph.

Notes: This formula is useful on for comparing the relative intensity of earthquakes. The actual energy of the earthquake is more complex.

Example:

The Richter magnitude scale uses logarithms to compare intensity of earthquakes.

True Intensity	Richter Scale Magnitude
10^1	$\log_{10}10^1 = 1$
10^4	$\log_{10}10^4 = 4$
$10^{5.8}$	$\log_{10}10^{5.8} = 5.8$

An earthquake of magnitude 2 is actually 10 times more intense than an earthquake of magnitude 1. The difference between the magnitudes of two earthquakes can be used to determine the difference in intensity. If the average earthquake measures 4.5 on the Richter scale, how much more intense is an earthquake that measures 8?

Practice: If earthquake A has a magnitude of 6.4 and it is 25 time more intense then earthquake B, what is the magnitude of earthquake B?

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Sound Loudness (decibel scale):

$\mathbf{L = 10log_{10}\left(\frac{I}{I_o}\right)}$,

where L is the loudness of the sound, I is the sound intensity (energy), and I_o is the threshold of human hearing.

Example:

The dynamic range of human hearing and sound intensity spans from 10^{-12} W/m^2 to about 10 W/m^2. The highest sound intensity that can be heard is 10 000 000 000 000 times as loud as the quietest! This span of sound intensity is impractical for normal use. A more convenient way to express loudness is a relative logarithmic scale, with the lowest sound that can be heard by the human ear, $I_0 = 10^{-12}$ W/m^2, given the measure of loudness of 0 dB.

Recall that the formula that is used to measure sound is $L = 10 \log \left(\frac{I}{I_0}\right)$, where L is the loudness measured in decibels, I is the intensity of the sound being measured, and I_0 is the intensity of sound at the threshold of hearing. The following table shows the loudness of a selection of sounds measured in decibels.

Sound	Loudness (dB)
soft whisper	30
normal conversation	60
shouting	80
subway	90
screaming	100
rock concert	120
jet engine	140
space-shuttle launch	180

How many times more intense is the sound of a rock concert than the sound of a subway?