Chapter 1

INTRODUCTION TO CALCULUS

In the English language, the rules of grammar are used to speak and write effectively. Asking for a cookie at the age of ten was much easier than when you were first learning to speak. These rules developed over time. Calculus developed in a similar way. Sir Isaac Newton and Gottfried Wilhelm von Leibniz independently organized an assortment of ideas and methods that were circulating among the mathematicians of their time. As a tool in the service of science, calculus served its purpose very well. More than two centuries passed, however, before mathematicians had identified and agreed on its underlying principles—its grammar. In this chapter, you will see some of the ideas that were brought together to form the underlying principles of calculus.

CHAPTER EXPECTATIONS

In this chapter, you will

- simplify radical expressions, **Section 1.1**
- use limits to determine the slope and the equation of the tangent to a graph, **Section 1.2**
- pose problems and formulate hypotheses regarding rates of change, **Section 1.3, Career Link**
- calculate and interpret average and instantaneous rates of change and relate these values to slopes of secants and tangents, **Section 1.3**
- understand and evaluate limits using appropriate properties, **Sections 1.4, 1.5**
- examine continuous functions and use limits to explain why a function is discontinuous, **Sections 1.5, 1.6**

Review of Prerequisite Skills

Before beginning this chapter, review the following concepts from previous courses:

- determining the slope of a line: $m = \frac{\Delta y}{\Delta x}$
- determining the equation of a line
- using function notation for substituting into and evaluating functions
- simplifying algebraic expressions
- factoring expressions
- finding the domain of functions
- calculating average rate of change and slopes of secant lines
- estimating instantaneous rate of change and slopes of tangent lines

Exercise

1. Determine the slope of the line passing through each of the following pairs of points:

a. $(2, 5)$ and $(6, -7)$

b. $(3, -4)$ and $(-1, 4)$

c. $(0, 0)$ and $(1, 4)$

d. $(0, 0)$ and $(-1, 4)$

e. $(-2.1, 4.41)$ and $(-2, 4)$

f. $\left(\frac{3}{4}, \frac{1}{4}\right)$ and $\left(\frac{7}{4}, -\frac{1}{4}\right)$

2. Determine the equation of a line for the given information.

a. slope 4, y-intercept -2

b. slope -2, y-intercept 5

c. through $(-1, 6)$ and $(4, 12)$

d. through $(-2, 4)$ and $(-6, 8)$

e. vertical, through $(-3, 5)$

f. horizontal, through $(-3, 5)$

3. Evaluate for $x = 2$.

a. $f(x) = -3x + 5$

b. $f(x) = (4x - 2)(3x - 6)$

c. $f(x) = -3x^2 + 2x - 1$

d. $f(x) = (5x + 2)^2$

4. For $f(x) = \frac{x}{x^2 + 4}$, determine each of the following values:

a. $f(-10)$

b. $f(-3)$

c. $f(0)$

d. $f(10)$

5. Consider the function f given by $f(x) = \begin{cases} \sqrt{3 - x}, \text{ if } x < 0 \\ \sqrt{3 + x}, \text{ if } x \geq 0 \end{cases}$

Calculate each of the following:

a. $f(-33)$

b. $f(0)$

c. $f(78)$

d. $f(3)$

6. A function s is defined for $t > -3$ by $s(t) = \begin{cases} \frac{1}{t}, \text{ if } -3 < t < 0 \\ 5, \text{ if } t = 0 \\ t^3, \text{ if } t > 0 \end{cases}$

Evaluate each of the following:

a. $s(-2)$ b. $s(-1)$ c. $s(0)$ d. $s(1)$ e. $s(100)$

7. Expand, simplify, and write each expression in standard form.

a. $(x - 6)(x + 2)$
b. $(5 - x)(3 + 4x)$
c. $x(5x - 3) - 2x(3x + 2)$
d. $(x - 1)(x + 3) - (2x + 5)(x - 2)$
e. $(a + 2)^3$
f. $(9a - 5)^3$

8. Factor each of the following:

a. $x^3 - x$
b. $x^2 + x - 6$
c. $2x^2 - 7x + 6$
d. $x^3 + 2x^2 + x$
e. $27x^3 - 64$
f. $2x^3 - x^2 - 7x + 6$

9. Determine the domain of each of the following:

a. $y = \sqrt{x + 5}$
b. $y = x^3$
c. $y = \frac{3}{x - 1}$
d. $h(x) = \frac{x^2 + 4}{x}$
e. $y = \frac{6x}{2x^2 - 5x - 3}$
f. $y = \frac{(x - 3)(x + 4)}{(x + 2)(x - 1)(x + 5)}$

10. The height of a model rocket in flight can be modelled by the equation $h(t) = -4.9t^2 + 25t + 2$, where h is the height in metres at t seconds. Determine the average rate of change in the model rocket's height with respect to time during

a. the first second
b. the second second

11. Sacha drains the water from a hot tub. The hot tub holds 1600 L of water. It takes 2 h for the water to drain completely. The volume of water in the hot tub is modelled by $V(t) = 1600 - \frac{t^2}{9}$, where V is the volume in litres at t minutes and $0 \leq t \leq 120$.

a. Determine the average rate of change in volume during the second hour.
b. Estimate the instantaneous rate of change in volume after exactly 60 min.
c. Explain why all estimates of the instantaneous rate of change in volume where $0 \leq t \leq 120$ result in a negative value.

12. a. Sketch the graph of $f(x) = -2(x - 3)^2 + 4$.
b. Draw a tangent line at the point $(5, f(5))$, and estimate its slope.
c. Estimate the instantaneous rate of change in $f(x)$ when $x = 5$.

CAREER LINK Investigate

CHAPTER 1: ASSESSING ATHLETIC PERFORMANCE

Differential calculus is fundamentally about the idea of instantaneous rate of change. A familiar rate of change is heart rate. Elite athletes are keenly interested in the analysis of heart rates. Sporting performance is enhanced when an athlete is able to increase his or her heart rate at a slower pace (that is, to get tired less quickly). A heart rate is described for an instant in time.

Time (s)	Time (min)	Number of Heartbeats
10	0.17	9
20	0.33	19
30	0.50	31
40	0.67	44
50	0.83	59
60	1.00	75

Heart rate is the instantaneous rate of change in the total number of heartbeats with respect to time. When nurses and doctors count heartbeats and then divide by the time elapsed, they are not determining the instantaneous rate of change but are calculating the average heart rate over a period of time (usually 10 s). In this chapter, the idea of the derivative will be developed, progressing from the average rate of change calculated over smaller and smaller intervals until a limiting value is reached at the instantaneous rate of change.

Case Study—Assessing Elite Athlete Performance

The table shows the number of heartbeats of an athlete who is undergoing a cardiovascular fitness test. Complete the discussion questions to determine if this athlete is under his or her maximum desired heart rate of 65 beats per minute at precisely 30 s.

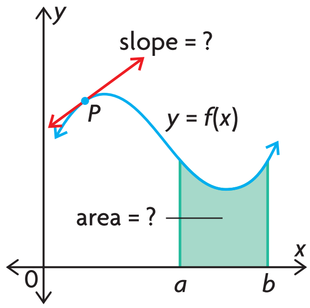
DISCUSSION QUESTIONS

1. Graph the number of heartbeats versus time (in minutes) on graph paper, joining the points to make a smooth curve. Draw a second relationship on the same set of axes, showing the resting heart rate of 50 beats per minute. Use the slopes of the two relationships graphed to explain why the test results indicate that the person must be exercising.
2. Discuss how the average rate of change in the number of heartbeats over an interval of time could be calculated using this graph. Explain your reasoning.
3. Calculate the athlete's average heart rate over the intervals of $[0\text{ s}, 60\text{ s}]$, $[10\text{ s}, 50\text{ s}]$, and $[20\text{ s}, 40\text{ s}]$. Show the progression of these average heart rate calculations on the graph as a series of secants.
4. Use the progression of these average heart-rate secants to make a graphical prediction of the instantaneous heart rate at $t = 30$ s. Is the athlete's heart rate less than 65 beats per minute at $t = 30$ s? Estimate the heart rate at $t = 60$ s.

What Is Calculus?

Two simple geometric problems originally led to the development of what is now called calculus. Both problems can be stated in terms of the graph of a function $y = f(x)$.

- The problem of tangents: What is the slope of the tangent to the graph of a function at a given point P?
- The problem of areas: What is the area under a graph of a function $y = f(x)$ between $x = a$ and $x = b$?



Interest in the problem of tangents and the problem of areas dates back to scientists such as Archimedes of Syracuse (287–212 BCE), who used his vast ingenuity to solve special cases of these problems. Further progress was made in the seventeenth century, most notably by Pierre de Fermat (1601–1665) and Isaac Barrow (1630–1677), a professor of Sir Isaac Newton (1642–1727) at the University of Cambridge, England. Professor Barrow recognized that there was a close connection between the problem of tangents and the problem of areas. However, it took the genius of both Newton and Gottfried Wilhelm von Leibniz (1646–1716) to show the way to handle both problems. Using the analytic geometry of Rene Descartes (1596–1650), Newton and Leibniz showed independently how these two problems could be solved by means of new operations on functions, called differentiation and integration. Their discovery is considered to be one of the major advances in the history of mathematics. Further research by mathematicians from many countries using these operations has created a problem-solving tool of immense power and versatility, which is known as calculus. It is a powerful branch of mathematics, used in applied mathematics, science, engineering, and economics.

We begin our study of calculus by discussing the meaning of a tangent and the related idea of rate of change. This leads us to the study of limits and, at the end of the chapter, to the concept of the derivative of a function.

Section 1.1—Radical Expressions: Rationalizing Denominators

Now that we have reviewed some concepts that will be needed before beginning the introduction to calculus, we have to consider simplifying expressions with radicals in the denominator of radical expressions. Recall that a rational number is a number that can be expressed as a fraction (quotient) containing integers. So the process of changing a denominator from a radical (square root) to a rational number (integer) is called **rationalizing the denominator**. The reason that we rationalize denominators is that dividing by an integer is preferable to dividing by a radical number.

In certain situations, it is useful to rationalize the numerator. Practice with rationalizing the denominator prepares you for rationalizing the numerator.

There are two situations that we need to consider: radical expressions with one-term denominators and those with two-term denominators. For both, the numerator and denominator will be multiplied by the same expression, which is the same as multiplying by one.

EXAMPLE 1 Selecting a strategy to rationalize the denominator

Simplify $\frac{3}{4\sqrt{5}}$ by rationalizing the denominator.

Solution

$$\frac{3}{4\sqrt{5}} = \frac{3}{4\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} \qquad \text{(Multiply both the numerator and denominator by } \sqrt{5}\text{)}$$

$$= \frac{3\sqrt{5}}{4 \times 5} \qquad \text{(Simplify)}$$

$$= \frac{3\sqrt{5}}{20}$$

When the denominator of a radical fraction is a two-term expression, you can rationalize the denominator by multiplying by the **conjugate**.

An expression such as $\sqrt{a} + \sqrt{b}$ has the conjugate $\sqrt{a} - \sqrt{b}$.

Why are conjugates important? Recall that the linear terms are eliminated when expanding a difference of squares. For example,

$$\begin{aligned}(a - b)(a + b) &= a^2 + ab - ab - b^2 \\ &= a^2 - b^2\end{aligned}$$

If a and b were radicals, squaring them would rationalize them.

Consider this product: $\left(\sqrt{m}+\sqrt{n}\right)\left(\sqrt{m}-\sqrt{n}\right)$, m, n rational

$$= \left(\sqrt{m}\right)^2 - \sqrt{mn} + \sqrt{mn} - \left(\sqrt{n}\right)^2$$

$$= m - n$$

Notice that the result is rational!

EXAMPLE 2 Creating an equivalent expression by rationalizing the denominator

Simplify $\frac{2}{\sqrt{6}+\sqrt{3}}$ by rationalizing the denominator.

Solution

$$\frac{2}{\sqrt{6}+\sqrt{3}} = \frac{2}{\sqrt{6}+\sqrt{3}} \times \frac{\sqrt{6}-\sqrt{3}}{\sqrt{6}-\sqrt{3}}$$

(Multiply both the numerator and denominator by $\sqrt{6}-\sqrt{3}$)

$$= \frac{2\left(\sqrt{6}-\sqrt{3}\right)}{6-3}$$

(Simplify)

$$= \frac{2\left(\sqrt{6}-\sqrt{3}\right)}{3}$$

EXAMPLE 3 Selecting a strategy to rationalize the denominator

Simplify the radical expression $\frac{5}{2\sqrt{6}+3}$ by rationalizing the denominator.

Solution

$$\frac{5}{2\sqrt{6}+3} = \frac{5}{2\sqrt{6}+3} \times \frac{2\sqrt{6}-3}{2\sqrt{6}-3}$$

(The conjugate $2\sqrt{6}+\sqrt{3}$ is $2\sqrt{6}-\sqrt{3}$)

$$= \frac{5\left(2\sqrt{6}-3\right)}{4\sqrt{36}-9}$$

(Simplify)

$$= \frac{5\left(2\sqrt{6}-3\right)}{24-9}$$

$$= \frac{5\left(2\sqrt{6}-3\right)}{15}$$

(Divide by the common factor of 5)

$$= \frac{2\sqrt{6}-3}{3}$$

The numerator can also be rationalized in the same way as the denominator was in the previous expressions.

EXAMPLE 4 Selecting a strategy to rationalize the numerator

Rationalize the numerator of the expression $\frac{\sqrt{7}-\sqrt{3}}{2}$.

Solution

$$\begin{aligned}
\frac{\sqrt{7}-\sqrt{3}}{2} &= \frac{\sqrt{7}-\sqrt{3}}{2} \times \frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}+\sqrt{3}} && \text{(Multiply the numerator and denominator by } \sqrt{7}+\sqrt{3}\text{)}\\
&= \frac{7-3}{2\left(\sqrt{7}+\sqrt{3}\right)} && \text{(Simplify)}\\
&= \frac{4}{2\left(\sqrt{7}+\sqrt{3}\right)} && \text{(Divide by the common factor of 2)}\\
&= \frac{2}{\sqrt{7}+\sqrt{3}}
\end{aligned}$$

IN SUMMARY

Key Ideas

- To rewrite a radical expression with a one-term radical in the denominator, multiply the numerator and denominator by the one-term denominator.

$$\begin{aligned}
\frac{\sqrt{a}}{\sqrt{b}} &= \frac{\sqrt{a}}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}}\\
&= \frac{\sqrt{ab}}{b}
\end{aligned}$$

- When the denominator of a radical expression is a two-term expression, rationalize the denominator by multiplying the numerator and denominator by the conjugate, and then simplify.

$$\begin{aligned}
\frac{1}{\sqrt{a}-\sqrt{b}} &= \frac{1}{\sqrt{a}-\sqrt{b}} \times \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}\\
&= \frac{\sqrt{a}+\sqrt{b}}{a-b}
\end{aligned}$$

Need to Know

- When you simplify a radical expression such as $\frac{\sqrt{3}}{5\sqrt{2}}$, multiply the numerator and denominator by the radical only.

$$\begin{aligned}
\frac{\sqrt{3}}{5\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} &= \frac{\sqrt{6}}{5(2)}\\
&= \frac{\sqrt{6}}{10}
\end{aligned}$$

- $\sqrt{a}+\sqrt{b}$ is the conjugate $\sqrt{a}-\sqrt{b}$, and vice versa.

Exercise 1.1

PART A

1. Write the conjugate of each radical expression.

a. $2\sqrt{3} - 4$ c. $-2\sqrt{3} - \sqrt{2}$ e. $\sqrt{2} - \sqrt{5}$

b. $\sqrt{3} + \sqrt{2}$ d. $3\sqrt{3} + \sqrt{2}$ f. $-\sqrt{5} + 2\sqrt{2}$

2. Rationalize the denominator of each expression. Write your answer in simplest form.

a. $\dfrac{\sqrt{3} + \sqrt{5}}{\sqrt{2}}$ c. $\dfrac{4\sqrt{3} + 3\sqrt{2}}{2\sqrt{3}}$

b. $\dfrac{2\sqrt{3} - 3\sqrt{2}}{\sqrt{2}}$ d. $\dfrac{3\sqrt{5} - \sqrt{2}}{2\sqrt{2}}$

PART B

K 3. Rationalize each denominator.

a. $\dfrac{3}{\sqrt{5} - \sqrt{2}}$ c. $\dfrac{\sqrt{3} - \sqrt{2}}{\sqrt{3} + \sqrt{2}}$ e. $\dfrac{2\sqrt{3} - \sqrt{2}}{5\sqrt{2} + \sqrt{3}}$

b. $\dfrac{2\sqrt{5}}{2\sqrt{5} + 3\sqrt{2}}$ d. $\dfrac{2\sqrt{5} - 8}{2\sqrt{5} + 3}$ f. $\dfrac{3\sqrt{3} - 2\sqrt{2}}{3\sqrt{3} + 2\sqrt{2}}$

4. Rationalize each numerator.

a. $\dfrac{\sqrt{5} - 1}{4}$ b. $\dfrac{2 - 3\sqrt{2}}{2}$ c. $\dfrac{\sqrt{5} + 2}{2\sqrt{5} - 1}$

C 5. a. Rationalize the denominator of $\dfrac{8\sqrt{2}}{\sqrt{20} - \sqrt{18}}$.

b. Rationalize the denominator of $\dfrac{8\sqrt{2}}{2\sqrt{5} - 3\sqrt{2}}$.

c. Why are your answers in parts a and b the same? Explain.

6. Rationalize each denominator.

a. $\dfrac{2\sqrt{2}}{2\sqrt{3} - \sqrt{8}}$ c. $\dfrac{2\sqrt{2}}{\sqrt{16} - \sqrt{12}}$ e. $\dfrac{3\sqrt{5}}{4\sqrt{3} - 5\sqrt{2}}$

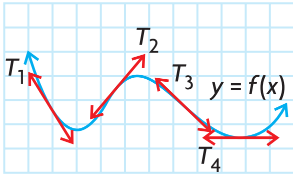
b. $\dfrac{2\sqrt{6}}{2\sqrt{27} - \sqrt{8}}$ d. $\dfrac{3\sqrt{2} + 2\sqrt{3}}{\sqrt{12} - \sqrt{8}}$ f. $\dfrac{\sqrt{18} + \sqrt{12}}{\sqrt{18} - \sqrt{12}}$

A 7. Rationalize the numerator of each of the following expressions:

a. $\dfrac{\sqrt{a} - 2}{a - 4}$ b. $\dfrac{\sqrt{x + 4} - 2}{x}$ c. $\dfrac{\sqrt{x + h} - x}{x}$

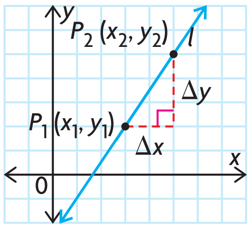
Section 1.2—The Slope of a Tangent

You are familiar with the concept of a **tangent** to a curve. What geometric interpretation can be given to a tangent to the graph of a function at a point P? A tangent is the straight line that most resembles the graph near a point. Its slope tells how steep the graph is at the point of tangency. In the figure below, four tangents have been drawn.



The goal of this section is to develop a method for determining the slope of a tangent at a given point on a curve. We begin with a brief review of lines and slopes.

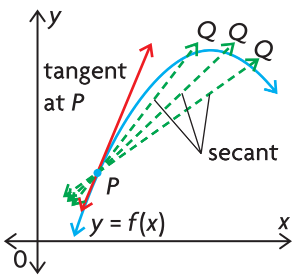
Lines and Slopes



The slope m of the line joining points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is defined as $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.

The equation of the line l in point-slope form is $\frac{y - y_1}{x - x_1} = m$ or $y - y_1 = m(x - x_1)$. The equation in slope–y-intercept form is $y = mx + b$, where b is the y-intercept of the line.

To determine the equation of a tangent to a curve at a given point, we first need to know the slope of the tangent. What can we do when we only have one point? We proceed as follows:



Consider a curve $y = f(x)$ and a point P that lies on the curve. Now consider another point Q on the curve. The line joining P and Q is called a **secant**. Think of Q as a moving point that slides along the curve toward P, so that the slope of the secant PQ becomes a progressively better estimate of the slope of the tangent at P.

This suggests the following definition of the slope of the tangent:

Slope of a Tangent

The slope of the tangent to a curve at a point P is the limiting slope of the secant PQ as the point Q slides along the curve toward P. In other words, the slope of the tangent is said to be the **limit** of the slope of the secant as Q approaches P along the curve.

We will illustrate this idea by finding the slope of the tangent to the parabola $y = x^2$ at $P(3, 9)$.

INVESTIGATION 1

A. Determine the y-coordinates of the following points that lie on the graph of the parabola $y = x^2$:

i) $Q_1(3.5, y)$ ii) $Q_2(3.1, y)$ iii) $Q_3(3.01, y)$ iv) $Q_4(3.001, y)$

B. Calculate the slopes of the secants through $P(3, 9)$ and each of the points Q_1, Q_2, Q_3, and Q_4.

C. Determine the y-coordinates of each point on the parabola, and then repeat part B using the following points.

i) $Q_5(2.5, y)$ ii) $Q_6(2.9, y)$ iii) $Q_7(2.99, y)$ iv) $Q_8(2.999, y)$

D. Use your results from parts B and C to estimate the slope of the tangent at point $P(3, 9)$.

E. Graph $y = x^2$ and the tangent to the graph at $P(3, 9)$.

In this investigation, you found the slope of the tangent by finding the limiting value of the slopes of a sequence of secants. Since we are interested in points Q that are close to $P(3, 9)$ on the parabola $y = x^2$ it is convenient to write Q as $(3 + h, (3 + h)^2)$, where h is a very small nonzero number. The variable h determines the position of Q on the parabola. As Q slides along the parabola toward P, h will take on values successively smaller and closer to zero. We say that "h approaches zero" and use the notation "$h \to 0$."

INVESTIGATION 2 A. Using technology or graph paper, draw the parabola $f(x) = x^2$.

B. Let P be the point $(1, 1)$.

C. Determine the slope of the secant through Q_1 and $P(1, 1)$, Q_2 and $P(1, 1)$ and so on, for points $Q_1(1.5, f(1.5))$, $Q_2(1.1, f(1.1))$, $Q_3(1.01, f(1.01))$, $Q_4(1.001, f(1.001))$, and $Q_5(1.0001, f(1.0001))$.

D. Draw these secants on the same graph you created in part A.

E. Use your results to estimate the slope of the tangent to the graph of f at point P.

F. Draw the tangent at point $P(1, 1)$.

INVESTIGATION 3 A. Determine an expression for the slope of the secant PQ through points $P(3, 9)$ and $Q(3 + h, (3 + h)^2)$.

B. Explain how you could use the expression in a part A to predict the slope of the tangent to the parabola $f(x) = x^2$ at point $P(3, 9)$.

The slope of the tangent to the parabola at point P is the limiting slope of the secant line PQ as point Q slides along the parabola; that is, as $h \to 0$, we write "$\lim\limits_{h\to 0}$" as the abbreviation for "limiting value as h approaches 0."

Therefore, from the investigation, the slope of the tangent at a point P is

$\lim\limits_{h\to 0}$ (slope of the secant PQ).

EXAMPLE 1 **Reasoning about the slope of a tangent as a limiting value**

Determine the slope of the tangent to the graph of the parabola $f(x) = x^2$ at $P(3, 9)$.

Solution

Using points $P(3, 9)$ and $Q(3 + h, (3 + h)^2)$, $h \neq 0$, the slope of the secant PQ is

$$\frac{\Delta x}{\Delta y} = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{(Substitute)}$$

$$= \frac{(3 + h)^2 - 9}{3 + h - 3} \qquad \text{(Expand)}$$

$$= \frac{9 + 6h + h^2 - 9}{h} \qquad \text{(Simplify and factor)}$$

$$= \frac{h(6 + h)}{h} \qquad \text{(Divide by the common factor of } h\text{)}$$

$$= (6 + h)$$

As $h \to 0$, the value of $(6 + h)$ approaches 6, and thus $\lim_{h \to 0}(6 + h) = 6$.

We conclude that the slope of the tangent at $P(3, 9)$ to the parabola $y = x^2$ is 6.

EXAMPLE 2 Selecting a strategy involving a series of secants to estimate the slope of a tangent

Tech | Support

For help graphing functions using a graphing calculator, see Technology Appendix p. 597.

a. Use your calculator to graph the parabola $y = -\frac{1}{8}(x + 1)(x - 7)$. Plot the points on the parabola from $x = -1$ to $x = 6$, where x is an integer.

b. Determine the slope of the secants using each point from part a and point $P(5, 1.5)$.

c. Use the result of part b to estimate the slope of the tangent at $P(5, 1.5)$.

Solution

a. Using the x-intercepts of -1 and 7, the equation of the axis of symmetry is $x = \frac{-1 + 7}{2} = 3$, so the x-coordinate of the vertex is 3.

Substitute $x = 3$ into $y = -\frac{1}{8}(x + 1)(x - 7)$.

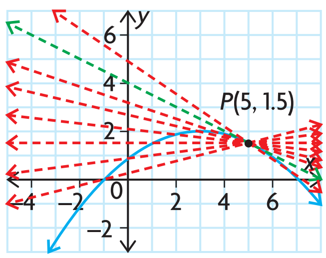
$y = -\frac{1}{8}(3 + 1)(3 - 7) = 2$

Therefore, the vertex is $(3, 2)$.

The y-intercept of the parabola is $\frac{7}{8}$.

The points on the parabola are $(-1, 0)$, $(0, 0.875)$, $(1, 1.5)$, $(2, 1.875)$, $(3, 2)$, $(4, 1.875)$, $(5, 1.5)$, and $(6, 0.875)$.

The parabola and the secants through each point and point $P(5, 1.5)$ are shown in red. The tangent through $P(5, 1.5)$ is shown in green.



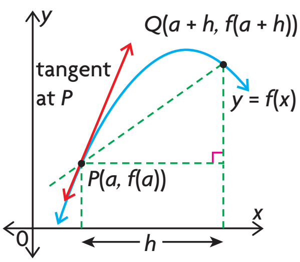
b. Using points $(-1, 0)$ and $P(5, 1.5)$, the slope is $m = \frac{1.5 - 0}{5 - (-1)} = 0.25$.

Using the other points and $P(5, 1.5)$, the slopes are 0.125, 0, -0.125, -0.25, -0.375, and -0.625, respectively.

c. The slope of the tangent at $P(5, 1.5)$ is between -0.375 and -0.625. It can be determined to be -0.5 using points closer and closer to $P(5, 1.5)$.

The Slope of a Tangent at an Arbitrary Point

We can now generalize the method used above to derive a formula for the slope of the tangent to the graph of any function $y = f(x)$.



Let $P(a, f(a))$ be a fixed point on the graph of $y = f(x)$, and let $Q(x, y) = Q(x, f(x))$ represent any other point on the graph. If Q is a horizontal distance of h units from P, then $x = a + h$ and $y = f(a + h)$. Point Q then has coordinates $Q(a + h, f(a + h))$.

The slope of the secant PQ is $\frac{\Delta y}{\Delta x} = \frac{f(a + h) - f(a)}{a + h - a} = \frac{f(a + h) - f(a)}{h}$.

This quotient is fundamental to calculus and is referred to as the **difference quotient**. Therefore, the slope m of the tangent at $P(a, f(a))$ is $\lim_{h \to 0}$ (slope of the secant PQ), which may be written as $m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$.

Slope of a Tangent as a Limit

The slope of the tangent to the graph $y = f(x)$ at point $P(a, f(a))$ is

$m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$, if this limit exists.

EXAMPLE 3 Connecting limits and the difference quotient to the slope of a tangent

a. Using the definition of the slope of a tangent, determine the slope of the tangent to the curve $y = -x^2 + 4x + 1$ at the point determined by $x = 3$.
b. Determine the equation of the tangent.
c. Sketch the graph of $y = -x^2 + 4x + 1$ and the tangent at $x = 3$.

Solution

a. The slope of the tangent can be determined using the expression above. In this example, $f(x) = -x^2 + 4x + 1$ and $a = 3$.

$$
\begin{aligned}
\text{Then} \quad f(3) &= -(3)^2 + 4(3) + 1 = 4 \\
\text{and} \quad f(3 + h) &= -(3 + h)^2 + 4(3 + h) + 1 \\
&= -9 - 6h - h^2 + 12 + 4h + 1 \\
&= -h^2 - 2h + 4
\end{aligned}
$$

The slope of the tangent at $(3, 4)$ is

$$m = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}$$ (Substitute)

$$= \lim_{h \to 0} \frac{[-h^2 - 2h + 4] - 4}{h}$$ (Simplify and factor)

$$= \lim_{h \to 0} \frac{h(-h - 2)}{h}$$ (Divide by the common factor)

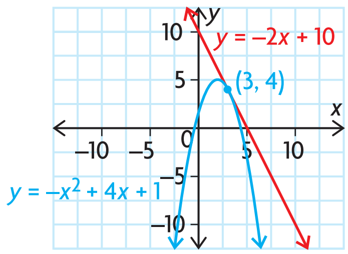
$$= \lim_{h \to 0} (-h - 2)$$ (Evaluate)

$$= -2$$

The slope of the tangent at $x = 3$ is -2.

b. The equation of the tangent at $(3, 4)$ is $\frac{y - 4}{x - 3} = -2$, or $y = -2x + 10$.

c. Using graphing software, we obtain



EXAMPLE 4 Selecting a limit strategy to determine the slope of a tangent

Determine the slope of the tangent to the rational function $f(x) = \frac{3x + 6}{x}$ at point $(2, 6)$.

Solution

Using the definition, the slope of the tangent at $(2, 6)$ is

$$m = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h}$$ (Substitute)

$$= \lim_{h \to 0} \frac{\frac{6 + 3h + 6}{2 + h} - 6}{h}$$ (Determine a common denominator)

$$= \lim_{h \to 0} \frac{\frac{6 + 3h + 6}{2 + h} - \frac{6(2 + h)}{2 + h}}{h}$$ (Simplify)

$$= \lim_{h \to 0} \frac{\frac{12 + 3h - 12 - 6h}{2 + h}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{-3h}{2+h}}{h}$$ (Multiply by the reciprocal)

$$= \lim_{h \to 0} \frac{-3h}{2+h} \times \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{-3}{2+h}$$ (Evaluate)

$$= -1.5$$

Therefore, the slope of the tangent to $f(x) = \frac{3x+6}{x}$ at $(2, 6)$ is -1.5.

EXAMPLE 5 Determining the slope of a line tangent to a root function

Find the slope of the tangent to $f(x) = \sqrt{x}$ at $x = 9$.

Solution

$$f(9) = \sqrt{9} = 3$$

$$f(9+h) = \sqrt{9+h}$$

Using the limit of the difference quotient, the slope of the tangent at $x = 9$ is

$$m = \lim_{h \to 0} \frac{f(9+h) - f(9)}{h}$$ (Substitute)

$$= \lim_{h \to 0} \frac{\sqrt{9+h} - 3}{h}$$ (Rationalize the numerator)

$$= \lim_{h \to 0} \frac{\sqrt{9+h} - 3}{h} \times \frac{\sqrt{9+h} + 3}{\sqrt{9+h} + 3}$$

$$= \lim_{h \to 0} \frac{(9+h) - 9}{h(\sqrt{9+h} + 3)}$$ (Simplify)

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{9+h} + 3)}$$ (Divide by the common factor of h)

$$= \lim_{h \to 0} \frac{1}{\sqrt{9+h} + 3}$$ (Evalute)

$$= \frac{1}{6}$$

Therefore, the slope of the tangent to $f(x) = \sqrt{x}$ at $x = 9$ is $\frac{1}{6}$.

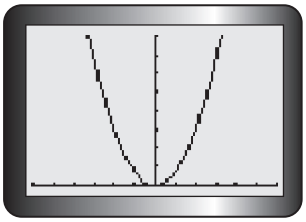
INVESTIGATION 4

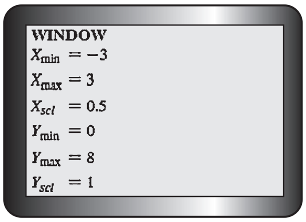
Tech | *Support*

For help graphing functions, tracing, and using the table feature on a graphing calculator, see Technology Appendices p. 597 and p. 599.

A graphing calculator can help us estimate the slope of a tangent at a point. The exact value can then be found using the definition of the slope of the tangent using the difference quotient. For example, suppose that we wish to find the slope of the tangent to $y = f(x) = x^3$ at $x = 1$.

A. Graph $Y_1 = \dfrac{((x + 0.01)^3 - x^3)}{0.01}$.





B. Explain why the values for the WINDOW were chosen.
Observe that the function entered in Y_1 is the difference quotient $\dfrac{f(a + h) - f(a)}{h}$ for $f(x) = x^3$ and $h = 0.01$. Remember that this approximates the slope of the tangent and not the graph of $f(x) = x^3$.

C. Use the TRACE function to find $X = 1.0212766$, $Y = 3.159756$.
This means that the slope of the secant passing through the points where $x = 1$ and $x = 1 + 0.01 = 1.01$ is about 3.2. The value 3.2 could be used as an approximation for the slope of the tangent at $x = 1$.

D. Can you improve this approximation? Explain how you could improve your estimate. Also, if you use different WINDOW values, you can see a different-sized, or differently centred, graph.

E. Try once again by setting $X_{min} = -9$, $X_{max} = 10$, and note the different appearance of the graph. Use the TRACE function to find $X = 0.90425532$, $Y = 2.4802607$, and then $X = 1.106383$, $Y = 3.7055414$. What is your guess for the slope of the tangent at $x = 1$ now? Explain why only estimation is possible.

F. Another way of using a graphing calculator to approximate the slope of the tangent is to consider h as the variable in the difference quotient. For this example, $f(x) = x^3$ at $x = 1$, look at $\dfrac{f(a + h) - f(a)}{h} = \dfrac{(1 + h)^3 - 1^3}{h}$.

G. Trace values of h as $h \to 0$. You can use the table or graph function of your calculator. Graphically, we say that we are looking at $\dfrac{(1 + h)^3 - 1}{h}$ in the neighbourhood of $h = 0$. To do this, graph $y = \dfrac{(1 + x)^3 - 1}{x}$ and examine the value of the function as $x \to 0$.

IN SUMMARY

Key Ideas

- The slope of the tangent to a curve at a point P is the limit of the slopes of the secants PQ as Q moves closer to P.

  $m_{\text{tangent}} = \lim_{Q \to P} (\textit{slope of secant } PQ)$

- The slope of the tangent to the graph of $y = f(x)$ at $P(a, f(a))$ is given by

  $m_{\text{tangent}} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$.

Need to Know

- To find the slope of the tangent at a point $P(a, f(a))$,
  - find the value of $f(a)$
  - find the value of $f(a + h)$
  - evaluate $\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$

Exercise 1.2

PART A

1. Calculate the slope of the line through each pair of points.
   a. $(2, 7), (-3, -8)$
   b. $\left(\frac{1}{2}, \frac{3}{2}\right), \left(\frac{7}{2}, -\frac{7}{2}\right)$
   c. $(6.3, -2.6), (1.5, -1)$
2. Determine the slope of a line perpendicular to each of the following:
   a. $y = 3x - 5$
   b. $13x - 7y - 11 = 0$
3. State the equation and sketch the graph of each line described below.
   a. passing through $(-4, -4)$ and $\left(\frac{5}{3}, -\frac{5}{3}\right)$
   b. having slope 8 and y-intercept 6
   c. having x-intercept 5 and y-intercept -3
   d. passing through $(5, 6)$ and $(5, -9)$

4. Simplify each of the following difference quotients:

a. $\dfrac{(5+h)^3-125}{h}$

b. $\dfrac{(3+h)^4-81}{h}$

c. $\dfrac{\frac{1}{1+h}-1}{h}$

d. $\dfrac{3(1+h)^2-3}{h}$

e. $\dfrac{\frac{3}{4+h}-\frac{3}{4}}{h}$

f. $\dfrac{\frac{-1}{2+h}+\frac{1}{2}}{h}$

5. Rationalize the numerator of each expression to obtain an equivalent expression.

a. $\dfrac{\sqrt{16+h}-4}{h}$

b. $\dfrac{\sqrt{h^2+5h+4}-2}{h}$

c. $\dfrac{\sqrt{5+h}-\sqrt{5}}{h}$

PART B

6. Determine an expression, in simplified form, for the slope of the secant PQ.

a. $P(1, 3)$, $Q(1+h, f(1+h))$, where $f(x) = 3x^2$

b. $P(1, 3)$, $Q(1+h, (1+h)^3+2)$

c. $P(9, 3)$, $Q(9+h, \sqrt{9+h})$

K 7. Consider the function $f(x) = x^3$.

a. Copy and complete the following table of values. P and Q are points on the graph of $f(x)$.

P	***Q***	**Slope of Line *PQ***
(2,)	(3,)	
(2,)	(2.5,)	
(2,)	(2.1,)	
(2,)	(2.01,)	
(2,)	(1,)	
(2,)	(1.5,)	
(2,)	(1.9,)	
(2,)	(1.99,)	

b. Use your results for part a to approximate the slope of the tangent to the graph of $f(x)$ at point P.

c. Calculate the slope of the secant PQ, where the x-coordinate of Q is $2+h$.

d. Use your result for part c to calculate the slope of the tangent to the graph of $f(x)$ at point P.

e. Compare your answers for parts b and d.

f. Sketch the graph of $f(x)$ and the tangent to the graph at point P.

8. Determine the slope of the tangent to each curve at the given value of x.

a. $y = 3x^2, x = -2$ b. $y = x^2 - x, x = 3$ c. $y = x^3, x = -2$

9. Determine the slope of the tangent to each curve at the given value of x.

a. $y = \sqrt{x - 2}, x = 3$

b. $y = \sqrt{x - 5}, x = 9$

c. $y = \sqrt{5x - 1}, x = 2$

10. Determine the slope of the tangent to each curve at the given value of x.

a. $y = \frac{8}{x}, x = 2$ b. $y = \frac{8}{3 + x}, x = 1$ c. $y = \frac{1}{x + 2}, x = 3$

11. Determine the slope of the tangent to each curve at the given point.

a. $y = x^2 - 3x, (2, -2)$

b. $f(x) = \frac{4}{x}, (-2, -2)$

c. $y = 3x^3, (1, 3)$

d. $y = \sqrt{x - 7}, (16, 3)$

e. $y = \sqrt{25 - x^2}, (3, 4)$

f. $y = \frac{4 + x}{x - 2}, (8, 2)$

12. Sketch the graph of the function in question 11, part e. Show that the slope of the tangent can be found using the properties of circles.

C 13. Explain how you would approximate the slope of the tangent at a point without using the definition of the slope of the tangent.

14. Using technology, sketch the graph of $y = \frac{3}{4}\sqrt{16 - x^2}$. Explain how the slope of the tangent at $P(0, 3)$ can be found without using the difference quotient.

15. Determine the equation of the tangent to $y = x^2 - 3x + 1$ at (3, 1).

16. Determine the equation of the tangent to $y = x^2 - 7x + 12$ where $x = 2$.

17. For $f(x) = x^2 - 4x + 1$, find

a. the coordinates of point A, where $x = 3$,

b. the coordinates of point B, where $x = 5$

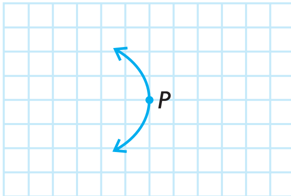
c. the equation of the secant AB

d. the equation of the tangent at A

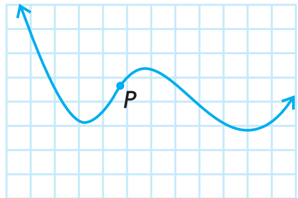
e. the equation of the tangent at B

18. Copy the following figures. Draw an approximate tangent for each curve at point P and estimate its slope.

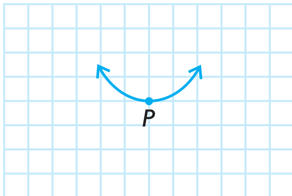
a.



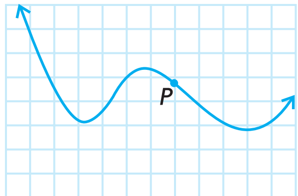
d.



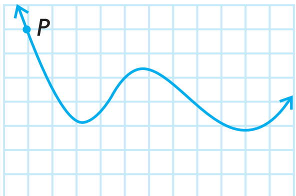
b.



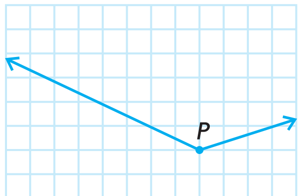
e.



c.



f.



19. Find the slope of the demand curve $D(p) = \frac{20}{\sqrt{p-1}}$, $p > 1$, at point $(5, 10)$.

A 20. It is projected that, t years from now, the circulation of a local newspaper will be $C(t) = 100t^2 + 400t + 5000$. Find how fast the circulation is increasing after 6 months. *Hint*: Find the slope of the tangent when $t = 0.5$.

T 21. Find the coordinates of the point on the curve $f(x) = 3x^2 - 4x$ where the tangent is parallel to the line $y = 8x$.

22. Find the points on the graph of $y = \frac{1}{3}x^3 - 5x - \frac{4}{x}$ at which the tangent is horizontal.

PART C

23. Show that, at the points of intersection of the quadratic functions $y = x^2$ and $y = \frac{1}{2} - x^2$, the tangents to the functions are perpendicular.

24. Determine the equation of the line that passes through $(2, 2)$ and is parallel to the line tangent to $y = -3x^3 - 2x$ at $(-1, 5)$.

25. a. Determine the slope of the tangent to the parabola $y = 4x^2 + 5x - 2$ at the point whose x-coordinate is a.

    b. At what point on the parabola is the tangent line parallel to the line $10x - 2y - 18 = 0$?

    c. At what point on the parabola is the tangent line perpendicular to the line $x - 35y + 7 = 0$?

Section 1.3—Rates of Change

Many practical relationships involve interdependent quantities. For example, the volume of a balloon varies with its height above the ground, air temperature varies with elevation, and the surface area of a sphere varies with the length of the radius.

These and other relationships can be described by means of a function, often of the form $y = f(x)$. The **dependent variable**, y, can represent quantities such as volume, air temperature, and area. The **independent variable**, x, can represent quantities such as height, elevation, and length.

We are often interested in how rapidly the dependent variable changes when there is a change in the independent variable. Recall that this concept is called **rate of change**. In this section, we show that an instantaneous rate of change can be calculated by finding the limit of a difference quotient in the same way that we find the slope of a tangent.

Velocity as a Rate of Change

An object moving in a straight line is an example of a rate-of-change model. It is customary to use either a horizontal or vertical line with a specified origin to represent the line of motion. On such a line, movement to the right or upward is considered to be in the positive direction, and movement to the left (or down) is considered to be in the negative direction. An example of an object moving along a line would be a vehicle entering a highway and travelling north 340 km in 4 h.

The average velocity would be $\frac{340}{4} = 85$ km/h, since

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}}$$

If $s(t)$ gives the position of the vehicle on a straight section of the highway at time t, then the average rate of change in the position of the vehicle over a time interval is average velocity $= \dfrac{\Delta s}{\Delta t}$.

INVESTIGATION

You are driving with a broken speedometer on a highway. At any instant, you do not know how fast the car is going. Your odometer readings are given

t **(h)**	0	1	2	2.5	3
***s*(*t*) (km)**	62	133	210	250	293

A. Determine the average velocity of the car over each interval.
B. The speed limit is 80 km/h. Do any of your results in part A suggest that you were speeding at any time? If so, when?
C. Explain why there may be other times when you were travelling above the posted speed limit.
D. Compute your average velocity over the interval $4 \leq t \leq 7$, if $s(4) = 375$ km and $s(7) = 609$ km.
E. After 3 h of driving, you decide to continue driving from Goderich to Huntsville, a distance of 345 km. Using the average velocity from part D, how long would it take you to make this trip?

EXAMPLE 1 **Reasoning about average velocity**

A pebble is dropped from a cliff, 80 m high. After t seconds, the pebble is s metres above the ground, where $s(t) = 80 - 5t^2$, $0 \leq t \leq 4$.

a. Calculate the average velocity of the pebble between the times $t = 1$ s and $t = 3$ s.
b. Calculate the average velocity of the pebble between the times $t = 1$ s and $t = 1.5$ s.
c. Explain why your answers for parts a and b are different.

Solution

a. $$\text{average velocity} = \frac{\Delta s}{\Delta t}$$

$$s(1) = 75$$
$$s(3) = 35$$

$$\begin{aligned}\text{average velocity} &= \frac{s(3) - s(1)}{3 - 1}\\ &= \frac{35 - 75}{2}\\ &= \frac{-40}{2}\\ &= -20 \text{ m/s}\end{aligned}$$

The average velocity in this 2 s interval is -20 m/s.

b. $$\begin{aligned}s(1.5) &= 80 - 5(1.5)^2\\ &= 68.75\end{aligned}$$

$$\begin{aligned}\text{average velocity} &= \frac{s(1.5) - s(1)}{1.5 - 1}\\ &= \frac{68.75 - 75}{0.5}\\ &= -12.5 \text{ m/s}\end{aligned}$$

The average velocity in this 0.5 s interval is -12.5 m/s.

c. Since gravity causes the velocity to increase with time, the smaller interval of 0.5 s gives a lower average velocity, as well as giving a value closer to the actual velocity at time $t = 1$.

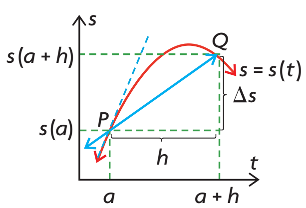
The following table shows the results of similar calculations of the average velocity over successively smaller time intervals:

Time Interval	Average Velocity (m/s)
$1 \leq t \leq 1.1$	-10.5
$1 \leq t \leq 1.01$	-10.05
$1 \leq t \leq 1.001$	-10.005

It appears that, as we shorten the time interval, the average velocity is approaching the value -10 m/s. The average velocity over the time interval $1 \leq t \leq 1 + h$ is

$$\begin{aligned}
\text{average velocity} &= \frac{s(1+h) - s(1)}{h} \\
&= \frac{[80 - 5(1+h)^2] - [80 - 5(1)^2]}{h} \\
&= \frac{75 - 10h - 5h^2 - 75}{h} \\
&= \frac{-10h - 5h^2}{h} \\
&= -10 - 5h, h \neq 0
\end{aligned}$$

If the time interval is very short, then h is small, so $5h$ is close to 0 and the average velocity is close to -10 m/s. The instantaneous velocity when $t = 1$ is defined to be the limiting value of these average values as h approaches 0. Therefore, the velocity (the word "instantaneous" is usually omitted) at time $t = 1$ s is $v = \lim_{h \to 0}(-10 - 5h) = -10$ m/s.



In general, suppose that the position of an object at time t is given by the function $s(t)$. In the time interval from $t = a$ to $t = a + h$, the change in position is $\Delta s = s(a + h) - s(a)$.

The average velocity over this time interval is $\frac{\Delta s}{\Delta t} = \frac{s(a+h) - s(a)}{h}$, which is the same as the slope of the secant PQ where $P(a, s(a))$ and $Q(a + h, s(a + h))$. The **velocity** at a particular time $t = a$ is calculated by finding the limiting value of the average velocity as $h \to 0$.

Instantaneous Velocity

The velocity of an object with position function $s(t)$, at time $t = a$, is

$$v(a) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$

Note that the velocity $v(a)$ is the slope of the tangent to the graph of $s(t)$ at $P(a, s(a))$. The speed of an object is the absolute value of its velocity. It indicates how fast an object is moving, whereas velocity indicates both speed and direction (relative to a given coordinate system).

EXAMPLE 2 Selecting a strategy to calculate velocity

A toy rocket is launched straight up so that its height s, in metres, at time t, in seconds, is given by $s(t) = -5t^2 + 30t + 2$. What is the velocity of the rocket at $t = 4$?

Solution

Since $s(t) = -5t^2 + 30t + 2$,

$$\begin{aligned} s(4+h) &= -5(4+h)^2 + 30(4+h) + 2 \\ &= -80 - 40h - 5h^2 + 120 + 30h + 2 \\ &= -5h^2 - 10h + 42 \\ s(4) &= -5(4)^2 + 30(4) + 2 \\ &= 42 \end{aligned}$$

The velocity at $t = 4$ is

$$v(4) = \lim_{h \to 0} \frac{s(4+h) - s(4)}{h}$$ (Substitute)

$$= \lim_{h \to 0} \frac{[-10h - 5h^2]}{h}$$ (Factor)

$$= \lim_{h \to 0} \frac{h(-10 - 5h)}{h}$$ (Simplify)

$$= \lim_{h \to 0} (-10 - 5h)$$ (Evaluate)

$$= -10$$

Therefore, the velocity of the rocket is 10 m/s downward at $t = 4$ s.

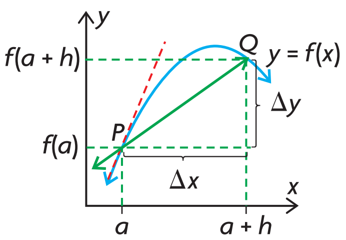
Comparing Average and Instantaneous Rates of Change

Velocity is only one example of the concept of rate of change. In general, suppose that a quantity y depends on x according to the equation $y = f(x)$. As the independent variable changes from a to $a + h$ ($\Delta x = a + h - a = h$), the corresponding change in the dependent variable y is $\Delta y = f(a + h) - f(a)$.

Average Rate of Change

The difference quotient $\frac{\Delta y}{\Delta x} = \frac{f(a + h) - f(a)}{h}$ is called the average rate of change in y with respect to x over the interval from $x = a$ to $x = a + h$.

From the diagram, it follows that the average rate of change equals the slope of the secant PQ of the graph of $f(x)$ where $P(a, f(a))$ and $Q(a + h, f(a + h))$. The instantaneous rate of change in y with respect to x when $x = a$ is defined to be the limiting value of the average rate of change as $h \to 0$.



Instantaneous Rates of Change

Therefore, we conclude that the instantaneous rate of change in $y = f(x)$ with respect to x when $x = a$ is $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim\limits_{h \to 0} \frac{f(a + h) - f(a)}{h}$, provided that the limit exists.

It should be noted that, as with velocity, the instantaneous rate of change in y with respect to x at $x = a$ equals the slope of the tangent to the graph of $y = f(x)$ at $x = a$.

EXAMPLE 3 **Selecting a strategy to calculate instantaneous rate of change**

The total cost, in dollars, of manufacturing x calculators is given by $C(x) = 10\sqrt{x} + 1000$.

a. What is the total cost of manufacturing 100 calculators?
b. What is the rate of change in the total cost with respect to the number of calculators, x, being produced when $x = 100$?

Solution

a. $C(100) = 10\sqrt{100} + 1000 = 1100$

Therefore, the total cost of manufacturing 100 calculators is \$1100.

b. The rate of change in the cost at $x = 100$ is given by

$$\lim_{h \to 0} \frac{C(100 + h) - C(100)}{h} \quad \text{(Substitute)}$$

$$= \lim_{h \to 0} \frac{10\sqrt{100 + h} + 1000 - 1100}{h}$$

$$= \lim_{h \to 0} \frac{10\sqrt{100 + h} - 100}{h} \times \frac{10\sqrt{100 + h} + 100}{10\sqrt{100 + h} + 100} \quad \text{(Rationalize the numerator)}$$

$$= \lim_{h \to 0} \frac{100(100 + h) - 10\,000}{h(10\sqrt{100 + h} + 100)} \quad \text{(Expand)}$$

$$= \lim_{h \to 0} \frac{100h}{h(10\sqrt{100 + h} + 100)} \quad \text{(Simplify)}$$

$$= \lim_{h \to 0} \frac{100}{(10\sqrt{100 + h} + 100)} \quad \text{(Evaluate)}$$

$$= \frac{100}{(10\sqrt{100 + 0} + 100)}$$

$$= 0.5$$

Therefore, the rate of change in the total cost with respect to the number of calculators being produced, when 100 calculators are being produced, is \$0.50 per calculator.

An Alternative Form for Finding Rates of Change

In Example 1, we determined the velocity of the pebble at $t = 1$ by taking the limit of the average velocity over the interval $1 \le t \le 1 + h$ as h approaches 0.

We can also determine the velocity at $t = 1$ by considering the average velocity over the interval from 1 to a general time t and letting t approach the value 1.

Then, $s(t) = 80 - 5t^2$

$s(1) = 75$

$$\begin{aligned} v(1) &= \lim_{t \to 1} \frac{s(t) - s(1)}{t - 1} \\ &= \lim_{t \to 1} \frac{5 - 5t^2}{t - 1} \\ &= \lim_{t \to 1} \frac{5(1 - t)(1 + t)}{t - 1} \\ &= \lim_{t \to 1} -5(1 + t) \\ &= -10 \end{aligned}$$

In general, the velocity of an object at time $t = a$ is $v(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}$. Similarly, the instantaneous rate of change in $y = f(x)$ with respect to x when $x = a$ is $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

IN SUMMARY

Key Ideas

- The average velocity can be found in the same way that we found the slope of the secant.

  $$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}}$$

- The instantaneous velocity is the slope of the tangent to the graph of the position function and is found in the same way that we found the slope of the tangent.

Need to Know

- To find the average velocity (average rate of change) from $t = a$ to $t = a + h$, we can use the difference quotient and the position function $s(t)$

  $$\frac{\Delta s}{\Delta t} = \frac{s(a + h) - s(a)}{h}$$

- The rate of change in the position function, $s(t)$, is the velocity at $t = a$, and we can find it by computing the limiting value of the average velocity as $h \to 0$:

  $$v(a) = \lim_{h \to 0} \frac{s(a + h) - s(a)}{h}$$

Exercise 1.3

PART A

1. The velocity of an object is given by $v(t) = t(t - 4)^2$. At what times, in seconds, is the object at rest?

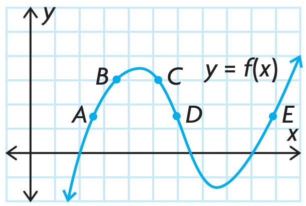
C 2. Give a geometrical interpretation of the following expressions, if s is a position function:

   a. $\dfrac{s(9) - s(2)$}{7}$

   b. $\lim\limits_{h \to 0} \dfrac{s(6 + h) - s(6)}{h}$

3. Give a geometrical interpretation of $\lim\limits_{h \to 0} \dfrac{\sqrt{4 + h} - 2}{h}$.

4. Use the graph to answer each question.



   a. Between which two consecutive points is the average rate of change in the function the greatest?

   b. Is the average rate of change in the function between A and B greater than or less than the instantaneous rate of change at B?

   c. Sketch a tangent to the graph somewhere between points D and E such that the slope of the tangent is the same as the average rate of change in the function between B and C.

5. What is wrong with the statement "The speed of the cheetah was 65 km/h north"?

6. Is there anything wrong with the statement "A school bus had a velocity of 60 km/h for the morning run, which is why it was late arriving"?

PART B

7. A construction worker drops a bolt while working on a high-rise building, 320 m above the ground. After t seconds, the bolt has fallen a distance of s metres, where $s(t) = 320 - 5t^2$, $0 \leq t \leq 8$.

   a. Calculate the average velocity during the first, third, and eighth seconds.

   b. Calculate the average velocity for the interval $3 \leq t \leq 8$.

   c. Calculate the velocity at $t = 2$.

K 8. The function $s(t) = 8t(t + 2)$ describes the distance s, in kilometres, that a car has travelled after a time t, in hours, for $0 \leq t \leq 5$.

a. Calculate the average velocity of the car during the following intervals:

i. from $t = 3$ to $t = 4$

ii. from $t = 3$ to $t = 3.1$

iii. from $t = 3$ to $t = 3.01$

b. Use your results for part a to approximate the instantaneous velocity of the car at $t = 3$.

c. Calculate the velocity at $t = 3$.

9. Suppose that a foreign-language student has learned $N(t) = 20t - t^2$ vocabulary terms after t hours of uninterrupted study, where $0 \leq t \leq 10$.

a. How many terms are learned between time $t = 2$ h and $t = 3$ h?

b. What is the rate, in terms per hour, at which the student is learning at time $t = 2$ h?

A 10. A medicine is administered to a patient. The amount of medicine M, in milligrams, in 1 mL of the patient's blood, t hours after the injection, is $M(t) = -\frac{1}{3}t^2 + t$, where $0 \leq t \leq 3$.

a. Find the rate of change in the amount M, 2 h after the injection.

b. What is the significance of the fact that your answer is negative?

11. The time t, in seconds, taken by an object dropped from a height of s metres to reach the ground is given by the formula $t = \sqrt{\frac{s}{5}}$. Determine the rate of change in time with respect to height when the object is 125 m above the ground.

12. Suppose that the temperature T, in degrees Celsius, varies with the height h, in kilometres, above Earth's surface according to the equation $T(h) = \frac{60}{h + 2}$. Find the rate of change in temperature with respect to height at a height of 3 km.

13. A spaceship approaching touchdown on a distant planet has height h, in metres, at time t, in seconds, given by $h = 25t^2 - 100t + 100$. When does the spaceship land on the surface? With what speed does it land (assuming it descends vertically)?

14. A manufacturer of soccer balls finds that the profit from the sale of x balls per week is given by $P(x) = 160x - x^2$ dollars.

a. Find the profit on the sale of 40 soccer balls per week.

b. Find the rate of change in profit at the production level of 40 balls per week.

c. Using a graphing calculator, graph the profit function and, from the graph, determine for what sales levels of x the rate of change in profit is positive.

15. Use the alternate definition $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ to calculate the instantaneous rate of change of $f(x)$ at the given point or value of x.

a. $f(x) = -x^2 + 2x + 3, (-2, -5)$

b. $f(x) = \frac{x}{x - 1}, x = 2$

c. $f(x) = \sqrt{x + 1}, x = 24$

16. The average annual salary of a professional baseball player can be modelled by the function $S(x) = 246 + 64x - 8.9x^2 + 0.95x^3$, where S represents the average annual salary, in thousands of dollars, and x is the number of years since 1982. Determine the rate at which the average salary was changing in 2005.

17. The motion of an avalanche is described by $s(t) = 3t^2$, where s is the distance, in metres, travelled by the leading edge of the snow at t seconds.

a. Find the distance travelled from 0 s to 5 s.

b. Find the rate at which the avalanche is moving from 0 s to 10 s.

c. Find the rate at which the avalanche is moving at 10 s.

d. How long, to the nearest second, does the leading edge of the snow take to move 600 m?

PART C

T

18. Let (a, b) be any point on the graph of $y = \frac{1}{x}, x \neq 0$. Prove that the area of the triangle formed by the tangent through (a, b) and the coordinate axes is 2.

19. MegaCorp's total weekly cost to produce x pencils can be written as $C(x) = F + V(x)$, where F, a constant, represents fixed costs such as rent and utilities and $V(x)$ represents variable costs, which depend on the production level x. Show that the rate of change in the weekly cost is independent of fixed costs.

20. A circular oil spill on the surface of the ocean spreads outward. Find the approximate rate of change in the area of the oil slick with respect to its radius when the radius is 100 m.

21. Show that the rate of change in the volume of a cube with respect to its edge length is equal to half the surface area of the cube.

22. Determine the instantaneous rate of change in

a. the surface area of a spherical balloon (as it is inflated) at the point in time when the radius reaches 10 cm

b. the volume of a spherical balloon (as it is deflated) at the point in time when the radius reaches 5 cm

Mid-Chapter Review

1. Calculate the product of each radical expression and its corresponding conjugate.
   a. $\sqrt{5} - \sqrt{2}$ b. $3\sqrt{5} + 2\sqrt{2}$ c. $9 + 2\sqrt{5}$ d. $3\sqrt{5} - 2\sqrt{10}$

2. Rationalize each denominator.
   a. $\dfrac{6 + \sqrt{2}}{\sqrt{3}}$
   b. $\dfrac{2\sqrt{3} + 4}{\sqrt{3}}$
   c. $\dfrac{5}{\sqrt{7} - 4}$
   d. $\dfrac{2\sqrt{3}}{\sqrt{3} - 2}$
   e. $\dfrac{5\sqrt{3}}{2\sqrt{3} + 4}$
   f. $\dfrac{3\sqrt{2}}{2\sqrt{3} - 5}$

3. Rationalize each numerator.
   a. $\dfrac{\sqrt{2}}{5}$
   b. $\dfrac{\sqrt{3}}{6 + \sqrt{2}}$
   c. $\dfrac{\sqrt{7} - 4}{5}$
   d. $\dfrac{2\sqrt{3} - 5}{3\sqrt{2}}$
   e. $\dfrac{\sqrt{3} - \sqrt{7}}{4}$
   f. $\dfrac{2\sqrt{3} + \sqrt{7}}{5}$

4. Determine the equation of the line described by the given information.
   a. slope $-\frac{2}{3}$, passing through point $(0, 6)$
   b. passing through points $(2, 7)$ and $(6, 11)$
   c. parallel to $y = 4x - 6$, passing through point $(2, 6)$
   d. perpendicular to $y = -5x + 3$, passing through point $(-1, -2)$

5. Find the slope of PQ, in simplified form, given $P(1, -1)$ and $Q(1 + h, f(1 + h))$, where $f(x) = -x^2$.

6. Consider the function $y = x^2 - 2x - 2$.
   a. Copy and complete the following tables of values. P and Q are points on the graph of $f(x)$.

P	***Q***	**Slope of Line *PQ***
$(-1, 1)$	$(-2, 6)$	$\frac{-5}{1} = -5$
$(-1, 1)$	$(-1.5, 3.25)$	
$(-1, 1)$	$(-1.1, \quad)$	
$(-1, 1)$	$(-1.01, \quad)$	
$(-1, 1)$	$(-1.001, \quad)$	

P	***Q***	**Slope of Line *PQ***
$(-1, 1)$	$(0, \quad)$	
$(-1, 1)$	$(-0.5, \quad)$	
$(-1, 1)$	$(-0.9, \quad)$	
$(-1, 1)$	$(-0.99, \quad)$	
$(-1, 1)$	$(-0.999, \quad)$	

b. Use your results for part a to approximate the slope of the tangent to the graph of $f(x)$ at point P.

c. Calculate the slope of the secant where the x-coordinate of Q is $-1 + h$.

d. Use your results for part c to calculate the slope of the tangent to the graph of $f(x)$ at point P.

e. Compare your answers for parts b and d.

7. Calculate the slope of the tangent to each curve at the given point or value of x.

a. $f(x) = x^2 + 3x - 5, (-3, -5)$

b. $y = \frac{1}{x}, x = \frac{1}{3}$

c. $y = \frac{4}{x - 2}, (6, 1)$

d. $f(x) = \sqrt{x + 4}, x = 5$

8. The function $s(t) = 6t(t + 1)$ describes the distance (in kilometres) that a car has travelled after a time t (in hours), for $0 \leq t \leq 6$.

a. Calculate the average velocity of the car during the following intervals.

i. from $t = 2$ to $t = 3$
ii. from $t = 2$ to $t = 2.1$
iii. from $t = 2$ to $t = 2.01$

b. Use your results for part a to approximate the instantaneous velocity of the car when $t = 2$.

c. Find the average velocity of the car from $t = 2$ to $t = 2 + h$.

d. Use your results for part c to find the velocity when $t = 2$.

9. Calculate the instantaneous rate of change of $f(x)$ with respect to x at the given value of x.

a. $f(x) = 5 - x^2, x = 2$

b. $f(x) = \frac{3}{x}, x = \frac{1}{2}$

10. An oil tank is being drained for cleaning. After t minutes, there are V litres of oil left in the tank, where $V(t) = 50(30 - t)^2, 0 \leq t \leq 30$.

a. Calculate the average rate of change in volume during the first 20 min.

b. Calculate the rate of change in volume at time $t = 20$.

11. Find the equation of the tangent at the given value of x.

a. $y = x^2 + x - 3, x = 4$

b. $y = 2x^2 - 7, x = -2$

c. $f(x) = 3x^2 + 2x - 5, x = -1$

d. $f(x) = 5x^2 - 8x + 3, x = 1$

12. Find the equation of the tangent to the graph of the function at the given value of x.

a. $f(x) = \frac{x}{x + 3}, x = -5$

b. $f(x) = \frac{2x + 5}{5x - 1}, x = -1$

Section 1.4—The Limit of a Function

The notation $\lim_{x \to a} f(x) = L$ is read "the limit of $f(x)$ as x approaches a equals L" and means that the value of $f(x)$ can be made arbitrarily close to L by choosing x sufficiently close to a (but not equal to a). But $\lim_{x \to a} f(x)$ exists if and only if the limiting value from the left equals the limiting value from the right. We shall use this definition to evaluate some limits.

Note: This is an intuitive explanation of the limit of a function. A more precise definition using inequalities is important for advanced work but is not necessary for our purposes.

INVESTIGATION 1 Determine the limit of $y = x^2 - 1$, as x approaches 2.

A. Copy and complete the table of values.

x	1	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5	3
$y = x^2 - 1$											

B. As x approaches 2 from the left, starting at $x = 1$, what is the approximate value of y?

C. As x approaches 2 from the right, starting at $x = 3$, what is the approximate value of y?

D. Graph $y = x^2 - 1$ using graphing software or graph paper.

E. Using arrows, illustrate that, as we choose a value of x that is closer and closer to $x = 2$, the value of y gets closer and closer to a value of 3.

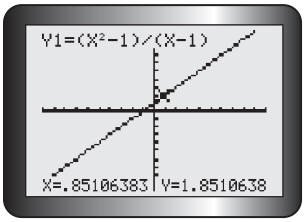
F. Explain why the limit of $y = x^2 - 1$ exists as x approaches 2, and give its approximate value.

EXAMPLE 1 Determine $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$ by graphing.

Solution

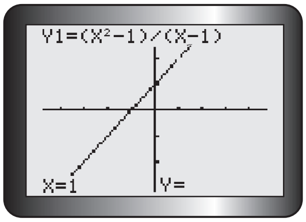
On a graphing calculator, display the graph of $f(x) = \frac{x^2 - 1}{x - 1}, x \neq 1$.

The graph shown on your calculator is a line $(f(x) = x + 1)$, whereas it should be a line with point (1, 2) deleted $(f(x) = x + 1, x \neq 1)$. The WINDOW used is $X_{min} = -10$, $X_{max} = 10$, $X_{scl} = 1$, and similarly for Y. Use the TRACE function to find $X = 0.85106383$, $Y = 1.8510638$ and $X = 1.0638298$, $Y = 2.0638298$.



Click ZOOM; select 4:ZDecimal, ENTER. Now, the graph of $f(x) = \frac{x^2 - 1}{x - 1}$ is displayed as a straight line with point (1, 2) deleted. The WINDOW has new values, too.

Use the TRACE function to find $X = 0.9$, $Y = 1.9$; $X = 1$, Y has no value given; and $X = 1.1$, $Y = 2.1$.



We can estimate $\lim_{x \to 1} f(x)$. As x approaches 1 from the left, written as "$x \to 1^-$", we observe that $f(x)$ approaches the value 2 from below. As x approaches 1 from the right, written as $x \to 1^+$, $f(x)$ approaches the value 2 from above.

We say that the limit at $x = 1$ exists only if the value approached from the left is the same as the value approached from the right. From this investigation, we conclude that $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$.

EXAMPLE 2 Selecting a table of values strategy to evaluate a limit

Determine $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$ by using a table.

Solution

We select sequences of numbers for $x \to 1^-$ and $x \to 1^+$.

x approaches 1 from the left $\rightarrow$						$\leftarrow$ x approaches 1 from the right				
x	0	0.5	0.9	0.99	0.999	1	1.001	1.01	1.1	1.5	2
$\frac{x^2 - 1}{x - 1}$	1	1.5	1.9	1.99	1.999	undefined	2.001	2.01	2.1	2.5	3
$f(x) = \frac{x^2 - 1}{x - 1}$ approaches 2 from below $\rightarrow$						$\leftarrow$ $f(x) = \frac{x^2 - 1}{x - 1}$ approaches 2 from above					

This pattern of numbers suggests that $\lim_{x\to 1} \frac{x^2 - 1}{x - 1} = 2$, as we found when graphing in Example 1.

EXAMPLE 3 Selecting a graphing strategy to evaluate a limit

Tech Support
For help graphing piecewise functions on a graphing calculator, see Technology Appendix p. 607.

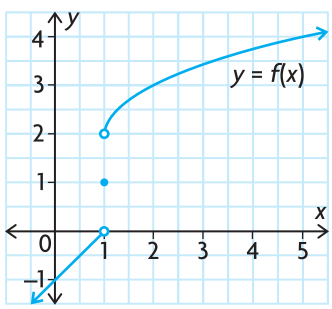
Sketch the graph of the piecewise function:

$$f(x) = \begin{cases} x - 1, \text{ if } x < 1 \\ 1, \text{ if } x = 1 \\ 2 + \sqrt{x - 1}, \text{ if } x > 1 \end{cases}$$

Determine $\lim_{x\to 1} f(x)$.

Solution

The graph of the function f consists of the line $y = x - 1$ for $x < 1$, the point $(1, 1)$ and the square root function $y = 2 + \sqrt{x - 1}$ for $x > 1$. From the graph of $f(x)$, observe that the limit of $f(x)$ as $x \to 1$ depends on whether $x < 1$ or $x > 1$. As $x \to 1^-$, $f(x)$ approaches the value of 0 from below. We write this as $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x - 1) = 0$.



Similarly, as $x \to 1^+$, $f(x)$ approaches the value 2 from above. We write this as $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \left(2 + \sqrt{x - 1}\right) = 2$. (This is the same when $x = 1$ is substituted into the expression $2 + \sqrt{x - 1}$.) These two limits are referred to as one-sided limits because, in each case, only values of x on one side of $x = 1$ are considered. How-ever, the one-sided limits are unequal—$\lim\limits_{x \to 1^-} f(x) = 0 \neq 2 = \lim\limits_{x \to 1^+} f(x)$—or more briefly, $\lim\limits_{x \to 1^-} f(x) \neq \lim\limits_{x \to 1^+} f(x)$. This implies that $f(x)$ does not approach a single value as $x \to 1$. We say "the limit of $f(x)$ as $x \to 1$ does not exist" and write "$\lim\limits_{x \to 1} f(x)$ does not exist." This may be surprising, since the function $f(x)$ was defined at $x = 1$—that is, $f(1) = 1$. We can now summarize the ideas introduced in these examples.

Limits and Their Existence

We say that the number L is the limit of a function $y = f(x)$ as x approaches the value a, written as $\lim\limits_{x \to a} f(x) = L$, if $\lim\limits_{x \to a^-} f(x) = L = \lim\limits_{x \to a^+} f(x)$. Otherwise, $\lim\limits_{x \to a} f(x)$ does not exist.

IN SUMMARY

Key Idea

- The limit of a function $y = f(x)$ at $x = a$ is written as $\lim\limits_{x \to a} f(x) = L$, which means that $f(x)$ approaches the value L as x approaches the value a from both the left and right side.

Need to Know

- $\lim\limits_{x \to a} f(x)$ may exist even if $f(a)$ is not defined.
- $\lim\limits_{x \to a} f(x)$ can be equal to $f(a)$. In this case, the graph of $f(x)$ passes through the point $(a, f(a))$.
- If $\lim\limits_{x \to a^-} f(x) = L$ and $\lim\limits_{x \to a^+} f(x) = L$, then L is the limit of $f(x)$ as x approaches a, that is $\lim\limits_{x \to a} f(x) = L$.

Exercise 1.4

PART A

1. What do you think is the appropriate limit of each sequence?
   a. 0.7, 0.72, 0.727, 0.7272, . . .
   b. 3, 3.1, 3.14, 3.141, 3.1415, 3.141 59, 3.141 592, . . .

C 2. Explain a process for finding a limit.

3. Write a concise description of the meaning of the following:
   a. a right-sided limit
   b. a left-sided limit
   c. a (two-sided) limit

4. Calculate each limit.

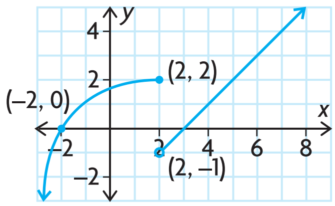
a. $\lim\limits_{x \to -5} x$
c. $\lim\limits_{x \to 10} x^2$
e. $\lim\limits_{x \to 1} 4$
b. $\lim\limits_{x \to 3} (x + 7)$
d. $\lim\limits_{x \to -2} (4 - 3x^2)$
f. $\lim\limits_{x \to 3} 2^x$

5. Determine $\lim\limits_{x \to 4} f(x)$, where $f(x) = \begin{cases} 1, \text{ if } x \neq 4 \\ -1, \text{ if } x = 4 \end{cases}$.

PART B

6. For the function $f(x)$ in the graph below, determine the following:

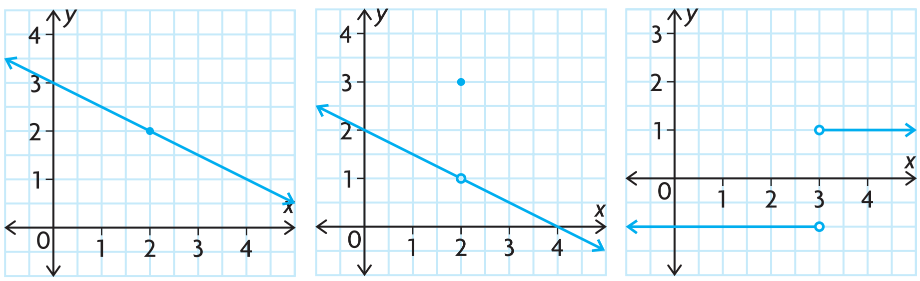
a. $\lim\limits_{x \to -2^+} f(x)$
b. $\lim\limits_{x \to 2^-} f(x)$
c. $\lim\limits_{x \to 2^+} f(x)$
d. $f(2)$



K

7. Use the graph to find the limit, if it exists.

a. $\lim\limits_{x \to 2} f(x)$
b. $\lim\limits_{x \to 2} f(x)$
c. $\lim\limits_{x \to 3} f(x)$



8. Evaluate each limit.

a. $\lim\limits_{x \to -1} (9 - x^2)$
b. $\lim\limits_{x \to 0} \sqrt{\dfrac{x + 20}{2x + 5}}$
c. $\lim\limits_{x \to 5} \sqrt{x - 1}$

9. Find $\lim\limits_{x \to 2} (x^2 + 1)$, and illustrate your result with a graph indicating the limiting value.

10. Evaluate each limit. If the limit does not exist, explain why.

a. $\lim\limits_{x \to 0^+} x^4$
c. $\lim\limits_{x \to 3^-} (x^2 - 4)$
e. $\lim\limits_{x \to 3^+} \dfrac{1}{x + 2}$
b. $\lim\limits_{x \to 2^-} (x^2 - 4)$
d. $\lim\limits_{x \to 1^+} \dfrac{1}{x - 3}$
f. $\lim\limits_{x \to 3} \dfrac{1}{x - 3}$

11. For each function, sketch the graph of the function. Determine the indicated limit if it exists.

a. $f(x) = \begin{cases} x + 2, \text{ if } x < -1 \\ -x + 2, \text{ if } x \geq -1 \end{cases}; \lim_{x \to -1} f(x)$

b. $f(x) = \begin{cases} -x + 4, \text{ if } x \leq 2 \\ -2x + 6, \text{ if } x > 2 \end{cases}; \lim_{x \to 2} f(x)$

c. $f(x) = \begin{cases} 4x, \text{ if } x \geq \frac{1}{2} \\ \frac{1}{x}, \text{ if } x < \frac{1}{2} \end{cases}; \lim_{x \to \frac{1}{2}} f(x)$

d. $f(x) = \begin{cases} 1, \text{ if } x < -0.5 \\ x^2 - 0.25, \text{ if } x \geq -0.5 \end{cases}; \lim_{x \to -0.5} f(x)$

A 12. Sketch the graph of any function that satisfies the given conditions.

a. $f(1) = 1, \lim_{x \to 1^+} f(x) = 3, \lim_{x \to 1^-} f(x) = 2$

b. $f(2) = 1, \lim_{x \to 2} f(x) = 0$

c. $f(x) = 1$, if $x < 1$ and $\lim_{x \to 1^+} f(x) = 2$

d. $f(3) = 0, \lim_{x \to 3^+} f(x) = 0$

13. Let $f(x) = mx + b$, where m and b are constants. If $\lim_{x \to 1} f(x) = -2$ and $\lim_{x \to -1} f(x) = 4$, find m and b.

PART C

T 14. Determine the real values of a, b, and c for the quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$, that satisfy the conditions $f(0) = 0$, $\lim_{x \to 1} f(x) = 5$, and $\lim_{x \to -2} f(x) = 8$.

15. The fish population, in thousands, in a lake at time t, in years, is modelled by the following function:

$$p(t) = \begin{cases} 3 + \frac{1}{12}t^2, \text{ if } 0 \leq t \leq 6 \\ 2 + \frac{1}{18}t^2, \text{ if } 6 < t \leq 12 \end{cases}$$

This function describes a sudden change in the population at time $t = 6$, due to a chemical spill.

a. Sketch the graph of $p(t)$.

b. Evaluate $\lim_{t \to 6^-} p(t)$ and $\lim_{t \to 6^+} p(t)$.

c. Determine how many fish were killed by the spill.

d. At what time did the population recover to the level before the spill?

Section 1.5—Properties of Limits

The statement $\lim_{x \to a} f(x) = L$ says that the values of $f(x)$ become closer and closer to the number L as x gets closer and closer to the number a (from either side of a), such that $x \neq a$. This means that when finding the limit of $f(x)$ as x approaches a, there is no need to consider $x = a$. In fact, $f(a)$ need not even be defined. The only thing that matters is the behaviour of $f(x)$ near $x = a$.

EXAMPLE 1 **Reasoning about the limit of a polynomial function**

Find $\lim_{x \to 2} (3x^2 + 4x - 1)$.

Solution

It seems clear that when x is close to 2, $3x^2$ is close to 12, and $4x$ is close to 8. Therefore, it appears that $\lim_{x \to 2} (3x^2 + 4x - 1) = 12 + 8 - 1 = 19$.

In Example 1, the limit was arrived at intuitively. It is possible to evaluate limits using the following properties of limits, which can be proved using the formal definition of limits. This is left for more advanced courses.

Properties of Limits

For any real number a, suppose that f and g both have limits that exist at $x = a$.

1. $\lim_{x \to a} k = k$, for any constant k
2. $\lim_{x \to a} x = a$
3. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
4. $\lim_{x \to a} [cf(x)] = c[\lim_{x \to a} f(x)]$, for any constant c
5. $\lim_{x \to a} [f(x)g(x)] = [\lim_{x \to a} f(x)][\lim_{x \to a} g(x)]$
6. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided that $\lim_{x \to a} g(x) \neq 0$
7. $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$, for any rational number n

EXAMPLE 2 Using the limit properties to evaluate the limit of a polynomial function

Evaluate $\lim_{x \to 2} (3x^2 + 4x - 1)$.

Solution

$$\begin{aligned}
\lim_{x \to 2} (3x^2 + 4x - 1) &= \lim_{x \to 2} (3x^2) + \lim_{x \to 2} (4x) - \lim_{x \to 2} (1) \\
&= 3 \lim_{x \to 2} (x^2) + 4 \lim_{x \to 2} (x) - 1 \\
&= 3[\lim_{x \to 2} x]^2 + 4(2) - 1 \\
&= 3(2)^2 + 8 - 1 \\
&= 19
\end{aligned}$$

Note: If f is a polynomial function, then $\lim_{x \to a} f(x) = f(a)$.

EXAMPLE 3 Using the limit properties to evaluate the limit of a rational function

Evaluate $\lim_{x \to -1} \frac{x^2 - 5x + 2}{2x^3 + 3x + 1}$.

Solution

$$\begin{aligned}
\lim_{x \to -1} \frac{x^2 - 5x + 2}{2x^3 + 3x + 1} &= \frac{\lim_{x \to -1} (x^2 - 5x + 2)}{\lim_{x \to -1} (2x^3 + 3x + 1)} \\
&= \frac{(-1)^2 - 5(-1) + 2}{2(-1)^3 + 3(-1) + 1} \\
&= \frac{8}{-4} \\
&= -2
\end{aligned}$$

EXAMPLE 4 Using the limit properties to evaluate the limit of a root function

Evaluate $\lim_{x \to 5} \sqrt{\frac{x^2}{x - 1}}$.

Solution

$$\begin{aligned}
\lim_{x \to 5} \sqrt{\frac{x^2}{x - 1}} &= \sqrt{\lim_{x \to 5} \frac{x^2}{x - 1}} \\
&= \sqrt{\frac{\lim_{x \to 5} x^2}{\lim_{x \to 5} (x - 1)}} \\
&= \sqrt{\frac{25}{4}} \\
&= \frac{5}{2}
\end{aligned}$$

Sometimes $\lim_{x \to a} f(x)$ cannot be found by direct substitution. This is particularly interesting when direct substitution results in an **indeterminate form** $\left(\frac{0}{0}\right)$. In such cases, we look for an equivalent function that agrees with f for all values except at $x = a$. Here are some examples.

EXAMPLE 5 **Selecting a factoring strategy to evaluate a limit**

Evaluate $\lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3}$.

Solution

Substitution produces the indeterminate form $\frac{0}{0}$. The next step is to simplify the function by factoring and reducing to see if the limit of the reduced form can be evaluated.

$$\lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3} = \lim_{x \to 3} \frac{(x + 1)(x - 3)}{x - 3} = \lim_{x \to 3} (x + 1)$$

The reduction is valid only if $x \neq 3$. This is not a problem, since $\lim_{x \to 3}$ requires values as x approaches 3, not when $x = 3$. Therefore,

$\lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3} = \lim_{x \to 3} (x + 1) = 4.$

EXAMPLE 6 **Selecting a rationalizing strategy to evaluate a limit**

Evaluate $\lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x}$.

Solution

A useful technique for finding a limit is to rationalize either the numerator or the denominator to obtain an algebraic form that is not indeterminate.

Substitution produces the indeterminate form $\frac{0}{0}$, so we will try rationalizing.

$$\begin{aligned}
\lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x} &= \lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x} \times \frac{\sqrt{x + 1} + 1}{\sqrt{x + 1} + 1} && \text{(Rationalize the numerator)} \\
&= \lim_{x \to 0} \frac{x + 1 - 1}{x(\sqrt{x + 1} + 1)} \\
&= \lim_{x \to 0} \frac{x}{x(\sqrt{x + 1} + 1)} && \text{(Simplify)} \\
&= \lim_{x \to 0} \frac{1}{\sqrt{x + 1} + 1} && \text{(Evaluate)} \\
&= \frac{1}{2}
\end{aligned}$$

INVESTIGATION

Here is an alternate technique for finding the value of a limit.

A. Find $\lim_{x\to 1} \frac{(x-1)}{\sqrt{x}-1}$ by rationalizing.

B. Let $u = \sqrt{x}$, and rewrite $\lim_{x\to 1} \frac{(x-1)}{\sqrt{x}-1}$ in terms of u. We know $x = u^2$, $\sqrt{x} \geq 0$, and $u \geq 0$. Therefore, as x approaches the value of 1, u approaches the value of 1. Use this substitution to find $\lim_{u\to 1} \frac{(u^2-1)}{u-1}$ by reducing the rational expression.

EXAMPLE 7 Selecting a substitution strategy to evaluate a limit

Evaluate $\lim_{x\to 0} \frac{(x+8)^{\frac{1}{3}}-2}{x}$.

Solution

This quotient is indeterminate $\left(\frac{0}{0}\right)$ when $x = 0$. Rationalizing the numerator $(x+8)^{\frac{1}{3}} - 2$ is not so easy. However, the expression can be simplified by substitution. Let $u = (x+8)^{\frac{1}{3}}$. Then $u^3 = x + 8$ and $x = u^3 - 8$. As x approaches the value 0, u approaches the value 2. The given limit becomes

$$\lim_{x\to 0} \frac{(x+8)^{\frac{1}{3}}-2}{x} = \lim_{u\to 2} \frac{u-2}{u^3-8} \quad \text{(Factor)}$$

$$= \lim_{u\to 2} \frac{u-2}{(u-2)(u^2+2u+4)} \quad \text{(Simplify)}$$

$$= \lim_{u\to 2} \frac{1}{u^2+2u+4} \quad \text{(Evaluate)}$$

$$= \frac{1}{12}$$

EXAMPLE 8 Evaluating a limit that involves absolute value

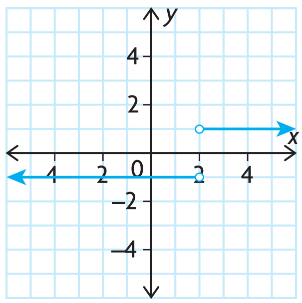
Evaluate $\lim_{x\to 2} \frac{|x-2|}{x-2}$. Illustrate with a graph.

Solution

Consider the following:

$$f(x) = \frac{|x-2|}{x-2} = \begin{cases} \dfrac{(x-2)}{x-2}, \text{ if } x > 2 \\ \dfrac{-(x-2)}{x-2}, \text{ if } x < 2 \end{cases}$$

$$= \begin{cases} 1, \text{ if } x > 2 \\ -1, \text{ if } x < 2 \end{cases}$$



Notice that $f(2)$ is not defined. Also note and that we must consider left-hand and right-hand limits.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(-1) = -1$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(1) = 1$$

Since the left-hand and right-hand limits are not the same, we conclude that $\lim_{x\to 2}\frac{|x-2|}{x-2}$ does not exist.

EXAMPLE 9 **Reasoning about the existence of a limit**

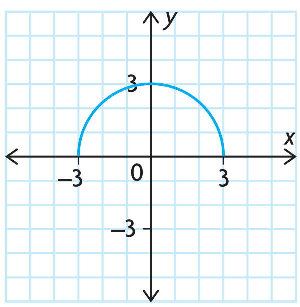
a. Evaluate $\lim_{x\to 3^-}\sqrt{9-x^2}$

b. Explain why the limit as x approaches 3^+ cannot be determined.

c. What can you conclude about $\lim_{x\to 3}\sqrt{9-x^2}$?

Solution

a. The graph of $f(x) = \sqrt{9-x^2}$ is the semicircle illustrated below.



From the graph, the left-hand limit at $x \to 3$ is 0. Therefore, $\lim_{x\to 3^-}\sqrt{9-x^2} = 0$.

b. The function is not defined for $x > 3$.

c. $\lim_{x\to 3}\sqrt{9-x^2}$ does not exist because the function is not defined on both sides of 3.

In this section, we learned the properties of limits and developed algebraic methods for evaluating limits. The examples in this section complement the table of values and graphing techniques introduced in previous sections.

IN SUMMARY

Key Ideas

- If f is a polynomial function, then $\lim_{x \to a} f(x) = f(a)$.
- Substituting $x = a$ into $\lim_{x \to a} f(x)$ can yield the indeterminate form $\frac{0}{0}$. If this happens, you may be able to find an equivalent function that is the same as the function f for all values except at $x = a$. Then, substitution can be used to find the limit.

Need to Know

To evaluate a limit algebraically, you can use the following techniques:

- direct substitution
- factoring
- rationalizing
- one-sided limits
- change of variable

For any of these techniques, a graph or table of values can be used to check your result.

Exercise 1.5

PART A

1. Are there different answers for $\lim_{x \to 2} (3 + x)$, $\lim_{x \to 2} 3 + x$, and $\lim_{x \to 2} (x + 3)$?
2. How do you find the limit of a rational function?

C 3. Once you know $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$, do you then know $\lim_{x \to a} f(x)$? Give reasons for your answer.

4. Evaluate each limit.

a. $\lim_{x \to 2} \frac{3x}{x^2 + 2}$

b. $\lim_{x \to -1} (x^4 + x^3 + x^2)$

c. $\lim_{x \to 9} \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$

d. $\lim_{x \to 2\pi} (x^3 + \pi^2 x - 5\pi^3)$

e. $\lim_{x \to 0} \left(\sqrt{3 + \sqrt{1 + x}}\right)$

f. $\lim_{x \to -3} \sqrt{\frac{x - 3}{2x + 4}}$

PART B

5. Use a graphing calculator to graph each function and estimate the limit. Then find the limit by substitution.

a. $\lim\limits_{x \to -2} \dfrac{x^3}{x - 2}$

b. $\lim\limits_{x \to 1} \dfrac{2x}{\sqrt{x^2 + 1}}$

6. Show that $\lim\limits_{t \to 1} \dfrac{t^3 - t^2 - 5t}{6 - t^2} = -1$.

K 7. Evaluate the limit of each indeterminate quotient.

a. $\lim\limits_{x \to 2} \dfrac{4 - x^2}{2 - x}$

b. $\lim\limits_{x \to -1} \dfrac{2x^2 + 5x + 3}{x + 1}$

c. $\lim\limits_{x \to 3} \dfrac{x^3 - 27}{x - 3}$

d. $\lim\limits_{x \to 0} \dfrac{2 - \sqrt{4 + x}}{x}$

e. $\lim\limits_{x \to 4} \dfrac{\sqrt{x} - 2}{x - 4}$

f. $\lim\limits_{x \to 0} \dfrac{\sqrt{7 - x} - \sqrt{7 + x}}{x}$

8. Evaluate the limit by using a change of variable.

a. $\lim\limits_{x \to 8} \dfrac{\sqrt[3]{x} - 2}{x - 8}$

b. $\lim\limits_{x \to 27} \dfrac{27 - x}{x^{\frac{1}{3}} - 3}$

c. $\lim\limits_{x \to 1} \dfrac{x^{\frac{1}{6}} - 1}{x - 1}$

d. $\lim\limits_{x \to 1} \dfrac{x^{\frac{1}{6}} - 1}{x^{\frac{1}{3}} - 1}$

e. $\lim\limits_{x \to 4} \dfrac{\sqrt{x} - 2}{\sqrt{x^3} - 8}$

f. $\lim\limits_{x \to 0} \dfrac{(x + 8)^{\frac{1}{3}} - 2}{x}$

9. Evaluate each limit, if it exists, using any appropriate technique.

a. $\lim\limits_{x \to 4} \dfrac{16 - x^2}{x^3 + 64}$

b. $\lim\limits_{x \to 4} \dfrac{x^2 - 16}{x^2 - 5x + 6}$

c. $\lim\limits_{x \to -1} \dfrac{x^2 + x}{x + 1}$

d. $\lim\limits_{x \to 0} \dfrac{\sqrt{x + 1} - 1}{x}$

e. $\lim\limits_{h \to 0} \dfrac{(x + h)^2 - x^2}{h}$

f. $\lim\limits_{x \to 1} \left[\left(\dfrac{1}{x - 1} \right) \left(\dfrac{1}{x + 3} - \dfrac{2}{3x + 5} \right) \right]$

10. By using one-sided limits, determine whether each limit exists. Illustrate your results geometrically by sketching the graph of the function.

a. $\lim\limits_{x \to 5} \dfrac{|x - 5|}{x - 5}$

b. $\lim\limits_{x \to \frac{5}{2}} \dfrac{|2x - 5|(x + 1)}{2x - 5}$

c. $\lim\limits_{x \to 2} \dfrac{x^2 - x - 2}{|x - 2|}$

d. $\lim\limits_{x \to -2} \dfrac{(x + 2)^3}{|x + 2|}$

A

11. Jacques Charles (1746–1823) discovered that the volume of a gas at a constant pressure increases linearly with the temperature of the gas. To obtain the data in the following table, one mole of hydrogen was held at a constant pressure of one atmosphere. The volume V was measured in litres, and the temperature T was measured in degrees Celsius.

T (°C)	−40	−20	0	20	40	60	80
V (L)	19.1482	20.7908	22.4334	24.0760	25.7186	27.3612	29.0038

a. Calculate first differences, and show that T and V are related by a linear relation.

b. Find the linear equation for V in terms of T.

c. Solve for T in terms of V for the equation in part b.

d. Show that $\lim_{V \to 0^+} T$ is approximately -273.15. *Note:* This represents the approximate number of degrees on the Celsius scale for absolute zero on the Kelvin scale (0 K).

e. Using the information you found in parts b and d, draw a graph of V versus T.

T

12. Show, using the properties of limits, that if $\lim_{x \to 5} f(x) = 3$, then $\lim_{x \to 5} \frac{x^2 - 4}{f(x)} = 7$.

13. If $\lim_{x \to 4} f(x) = 3$, use the properties of limits to evaluate each limit.

a. $\lim_{x \to 4} [f(x)]^3$

b. $\lim_{x \to 4} \frac{[f(x)]^2 - x^2}{f(x) + x}$

c. $\lim_{x \to 4} \sqrt{3f(x) - 2x}$

PART C

14. If $\lim_{x \to 0} \frac{f(x)}{x} = 1$ and $\lim_{x \to 0} g(x)$ exists and is nonzero, then evaluate each limit.

a. $\lim_{x \to 0} f(x)$

b. $\lim_{x \to 0} \frac{f(x)}{g(x)}$

15. If $\lim_{x \to 0} \frac{f(x)}{x} = 1$ and $\lim_{x \to 0} \frac{g(x)}{x} = 2$, then evaluate each limit.

a. $\lim_{x \to 0} g(x)$

b. $\lim_{x \to 0} \frac{f(x)}{g(x)}$

16. Evaluate $\lim_{x \to 0} \frac{\sqrt{x + 1} - \sqrt{2x + 1}}{\sqrt{3x + 4} - \sqrt{2x + 4}}$.

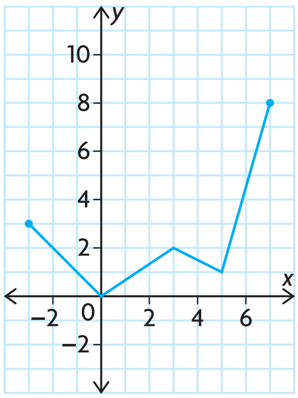
17. Does $\lim_{x \to 1} \frac{x^2 + |x - 1| - 1}{|x - 1|}$ exist? Illustrate your answer by sketching a graph of the function.

Section 1.6—Continuity

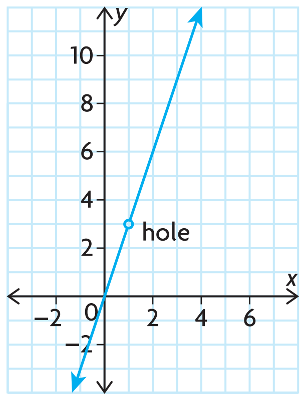
The idea of continuity may be thought of informally as the idea of being able to draw a graph without lifting one's pencil. The concept arose from the notion of a graph "without breaks or jumps or gaps."

When we talk about a function being continuous at a point, we mean that the graph passes through the point without a break. A graph that is not continuous at a point (sometimes referred to as being discontinuous at a point) has a break of some type at the point. The following graphs illustrate these ideas:

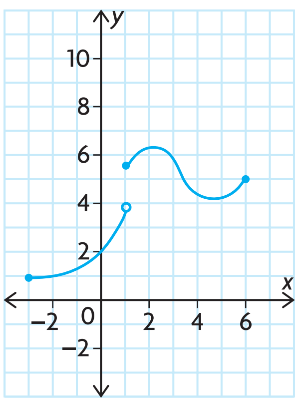
A. Continuous for all values of the domain



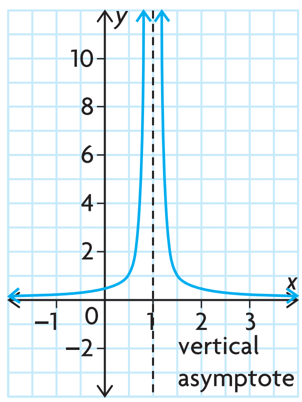
B. Discontinuous at $x = 1$ (point discontinuity)



C. Discontinuous at $x = 1$ (jump discontinuity)



D. Discontinuous at $x = 1$ (infinite discontinuity)



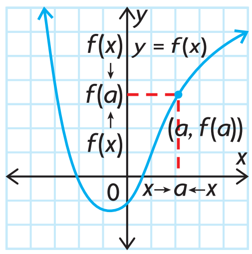
What conditions must be satisfied for a function f to be continuous at a? First, $f(a)$ must be defined. The curves in figure B and figure D above are not continuous at $x = 1$ because they are not defined at $x = 1$.

A second condition for continuity at a point $x = a$ is that the function makes no jumps there. This means that, if "x is close to a," then $f(x)$ must be close to $f(a)$. This condition is satisfied if $\lim_{x \to a} f(x) = f(a)$. Looking at the graph in figure C, on the previous page, we see that $\lim_{x \to 1} f(x)$ does not exist, and the function is therefore not continuous at $x = 1$.

We can now define the continuity of a function at a point.

Continuity at a Point

The function $f(x)$ is continuous at $x = a$ if $f(a)$ is defined and if $\lim_{x \to a} f(x) = f(a)$.



Otherwise, $f(x)$ is discontinuous at $x = a$.

The geometrical meaning of f being continuous at $x = a$ can be stated as follows: As $x \to a$, the points $(x, f(x))$ on the graph of f converge at the point $(a, f(a))$, ensuring that the graph of f is unbroken at $(a, f(a))$.

EXAMPLE 1 Reasoning about continuity at a point

a. Graph the following function:

$$f(x) = \begin{cases} x^2 - 3, \text{ if } x \le -1 \\ x - 1, \text{ if } x > -1 \end{cases}$$

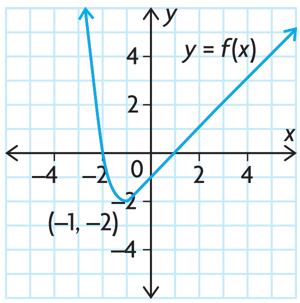
b. Determine $\lim_{x \to -1} f(x)$

c. Determine $f(-1)$.

d. Is f continuous at $x = -1$? Explain.

Solution

a.



b. From the graph, $\lim_{x \to -1} f(x) = -2$. *Note:* Both the left-hand and right-hand limits are equal.

c. $f(-1) = -2$

d. Therefore, $f(x)$ is continuous at $x = -1$, since $f(-1) = \lim_{x \to -1} f(x)$.

EXAMPLE 2

Reasoning whether a function is continuous or discontinuous at a point

Test the continuity of each of the following functions at $x = 2$. If a function is not continuous at $x = 2$, give a reason why it is not continuous.

a. $f(x) = x^3 - x$

b. $g(x) = \dfrac{x^2 - x - 2}{x - 2}$

c. $h(x) = \dfrac{x^2 - x - 2}{x - 2}$, if $x \neq 2$ and $h(2) = 3$

d. $F(x) = \dfrac{1}{(x - 2)^2}$

e. $G(x) = \begin{cases} 4 - x^2, \text{if } x < 2 \\ 3, \text{if } x \geq 2 \end{cases}$

Solution

a. The function f is continuous at $x = 2$ since $f(2) = 6 = \lim_{x \to 2} f(x)$. (Polynomial functions are continuous at all real values of x.)

b. The function g is not continuous at $x = 2$ because g is not defined at this value.

c.
$$\begin{aligned}\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} &= \lim_{x \to 2} \frac{(x - 2)(x + 1)}{(x - 2)} \\ &= \lim_{x \to 2} (x + 1) \\ &= 3 \\ &= h(2)\end{aligned}$$

Therefore, $h(x)$ is continuous at $x = 2$.

d. The function F is not continuous at $x = 2$ because $F(2)$ is not defined.

e. $\lim_{x \to 2^-} G(x) = \lim_{x \to 2^-} (4 - x^2) = 0$ and $\lim_{x \to 2^+} G(x) = \lim_{x \to 2^+} (3) = 3$

Therefore, since $\lim_{x \to 2} G(x)$ does not exist, the function is not continuous at $x = 2$.

INVESTIGATION To test the definition of continuity by graphing, investigate the following:

A. Draw the graph of each function in Example 2.

B. Which of the graphs are continuous, contain a hole or a jump, or have a vertical asymptote?

C. Given only the defining rule of a function $y = f(x)$, such as $f(x) = \frac{8x^3 - 9x + 5}{x^2 + 300x}$, explain why the graphing technique to test for continuity on an interval may be less suitable.

D. Determine where $f(x) = \frac{8x^3 - 9x + 5}{x^2 + 300x}$ is not defined and where it is continuous.

IN SUMMARY

Key Ideas

- A function f is continuous at $x = a$ if
  - $f(a)$ is defined
  - $\lim_{x \to a} f(x)$ exists
  - $\lim_{x \to a} f(x) = f(a)$
- A function that is not continuous has some type of break in its graph. This break is the result of a hole, jump, or vertical asymptote.

Need to Know

- All polynomial functions are continuous for all real numbers.
- A rational function $h(x) = \frac{f(x)}{g(x)}$ is continuous at $x = a$ if $g(a) \neq 0$.
- A rational function in simplified form has a discontinuity at the zeros of the denominator.
- When the one-sided limits are not equal to each other, then the limit at this point does not exist and the function is not continuous at this point.

Exercise 1.6

PART A

C 1. How can looking at a graph of a function help you tell where the function is continuous?

2. What does it mean for a function to be continuous over a given domain?

3. What are the basic types of discontinuity? Give an example of each.

4. Find the value(s) of x at which each function is discontinuous.

a. $f(x) = \dfrac{9 - x^2}{x - 3}$

b. $g(x) = \dfrac{7x - 4}{x}$

c. $h(x) = \dfrac{x^2 + 1}{x^3}$

d. $f(x) = \dfrac{x - 4}{x^2 - 9}$

e. $g(x) = \dfrac{13x}{x^2 + x - 6}$

f. $h(x) = \begin{cases} -x, \text{ if } x \le 3 \\ 1 - x, \text{ if } x > 3 \end{cases}$

PART B

K 5. Determine all the values of x for which each function is continuous.

a. $f(x) = 3x^5 + 2x^3 - x$

b. $g(x) = \pi x^2 - 4.2x + 7$

c. $h(x) = \dfrac{x^2 + 16}{x^2 - 5x}$

d. $f(x) = \sqrt{x + 2}$

e. $g(x) = 10^x$

f. $h(x) = \dfrac{16}{x^2 + 65}$

6. Examine the continuity of $g(x) = x + 3$ when $x = 2$.

7. Sketch a graph of the following function:

$$h(x) = \begin{cases} x - 1, \text{ if } x < 3 \\ 5 - x, \text{ if } x \ge 3 \end{cases}$$

Determine if the function is continuous everywhere.

8. Sketch a graph of the following function:

$$f(x) = \begin{cases} x^2, \text{ if } x < 0 \\ 3, \text{ if } x \ge 0 \end{cases}$$

Is the function continuous?

A 9. Recent postal rates for non-standard and oversized letter mail within Canada are given in the following table. Maximum dimensions for this type of letter mail are 380 mm by 270 mm by 20 mm.

100 g or Less	Between 100 g and 200 g	Between 200 g and 500 g
\$1.10	\$1.86	\$2.55

Draw a graph of the cost, in dollars, to mail a non-standard envelope as a function of its mass in grams. Where are the discontinuities of this function?

10. Determine whether $f(x) = \dfrac{x^2 - x - 6}{x - 3}$ is continuous at $x = 3$.

11. Examine the continuity of the following function:

$$f(x) = \begin{cases} x, \text{ if } x \le 1 \\ 1, \text{ if } 1 < x \le 2 \\ 3, \text{ if } x > 2 \end{cases}$$

12. $g(x) = \begin{cases} x + 3, \text{ if } x \neq 3 \\ 2 + \sqrt{k}, \text{ if } x = 3 \end{cases}$

Find k, if $g(x)$ is continuous.

13. The signum function is defined as follows:

$$f(x) = \begin{cases} -1, \text{ if } x < 0 \\ 0, \text{ if } x = 0 \\ 1, \text{ if } x > 0 \end{cases}$$

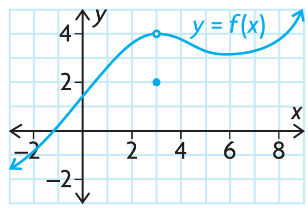
a. Sketch the graph of the signum function.

b. Find each limit, if it exists.

i. $\lim_{x \to 0^-} f(x)$ ii. $\lim_{x \to 0^+} f(x)$ iii. $\lim_{x \to 0} f(x)$

c. Is $f(x)$ continuous? Explain.

14. Examine the graph of $f(x)$.



a. Find $f(3)$.

b. Evaluate $\lim_{x \to 3^-} f(x)$.

c. Is $f(x)$ continuous on the interval $-3 < x < 8$? Explain.

15. What must be true about A and B for the function

$$f(x) = \begin{cases} \dfrac{Ax - B}{x - 2}, \text{ if } x \leq 1 \\ 3x, \text{ if } 1 < x < 2 \\ Bx^2 - A, \text{ if } x \geq 2 \end{cases}$$

if the function is continuous at $x = 1$ but discontinuous at $x = 2$?

PART C

16. Find constants a and b, such that the function

$$f(x) = \begin{cases} -x, \text{ if } -3 \leq x \leq -2 \\ ax^2 + b, \text{ if } -2 < x < 0 \\ 6, \text{ if } x = 0 \end{cases}$$

is continuous for $-3 \leq x \leq 0$.

17. Consider the following function:

$$g(x) = \begin{cases} \dfrac{x|x - 1|}{x - 1}, \text{ if } x \neq 1 \\ 0, \text{ if } x = 1 \end{cases}$$

a. Evaluate $\lim_{x \to 1^+} g(x)$ and $\lim_{x \to 1^-} g(x)$, and then determine whether $\lim_{x \to 1} g(x)$ exists.

b. Sketch the graph of $g(x)$, and identify any points of discontinuity.

CAREER LINK WRAP-UP | Investigate and Apply

CHAPTER 1: ASSESSING ATHLETIC PERFORMANCE

An Olympic coach has developed a 6 min fitness test for her team members that sets target values for heart rates. The monitor they have available counts the total number of heartbeats, starting from a rest position at "time zero." The results for one of the team members are given in the table below.

Time (min)	Number of Heartbeats
0.0	0
1.0	55
2.0	120
3.0	195
4.0	280
5.0	375
6.0	480

a. The coach has established that each athlete's heart rate must not exceed 100 beats per minute at exactly 3 min. Using a graphical technique, determine if this athlete meets the coach's criterion.

b. The coach needs to know the instant in time when an athlete's heart rate actually exceeds 100 beats per minute. Explain how you would solve this problem graphically. Is a graphical solution an efficient method? Explain. How is this problem different from part a?

c. Build a mathematical model with the total number of heartbeats as a function of time ($n = f(t)$). First determine the degree of the polynomial, and then use a graphing calculator to obtain an algebraic model.

d. Solve part b algebraically by obtaining an expression for the instantaneous rate of change in the number of heartbeats (heart rate) as a function of time ($r = g(t)$) using the methods presented in the chapter. Compare the accuracy and efficiency of solving this problem graphically and algebraically.

Key Concepts Review

We began our introduction to calculus by considering the slope of a tangent and the related concept of rate of change. This led us to the study of limits and has laid the groundwork for Chapter 2 and the concept of the derivative of a function. Consider the following brief summary to confirm your understanding of the key concepts covered in Chapter 1:

- slope of the tangent as the limit of the slope of the secant as Q approaches P along the curve
- slope of a tangent at an arbitrary point
- average and instantaneous rates of change, average velocity, and (instantaneous) velocity
- the limit of a function at a value of the independent variable, which exists when the limiting value from the left equals the limiting value from the right
- properties of limits and the indeterminate form $\frac{0}{0}$
- continuity as a property of a graph "without breaks or jumps or gaps"

Formulas

- The slope of the tangent to the graph $y = f(x)$ at point $P(a, f(a))$ is $m = \lim_{x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$.
- Average velocity $= \frac{\text{change in position}}{\text{change in time}}$
- The (instantaneous) velocity of an object, represented by position function $s(t)$, at time $t = a$, is $v(a) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{h \to 0} \frac{s(a + h) - s(a)}{h}$.
- If f is a polynomial function, then $\lim_{x \to a} f(x) = f(a)$.
- The function $f(x)$ is continuous at $x = a$ if $f(a)$ is defined and if $\lim_{x \to a} f(x) = f(a)$.

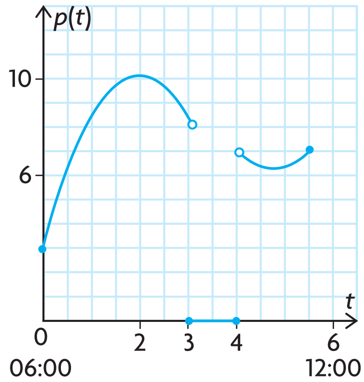
Review Exercise

1. Consider the graph of the function $f(x) = 5x^2 - 8x$.
   a. Find the slope of the secant that joins the points on the graph given by $x = -2$ and $x = 3$.
   b. Determine the average rate of change as x changes from -1 to 4.
   c. Find an equation for the line that is tangent to the graph of the function at $x = 1$.
2. Calculate the slope of the tangent to the given function at the given point or value of x.
   a. $f(x) = \frac{3}{x+1}$, $P(2, 1)$
   b. $g(x) = \sqrt{x+2}$, $P(-1, 1)$
   c. $h(x) = \frac{2}{\sqrt{x+5}}$, $P\left(4, \frac{2}{3}\right)$
   d. $f(x) = \frac{5}{x-2}$, $P\left(4, \frac{5}{2}\right)$
3. Calculate the slope of the graph of $f(x) = \begin{cases} 4 - x^2, \text{ if } x \leq 1 \\ 2x + 1, \text{ if } x > 1 \end{cases}$ at each of the following points:
   a. $P(-1, 3)$
   b. $P(2, 5)$
4. The height, in metres, of an object that has fallen from a height of 180 m is given by the position function $s(t) = -5t^2 + 180$, where $t \geq 0$ and t is in seconds.
   a. Find the average velocity during each of the first two seconds.
   b. Find the velocity of the object when $t = 4$.
   c. At what velocity will the object hit the ground?
5. After t minutes of growth, a certain bacterial culture has a mass, in grams, of $M(t) = t^2$.
   a. How much does the bacterial culture grow during the time $3 \leq t \leq 3.01$?
   b. What is its average rate of growth during the time interval $3 \leq t \leq 3.01$?
   c. What is its rate of growth when $t = 3$?
6. It is estimated that, t years from now, the amount of waste accumulated Q, in tonnes, will be $Q(t) = 10^4(t^2 + 15t + 70)$, $0 \leq t \leq 10$.
   a. How much waste has been accumulated up to now?
   b. What will be the average rate of change in this quantity over the next three years?

c. What is the present rate of change in this quantity?

d. When will the rate of change reach 3.0×10^5 per year?

7. The electrical power $p(t)$, in kilowatts, being used by a household as a function of time t, in hours, is modelled by a graph where $t = 0$ corresponds to 06:00. The graph indicates peak use at 08:00 and a power failure between 09:00 and 10:00.



a. Determine $\lim_{t \to 2} p(t)$.

b. Determine $\lim_{t \to 4^+} p(t)$ and $\lim_{t \to 4^-} p(t)$.

c. For what values of t is $p(t)$ discontinuous?

8. Sketch a graph of any function that satisfies the given conditions.

a. $\lim_{x \to -1} f(x) = 0.5$, f is discontinuous at $x = -1$

b. $f(x) = -4$ if $x < 3$, f is an increasing function when $x > 3$, $\lim_{x \to 3^+} f(x) = 1$

9. a. Sketch the graph of the following function:

$$f(x) = \begin{cases} x + 1, \text{ if } x < -1 \\ -x + 1, \text{ if } -1 \le x < 1 \\ x - 2, \text{ if } x > 1 \end{cases}$$

b. Find all values at which the function is discontinuous.

c. Find the limits at those values, if they exist.

10. Determine whether $f(x) = \dfrac{x^2 + 2x - 8}{x + 4}$ is continuous at $x = -4$.

11. Consider the function $f(x) = \dfrac{2x - 2}{x^2 + x - 2}$.

a. For what values of x is f discontinuous?

b. At each point where f is discontinuous, determine the limit of $f(x)$, if it exists.

12. Use a graphing calculator to graph each function and estimate the limits, if they exist.
   a. $f(x) = \frac{1}{x^2}, \lim_{x \to 0} f(x)$
   b. $g(x) = x(x - 5), \lim_{x \to 0} g(x)$
   c. $h(x) = \frac{x^3 - 27}{x^2 - 9}, \lim_{x \to 4} h(x)$ and $\lim_{x \to -3} h(x)$
13. Copy and complete each table, and use your results to estimate the limit. Use a graphing calculator to graph the function to confirm your result.
   a. $\lim_{x \to 2} \frac{x - 2}{x^2 - x - 2}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$\frac{x-2}{x^2-x-2}$						

   b. $\lim_{x \to 1} \frac{x - 1}{x^2 - 1}$

x	0.9	0.99	0.999	1.001	1.01	1.1
$\frac{x-1}{x^2-1}$						

14. Copy and complete the table, and use your results to estimate the limit.
   $\lim_{x \to 0} \frac{\sqrt{x + 3} - \sqrt{3}}{x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$\frac{\sqrt{x+3}-\sqrt{3}}{x}$						

   Then determine the limit using an algebraic technique, and compare your answer with your estimate.
15. a. Copy and complete the table to approximate the limit of $f(x) = \frac{\sqrt{x + 2} - 2}{x - 2}$ as $x \to 2$.

x	2.1	2.01	2.001	2.0001
$f(x) = \frac{\sqrt{x+2}-2}{x-2}$				

   b. Use a graphing calculator to graph f, and use the graph to approximate the limit.
   c. Use the technique of rationalizing the numerator to find $\lim_{x \to 2} \frac{\sqrt{x + 2} - 2}{x - 2}$.

16. Evaluate the limit of each difference quotient. Interpret the limit as the slope of the tangent to a curve at a specific point.

   a. $\lim_{h\to 0} \frac{(5+h)^2 - 25}{h}$

   b. $\lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h}$

   c. $\lim_{h\to 0} \frac{\frac{1}{(4+h)} - \frac{1}{4}}{h}$

17. Evaluate each limit using one of the algebraic methods discussed in this chapter, if the limit exists.

   a. $\lim_{x\to -4} \frac{x^2 + 12x + 32}{x + 4}$

   b. $\lim_{x\to a} \frac{(x + 4a)^2 - 25a^2}{x - a}$

   c. $\lim_{x\to 0} \frac{\sqrt{x+5} - \sqrt{5-x}}{x}$

   d. $\lim_{x\to 2} \frac{x^2 - 4}{x^3 - 8}$

   e. $\lim_{x\to 4} \frac{4 - \sqrt{12 + x}}{x - 4}$

   f. $\lim_{x\to 0} \frac{1}{x}\left(\frac{1}{2+x} - \frac{1}{2}\right)$

18. Explain why the given limit does not exist.

   a. $\lim_{x\to 3} \sqrt{x - 3}$

   b. $\lim_{x\to 2} \frac{x^2 - 4}{x^2 - 4x + 4}$

   c. $f(x) = \begin{cases} -5, \text{ if } x < 1 \\ 2, \text{ if } x \ge 1 \end{cases}; \lim_{x\to 1} f(x)$

   d. $\lim_{x\to 2} \frac{1}{\sqrt{x - 2}}$

   e. $\lim_{x\to 0} \frac{|x|}{x}$

   f. $f(x) = \begin{cases} 5x^2, \text{ if } x < -1 \\ 2x + 1, \text{ if } x \ge -1 \end{cases}; \lim_{x\to -1} f(x)$

19. Determine the equation of the tangent to the curve of each function at the given value of x.

   a. $y = -3x^2 + 6x + 4$ where $x = 1$

   b. $y = x^2 - x - 1$ where $x = -2$

   c. $f(x) = 6x^3 - 3$ where $x = -1$

   d. $f(x) = -2x^4$ where $x = 3$

20. The estimated population of a bacteria colony is $P(t) = 20 + 61t + 3t^2$, where the population, P, is measured in thousands at t hours.

   a. What is the estimated population of the colony at 8 h?

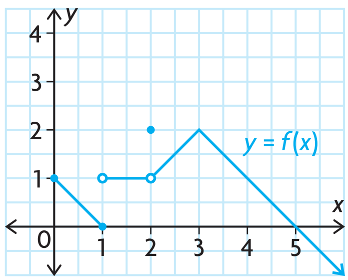
   b. At what rate is the population changing at 8 h?

CHAPTER 1 TEST

1. Explain why $\lim_{x \to 1} \frac{1}{x - 1}$ does not exist.

2. Consider the graph of the function $f(x) = 5x^2 - 8x$. Calculate the slope of the secant that joins the points on the graph given by $x = -2$ and $x = 1$.

3. For the function shown below, determine the following:



a. $\lim_{x \to 1} f(x)$

b. $\lim_{x \to 2} f(x)$

c. $\lim_{x \to 4^-} f(x)$

d. values of x for which f is discontinuous

4. A weather balloon is rising vertically. After t hours, its distance above the ground, measured in kilometres, is given by the formula $s(t) = 8t - t^2$.

   a. Determine the average velocity of the weather balloon from $t = 2$ h to $t = 5$ h.

   b. Determine its velocity at $t = 3$ h.

5. Determine the average rate of change in $f(x) = \sqrt{x + 11}$ with respect to x from $x = 5$ to $x = 5 + h$.

6. Determine the slope of the tangent at $x = 4$ for $f(x) = \frac{x}{x^2 - 15}$.

7. Evaluate the following limits:

   a. $\lim_{x \to 3} \frac{4x^2 - 36}{2x - 6}$

   b. $\lim_{x \to 2} \frac{2x^2 - x - 6}{3x^2 - 7x + 2}$

   c. $\lim_{x \to 5} \frac{x - 5}{\sqrt{x - 1} - 2}$

   d. $\lim_{x \to -1} \frac{x^3 + 1}{x^4 - 1}$

   e. $\lim_{x \to 3} \left(\frac{1}{x - 3} - \frac{6}{x^2 - 9} \right)$

   f. $\lim_{x \to 0} \frac{(x + 8)^{\frac{1}{3}} - 2}{x}$

8. Determine constants a and b such that $f(x)$ is continuous for all values of x.

$$f(x) = \begin{cases} ax + 3, \text{ if } x > 5 \\ 8, \text{ if } x = 5 \\ x^2 + bx + a, \text{ if } x < 5 \end{cases}$$

Chapter 2

DERIVATIVES

Imagine a driver speeding down a highway, at 140 km/h. He hears a police siren and is quickly pulled over. The police officer tells him that he was speeding, but the driver argues that because he has travelled 200 km from home in two hours, his average speed is within the 100 km/h limit. The driver's argument fails because police officers charge speeders based on their instantaneous speed, not their average speed.

There are many other situations in which the instantaneous rate of change is more important than the average rate of change. In calculus, the derivative is a tool for finding instantaneous rates of change. This chapter shows how the derivative can be determined and applied in a great variety of situations.

CHAPTER EXPECTATIONS

In this chapter, you will

- understand and determine derivatives of polynomial and simple rational functions from first principles, **Section 2.1**
- identify examples of functions that are not differentiable, **Section 2.1**
- justify and use the rules for determining derivatives, **Sections 2.2, 2.3, 2.4, 2.5**
- identify composition as two functions applied in succession, **Section 2.5**
- determine the composition of two functions expressed in notation, and decompose a given composite function into its parts, **Section 2.5**
- use the derivative to solve problems involving instantaneous rates of change, **Sections 2.2, 2.3, 2.4, 2.5**

Review of Prerequisite Skills

Before beginning your study of derivatives, it may be helpful to review the following concepts from previous courses and the previous chapter:

- Working with the properties of exponents
- Simplifying radical expressions
- Finding the slopes of parallel and perpendicular lines
- Simplifying rational expressions
- Expanding and factoring algebraic expressions
- Evaluating expressions
- Working with the difference quotient

Exercise

1. Use the exponent laws to simplify each of the following expressions. Express your answers with positive exponents.

a. $a^5 \times a^3$

b. $(-2a^2)^3$

c. $\dfrac{4p^7 \times 6p^9}{12p^{15}}$

d. $(a^4b^{-5})(a^{-6}b^{-2})$

e. $(3e^6)(2e^3)^4$

f. $\dfrac{(3a^{-4})[2a^3(-b)^3]}{12a^5b^2}$

2. Simplify and write each expression in exponential form.

a. $(x^{\frac{1}{2}})(x^{\frac{2}{3}})$

b. $(8x^6)^{\frac{2}{3}}$

c. $\dfrac{\sqrt{a}\sqrt[3]{a}}{\sqrt{a}}$

3. Determine the slope of a line that is perpendicular to a line with each given slope.

a. $\dfrac{2}{3}$

b. $-\dfrac{1}{2}$

c. $\dfrac{5}{3}$

d. -1

4. Determine the equation of each of the following lines:

a. passing through points $A(-3, -4)$ and $B(9, -2)$

b. passing through point $A(-2, -5)$ and parallel to the line $3x - 2y = 5$

c. perpendicular to the line $y = \frac{3}{4}x - 6$ and passing through point $A(4, -3)$

5. Expand, and collect like terms.

a. $(x - 3y)(2x + y)$

b. $(x - 2)(x^2 - 3x + 4)$

c. $(6x - 3)(2x + 7)$

d. $2(x + y) - 5(3x - 8y)$

e. $(2x - 3y)^2 + (5x + y)^2$

f. $3x(2x - y)^2 - x(5x - y)(5x + y)$

6. Simplify each expression.

a. $\dfrac{3x(x + 2)}{x^2} \times \dfrac{5x^3}{2x(x + 2)}$

b. $\dfrac{y}{(y + 2)(y - 5)} \times \dfrac{(y - 5)^2}{4y^3}$

c. $\dfrac{4}{(h + k)} \div \dfrac{9}{2(h + k)}$

d. $\dfrac{(x + y)(x - y)}{5(x - y)} \div \dfrac{(x + y)^3}{10}$

e. $\dfrac{x - 7}{2x} + \dfrac{5x}{x - 1}$

f. $\dfrac{x + 1}{x - 2} - \dfrac{x + 2}{x + 3}$

7. Factor each expression completely.

a. $4k^2 - 9$

b. $x^2 + 4x - 32$

c. $3a^2 - 4a - 7$

d. $x^4 - 1$

e. $x^3 - y^3$

f. $r^4 - 5r^2 + 4$

8. Use the factor theorem to factor the following expressions:

a. $a^3 - b^3$ b. $a^5 - b^5$ c. $a^7 - b^7$ d. $a^n - b^n$

9. If $f(x) = -2x^4 + 3x^2 + 7 - 2x$, evaluate

a. $f(2)$ b. $f(-1)$ c. $f\left(\dfrac{1}{2}\right)$ d. $f(-0.25)$

10. Rationalize the denominator in each of the following expressions:

a. $\dfrac{3}{\sqrt{2}}$ b. $\dfrac{4 - \sqrt{2}}{\sqrt{3}}$ c. $\dfrac{2 + 3\sqrt{2}}{3 - 4\sqrt{2}}$ d. $\dfrac{3\sqrt{2} - 4\sqrt{3}}{3\sqrt{2} + 4\sqrt{3}}$

11. a. If $f(x) = 3x^2 - 2x$, determine the expression for the difference quotient $\dfrac{f(a + h) - f(a)}{h}$ when $a = 2$. Explain what this expression can be used for.

b. Evaluate the expression you found in part a. for a small value of h where $h = 0.01$.

c. Explain what the value you determined in part b. represents.

CAREER LINK Investigate

CHAPTER 2: THE ELASTICITY OF DEMAND

Have you ever wondered how businesses set prices for their goods and services? An important idea in marketing is *elasticity of demand*, or the response of consumers to a change in price. Consumers respond differently to a change in the price of a staple item, such as bread, than they do to a change in the price of a luxury item, such as jewellery. A family would probably still buy the same quantity of bread if the price increased by 20%. This is called *inelastic* demand. If the price of a gold chain, however, increased by 20%, sales would likely decrease 40% or more. This is called *elastic* demand. Mathematically, elasticity is defined as the negative of the relative (percent) change in the number demanded $\left(\frac{\Delta n}{n}\right)$ divided by the relative (percent) change in the price $\left(\frac{\Delta p}{p}\right)$:

$$E = -\left[\left(\frac{\Delta n}{n}\right) \div \left(\frac{\Delta p}{p}\right)\right]$$

For example, if a store increased the price of a CD from \$17.99 to \$19.99, and the number sold per week went from 120 to 80, the elasticity would be

$$E = -\left[\left(\frac{80 - 120}{120}\right) \div \left(\frac{19.99 - 17.99}{17.99}\right)\right] \doteq 3.00$$

An elasticity of about 3 means that the change in demand is three times as large, in percent terms, as the change in price. The CDs have an elastic demand because a small change in price can cause a large change in demand. In general, goods or services with elasticities greater than one ($E > 1$) are considered elastic (e.g., new cars), and those with elasticities less than one ($E < 1$) are considered inelastic (e.g., milk). In our example, we calculated the average elasticity between two price levels, but, in reality, businesses want to know the elasticity at a specific, or *instantaneous*, price level. In this chapter, you will develop the rules of differentiation that will enable you to calculate the instantaneous rate of change for several classes of functions.

Case Study—Marketer: Product Pricing

In addition to developing advertising strategies, marketing departments also conduct research into, and make decisions on, pricing. Suppose that the demand–price relationship for weekly movie rentals at a convenience store is $n(p) = \frac{500}{p}$, where $n(p)$ is demand and p is price.

DISCUSSION QUESTIONS

1. Generate two lists, each with at least five goods and services that you have purchased recently, classifying each of the goods and services as having elastic or inelastic demand.
2. Calculate and discuss the elasticity if a movie rental fee increases from \$1.99 to \$2.99.

Section 2.1—The Derivative Function

In this chapter, we will extend the concepts of the slope of a tangent and the rate of change to introduce the **derivative**. We will examine the methods of differentiation, which we can use to determine the derivatives of polynomial and rational functions. These methods include the use of the power rule, sum and difference rules, and product and quotient rules, as well as the chain rule for the composition of functions.

The Derivative at a Point

In the previous chapter, we encountered limits of the form $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$. This limit has two interpretations: the slope of the tangent to the graph $y = f(x)$ at the point $(a, f(a))$, and the instantaneous rate of change of $y = f(x)$ with respect to x at $x = a$. Since this limit plays a central role in calculus, it is given a name and a concise notation. It is called the **derivative of $f(x)$ at $x = a$**. It is denoted by $f'(a)$ and is read as "f prime of a."

The **derivative of f at the number a** is given by $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, provided that this limit exists.

EXAMPLE 1 **Selecting a limit strategy to determine the derivative at a number**

Determine the derivative of $f(x) = x^2$ at $x = -3$.

Solution

Using the definition, the derivative at $x = -3$ is given by

$$\begin{aligned}
f'(-3) &= \lim_{h\to 0}\frac{f(-3+h)-f(-3)}{h} \\
&= \lim_{h\to 0}\frac{(-3+h)^2-(-3)^2}{h} && \text{(Expand)} \\
&= \lim_{h\to 0}\frac{9-6h+h^2-9}{h} && \text{(Simplify and factor)} \\
&= \lim_{h\to 0}\frac{h(-6+h)}{h} \\
&= \lim_{h\to 0}(-6+h) \\
&= -6
\end{aligned}$$

Therefore, the derivative of $f(x) = x^2$ at $x = -3$ is -6.

An alternative way of writing the derivative of f at the number a is $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

In applications where we are required to find the value of the derivative for a number of particular values of x, using the definition repeatedly for each value is tedious.

The next example illustrates the efficiency of calculating the derivative of $f(x)$ at an arbitrary value of x and using the result to determine the derivatives at a number of particular x-values.

EXAMPLE 2 Connecting the derivative of a function to an arbitrary value

a. Determine the derivative of $f(x) = x^2$ at an arbitrary value of x.
b. Determine the slopes of the tangents to the parabola $y = x^2$ at $x = -2$, 0, and 1.

Solution

a. Using the definition,

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} && \text{(Expand)} \\
&= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} && \text{(Simplify and factor)} \\
&= \lim_{h \to 0} \frac{h(2x + h)}{h} \\
&= \lim_{h \to 0} (2x + h) \\
&= 2x
\end{aligned}
$$

The derivative of $f(x) = x^2$ at an arbitrary value of x is $f'(x) = 2x$.

b. The required slopes of the tangents to $y = x^2$ are obtained by evaluating the derivative $f'(x) = 2x$ at the given x-values. We obtain the slopes by substituting for x:

$f'(-2) = -4$ $\quad$ $f'(0) = 0$ $\quad$ $f'(1) = 2$

The slopes are -4, 0, and 2, respectively.

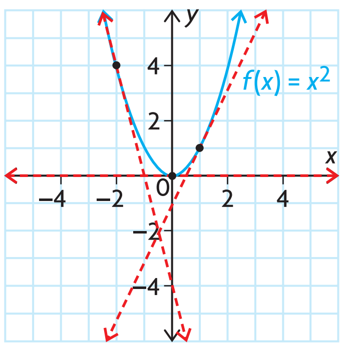
In fact, knowing the x-coordinate of a point on the parabola $y = x^2$, we can easily find the slope of the tangent at that point. For example, given the x-coordinates of points on the curve, we can produce the following table.

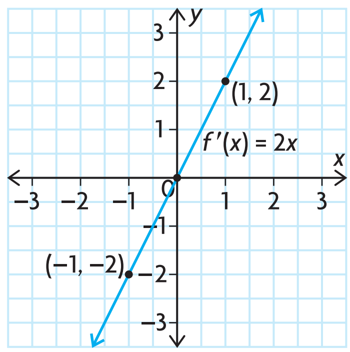
The slope of the tangent to the curve $f(x) = x^2$ at a point $P(x, y)$ is given by the derivative $f'(x) = 2x$. For each x-value, there is an associated value $2x$.

For the Parabola $f(x) = x^2$

$P(x, y)$	x-Coordinate of P	Slope of Tangent at P
$(-2, 4)$	-2	$2(-2) = -4$
$(-1, 1)$	-1	-2
$(0, 0)$	0	0
$(1, 1)$	1	2
$(2, 4)$	2	4
(a, a^2)	a	$2a$

The graphs of $f(x) = x^2$ and the derivative function $f'(x) = 2x$ are shown below. The tangents at $x = -2$, 0, and 1 are shown on the graph of $f(x) = x^2$.





Notice that the graph of the derivative function of the quadratic function (of degree two) is a linear function (of degree one).

INVESTIGATION

A. Determine the derivative with respect to x of each of the following functions:

a. $f(x) = x^3$ b. $f(x) = x^4$ c. $f(x) = x^5$

B. In Example 2, we showed that the derivative of $f(x) = x^2$ is $f'(x) = 2x$. Referring to step 1, what pattern do you see developing?

C. Use the pattern from step 2 to predict the derivative of $f(x) = x^{39}$.

D. What do you think $f'(x)$ would be for $f(x) = x^n$, where n is a positive integer?

The Derivative Function

The derivative of f at $x = a$ is a number $f'(a)$. If we let a be arbitrary and assume a general value in the domain of f, the derivative f' is a function. For example, if $f(x) = x^2$, $f'(x) = 2x$, which is itself a function.

The Definition of the Derivative Function

The derivative of $f(x)$ with respect to x is the function $f'(x)$, where $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$, provided that this limit exists.

The $f'(x)$ notation for this limit was developed by Joseph Louis Lagrange (1736–1813), a French mathematician. When you use this limit to determine the derivative of a function, it is called determining the derivative from first principles.

In Chapter 1, we discussed velocity at a point. We can now define (instantaneous) velocity as the derivative of position with respect to time. If the position of a body at time t is $s(t)$, then the velocity of the body at time t is

$$v(t) = s'(t) = \lim_{h \to 0} \frac{s(t + h) - s(t)}{h}.$$

Likewise, the (instantaneous) rate of change of $f(x)$ with respect to x is the function $f'(x)$, whose value is $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$.

EXAMPLE 3 Determining the derivative from first principles

Determine the derivative $f'(t)$ of the function $f(t) = \sqrt{t}, t \geq 0$.

Solution

Using the definition,
$$\begin{aligned}
f'(t) &= \lim_{h \to 0} \frac{f(t + h) - f(t)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{t + h} - \sqrt{t}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{t + h} - \sqrt{t}}{h} \left(\frac{\sqrt{t + h} + \sqrt{t}}{\sqrt{t + h} + \sqrt{t}} \right) \quad \text{(Rationalize the numerator)} \\
&= \lim_{h \to 0} \frac{(t + h) - t}{h(\sqrt{t + h} + \sqrt{t})} \\
&= \lim_{h \to 0} \frac{h}{h(\sqrt{t + h} + \sqrt{t})} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{t + h} + \sqrt{t}} \\
&= \frac{1}{2\sqrt{t}}, \text{ for } t > 0
\end{aligned}$$

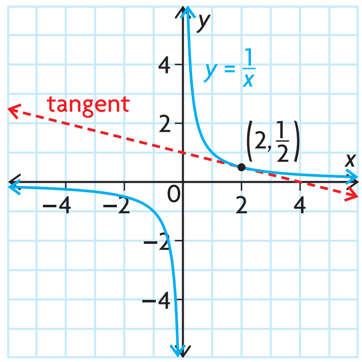
Note that $f(t) = \sqrt{t}$ is defined for all instances of $t \geq 0$, whereas its derivative $f'(t) = \frac{1}{2\sqrt{t}}$ is defined only for instances when $t > 0$. From this, we can see that a function need not have a derivative throughout its entire domain.

EXAMPLE 4 Selecting a strategy involving the derivative to determine the equation of a tangent

Determine an equation of the tangent to the graph of $f(x) = \frac{1}{x}$ at the point where $x = 2$.

Solution

When $x = 2$, $y = \frac{1}{2}$. The graph of $y = \frac{1}{x}$, the point $\left(2, \frac{1}{2}\right)$, and the tangent at the point are shown.



First find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h}$$ (Simplify the fraction)

$$= \lim_{h \to 0} \frac{x - (x+h)}{h(x+h)x}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h)x}$$

$$= -\frac{1}{x^2}$$

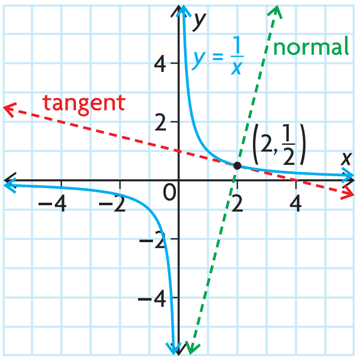
The slope of the tangent at $x = 2$ is $m = f'(2) = -\frac{1}{4}$. The equation of the tangent is $y - \frac{1}{2} = -\frac{1}{4}(x - 2)$ or, in standard form, $x + 4y - 4 = 0$.

EXAMPLE 5 Selecting a strategy involving the derivative to solve a problem

Determine an equation of the line that is perpendicular to the tangent to the graph of $f(x) = \frac{1}{x}$ at $x = 2$ and that intersects it at the point of tangency.

Solution

In Example 4, we found that the slope of the tangent at $x = 2$ is $f'(2) = -\frac{1}{4}$, and the point of tangency is $\left(2, \frac{1}{2}\right)$. The perpendicular line has slope 4, the negative reciprocal of $-\frac{1}{4}$. Therefore, the required equation is $y - \frac{1}{2} = 4(x - 2)$, or $8x - 2y - 15 = 0$.

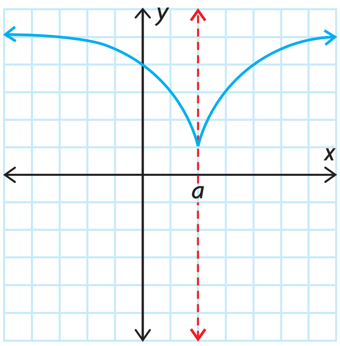


The line whose equation we found in Example 5 has a name.

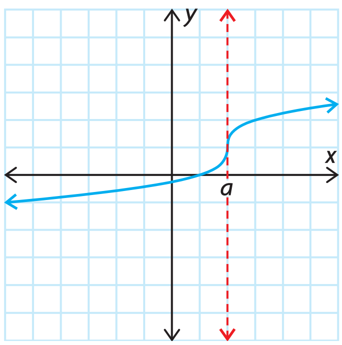
The **normal** to the graph of f at point P is the line that is perpendicular to the tangent at P.

The Existence of Derivatives

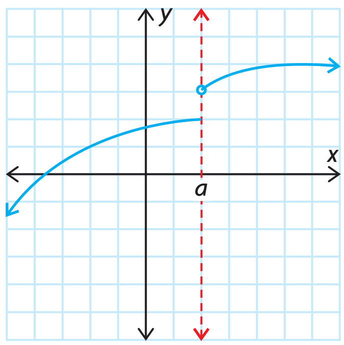
A function f is said to be **differentiable** at a if $f'(a)$ exists. At points where f is not differentiable, we say that the *derivative does not exist*. Three common ways for a derivative to fail to exist are shown.



Cusp



Vertical Tangent



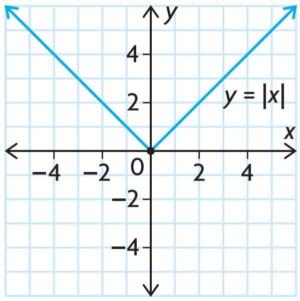
Discontinuity

EXAMPLE 6 Reasoning about differentiability at a point

Show that the absolute value function $f(x) = |x|$ is not differentiable at $x = 0$.

Solution

The graph of $f(x) = |x|$ is shown. Because the slope for $x < 0$ is -1, whereas the slope for $x > 0$ is $+1$, the graph has a "corner" at $(0, 0)$, which prevents a unique tangent from being drawn there. We can show this using the definition of a derivative.



$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{f(h) - 0}{h} \\ &= \lim_{h \to 0} \frac{|h|}{h} \end{aligned}$$

Now, we will consider one-sided limits.

$|h| = h$ when $h > 0$ and $|h| = -h$ when $h < 0$.

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$$

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} (1) = 1$$

Since the left-hand limit and the right-hand limit are not the same, the derivative does not exist at $x = 0$.

From Example 6, we conclude that it is possible for a function to be **continuous** at a point and yet *not differentiable* at this point. However, if a function is differentiable at a point, then it is also continuous at this point.

Other Notation for Derivatives

Symbols other than $f'(x)$ are often used to denote the derivative. If $y = f(x)$, the symbols y' and $\frac{dy}{dx}$ are used instead of $f'(x)$. The notation $\frac{dy}{dx}$ was originally used by Leibniz and is read "dee y by dee x." For example, if $y = x^2$, the derivative is $y' = 2x$ or, in Leibniz notation, $\frac{dy}{dx} = 2x$. Similarly, in Example 4, we showed that if $y = \frac{1}{x}$, then $\frac{dy}{dx} = -\frac{1}{x^2}$. The Leibniz notation reminds us of the process by which the derivative is obtained—namely, as the limit of a difference quotient:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

By omitting y and f altogether, we can combine these notations and write $\frac{d}{dx}(x^2) = 2x$, which is read "the derivative of x^2 with respect to x is $2x$." It is important to note that $\frac{dy}{dx}$ is *not a fraction*.

IN SUMMARY

Key Ideas

- The derivative of a function f at a point $(a, f(a))$ is $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, or $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ if the limit exists.
- A function is said to be **differentiable** at a if $f'(a)$ exists. A function is differentiable on an interval if it is differentiable at every number in the interval.
- The derivative function for any function $f(x)$ is given by $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, if the limit exists.

Need to Know

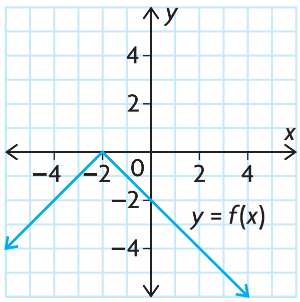
- To find the derivative at a point $x = a$, you can use $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.
- The derivative $f'(a)$ can be interpreted as either
  - the slope of the tangent at $(a, f(a))$, or
  - the instantaneous rate of change of $f(x)$ with respect to x when $x = a$.
- Other notations for the derivative of the function $y = f(x)$ are $f'(x)$, y', and $\frac{dy}{dx}$.
- The normal to the graph of a function at point P, is a line that is perpendicular to the tangent line that passes through point P.

Exercise 2.1

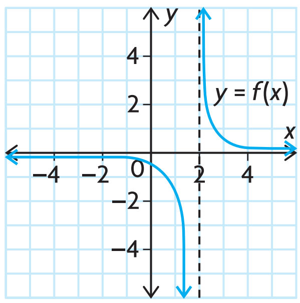
PART A

1. State the domain on which f is differentiable.

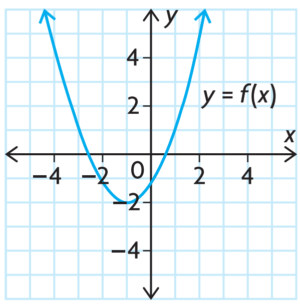
a.



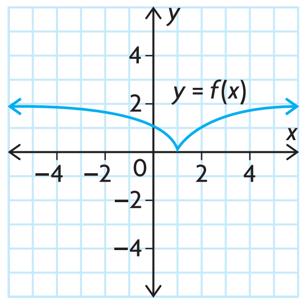
b.



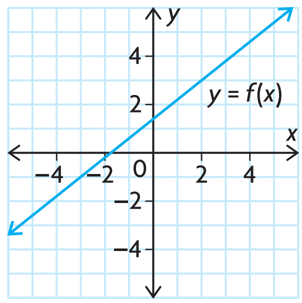
c.



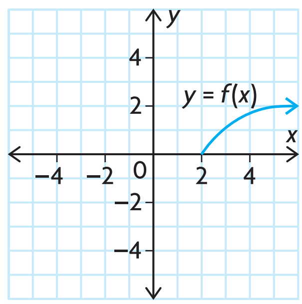
d.



e.



f.



C 2. Explain what the derivative of a function represents.

3. Illustrate two situations in which a function does not have a derivative at $x = 1$.

4. For each function, find $f(a + h)$ and $f(a + h) - f(a)$.

a. $f(x) = 5x - 2$

b. $f(x) = x^2 + 3x - 1$

c. $f(x) = x^3 - 4x + 1$

d. $f(x) = x^2 + x - 6$

e. $f(x) = -7x + 4$

f. $f(x) = 4 - 2x - x^2$

PART B

K 5. For each function, find the value of the derivative $f'(a)$ for the given value of a.

a. $f(x) = x^2, a = 1$

b. $f(x) = x^2 + 3x + 1, a = 3$

c. $f(x) = \sqrt{x + 1}, a = 0$

d. $f(x) = \frac{5}{x}, a = -1$

6. Use the definition of the derivative to find $f'(x)$ for each function.

a. $f(x) = -5x - 8$

b. $f(x) = 2x^2 + 4x$

c. $f(x) = 6x^3 - 7x$

d. $f(x) = \sqrt{3x + 2}$

7. In each case, find the derivative $\frac{dy}{dx}$ from first principles.

a. $y = 6 - 7x$

b. $y = \frac{x + 1}{x - 1}$

c. $y = 3x^2$

8. Determine the slope of the tangents to $y = 2x^2 - 4x$ when $x = 0$, $x = 1$, and $x = 2$. Sketch the graph, showing these tangents.

9. a. Sketch the graph of $f(x) = x^3$.

b. Calculate the slopes of the tangents to $f(x) = x^3$ at points with x-coordinates $-2, -1, 0, 1, 2$.

c. Sketch the graph of the derivative function $f'(x)$.

d. Compare the graphs of $f(x)$ and $f'(x)$.

A 10. An object moves in a straight line with its position at time t seconds given by $s(t) = -t^2 + 8t$, where s is measured in metres. Find the velocity when $t = 0$, $t = 4$, and $t = 6$.

11. Determine an equation of the line that is tangent to the graph of $f(x) = \sqrt{x + 1}$ and parallel to $x - 6y + 4 = 0$.

12. For each function, use the definition of the derivative to determine $\frac{dy}{dx}$, where $a, b, c,$ and m are constants.

a. $y = c$

b. $y = x$

c. $y = mx + b$

d. $y = ax^2 + bx + c$

13. Does the function $f(x) = x^3$ ever have a negative slope? If so, where? Give reasons for your answer.

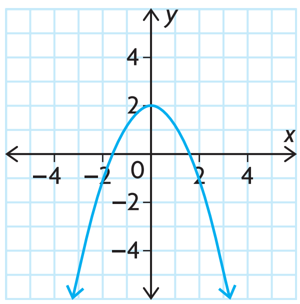
14. A football is kicked up into the air. Its height, h, above the ground, in metres, at t seconds can be modelled by $h(t) = 18t - 4.9t^2$.

a. Determine $h'(2)$.

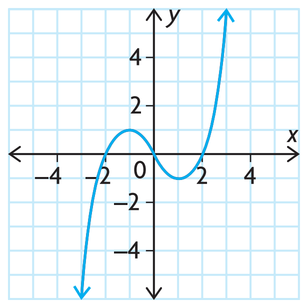
b. What does $h'(2)$ represent?

T 15. Match each function in graphs **a**, **b**, and **c** with its corresponding derivative, graphed in **d**, **e**, and **f**.

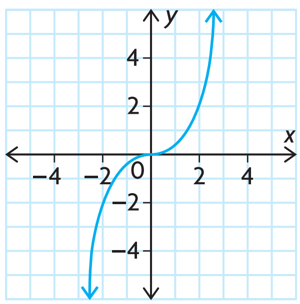
a.



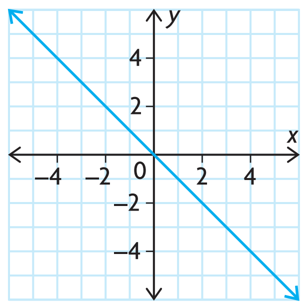
d.



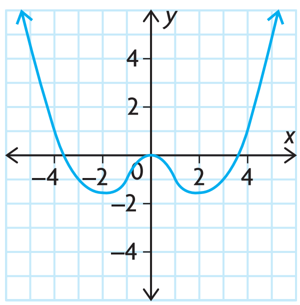
b.



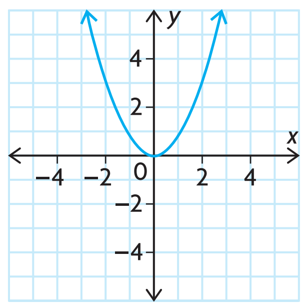
e.



c.



f.



PART C

16. For the function $f(x) = x|x|$, show that $f'(0)$ exists. What is the value?

17. If $f(a) = 0$ and $f'(a) = 6$, find $\lim\limits_{h \to 0} \frac{f(a + h)}{2h}$.

18. Give an example of a function that is continuous on $-\infty < x < \infty$ but is not differentiable at $x = 3$.

19. At what point on the graph of $y = x^2 - 4x - 5$ is the tangent parallel to $2x - y = 1$?

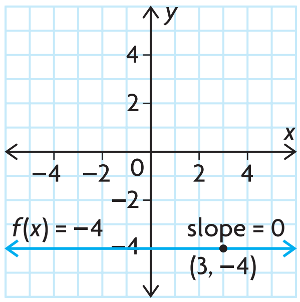
20. Determine the equations of both lines that are tangent to the graph of $f(x) = x^2$ and pass through point $(1, -3)$.

Section 2.2—The Derivatives of Polynomial Functions

We have seen that derivatives of functions are of practical use because they represent instantaneous rates of change.

Computing derivatives from the limit definition, as we did in Section 2.1, is tedious and time-consuming. In this section, we will develop some rules that simplify the process of differentiation.

We will begin developing the rules of differentiation by looking at the constant function, $f(x) = k$. Since the graph of any constant function is a horizontal line with slope zero at each point, the derivative should be zero. For example, if $f(x) = -4$, then $f'(3) = 0$. Alternatively, we can write $\frac{d}{dx}(-4) = 0$.



The Constant Function Rule

If $f(x) = k$, where k is a constant, then $f'(x) = 0$.

In Leibniz notation, $\frac{d}{dx}(k) = 0$.

Proof:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0}\frac{k-k}{h} && \text{(Since } f(x) = k \text{ and } f(x+h) = k \text{ for all } h\text{)} \\ &= \lim_{h\to 0} 0 \\ &= 0 \end{aligned}$$

EXAMPLE 1 **Applying the constant function rule**

a. If $f(x) = 5, f'(x) = 0$.

b. If $y = \frac{\pi}{2}, \frac{dy}{dx} = 0$.

A **power function** is a function of the form $f(x) = x^n$, where n is a real number. In the previous section, we observed that for $f(x) = x^2, f'(x) = 2x$; for $g(x) = \sqrt{x} = x^{\frac{1}{2}}, g'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$; and for $h(x) = \frac{1}{x} = x^{-1}, h'(x) = -x^{-2}$. As well, we hypothesized that $\frac{d}{dx}(x^n) = nx^{n-1}$. In fact, this is true and is called the **power rule**.

The Power Rule

If $f(x) = x^n$, where n is a real number, then $f'(x) = nx^{n-1}$.
In Leibniz notation, $\frac{d}{dx}(x^n) = nx^{n-1}$.

Proof:
(*Note:* n is a positive integer.)
Using the definition of the derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \text{ where } f(x) = x^n$$

$$= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \qquad \text{(Factor)}$$

$$= \lim_{h \to 0} \frac{(x+h-x)[(x+h)^{n-1} + (x+h)^{n-2}x + \ldots + (x+h)x^{n-2} + x^{n-1}]}{h}$$

$$= \lim_{h \to 0}[(x+h)^{n-1} + (x+h)^{n-2}x + \ldots + (x+h)x^{n-2} + x^{n-1}] \qquad \text{(Divide out } h\text{)}$$

$$= x^{n-1} + x^{n-2}(x) + \ldots + (x)x^{n-2} + x^{n-1}$$

$$= x^{n-1} + x^{n-1} + \ldots + x^{n-1} + x^{n-1} \qquad \text{(Since there are } n \text{ terms)}$$

$$= nx^{n-1}$$

EXAMPLE 2 Applying the power rule

a. If $f(x) = x^7$, then $f'(x) = 7x^6$.

b. If $g(x) = \frac{1}{x^3} = x^{-3}$, then $g'(x) = -3x^{-3-1} = -3x^{-4} = -\frac{3}{x^4}$.

c. If $s = t^{\frac{3}{2}}$, $\frac{ds}{dt} = \frac{3}{2}t^{\frac{1}{2}} = \frac{3}{2}\sqrt{t}$.

d. $\frac{d}{dx}(x) = 1x^{1-1} = x^0 = 1$

The Constant Multiple Rule

If $f(x) = kg(x)$, where k is a constant, then $f'(x) = kg'(x)$.
In Leibniz notation, $\frac{d}{dx}(ky) = k\frac{dy}{dx}$.

Proof:
Let $f(x) = kg(x)$. By the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{kg(x+h)-kg(x)}{h} && \text{(Factor)} \\ &= \lim_{h\to 0}k\left[\frac{g(x+h)-g(x)}{h}\right] \\ &= k\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right] && \text{(Property of limits)} \\ &= kg'(x) \end{aligned}$$

EXAMPLE 3 Applying the constant multiple rule

Differentiate the following functions:

a. $f(x) = 7x^3$

b. $y = 12x^{\frac{4}{3}}$

Solution

a. $f(x) = 7x^3$

$$f'(x) = 7\frac{d}{dx}(x^3) = 7(3x^2) = 21x^2$$

b. $y = 12x^{\frac{4}{3}}$

$$\frac{dy}{dx} = 12\frac{d}{dx}(x^{\frac{4}{3}}) = 12\left(\frac{4}{3}x^{(\frac{4}{3}-1)}\right) = 16x^{\frac{1}{3}}$$

We conclude this section with the sum and difference rules.

The Sum Rule

If functions $p(x)$ and $q(x)$ are differentiable, and $f(x) = p(x) + q(x)$, then $f'(x) = p'(x) + q'(x)$.

In Leibniz notation, $\frac{d}{dx}(f(x)) = \frac{d}{dx}(p(x)) + \frac{d}{dx}(q(x))$.

Proof:

Let $f(x) = p(x) + q(x)$. By the definition of the derivative,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0}\frac{[p(x+h) + q(x+h)] - [p(x) + q(x)]}{h} \\
&= \lim_{h\to 0}\left\{\frac{[p(x+h) - p(x)]}{h} + \frac{[q(x+h) - q(x)]}{h}\right\} \\
&= \lim_{h\to 0}\left\{\frac{[p(x+h) - p(x)]}{h}\right\} + \lim_{h\to 0}\left\{\frac{[q(x+h) - q(x)]}{h}\right\} \\
&= p'(x) + q'(x)
\end{aligned}$$

The Difference Rule

If functions $p(x)$ and $q(x)$ are differentiable, and $f(x) = p(x) - q(x)$, then $f'(x) = p'(x) - q'(x)$.

In Leibniz notation, $\frac{d}{dx}(f(x)) = \frac{d}{dx}(p(x)) - \frac{d}{dx}(q(x))$.

The proof for the difference rule is similar to the proof for the sum rule.

EXAMPLE 4 Selecting appropriate rules to determine the derivative

Differentiate the following functions:

a. $f(x) = 3x^2 - 5\sqrt{x}$

b. $y = (3x + 2)^2$

Solution

We apply the constant multiple, power, sum, and difference rules.

a. $f(x) = 3x^2 - 5\sqrt{x}$

$$f'(x) = \frac{d}{dx}(3x^2) - \frac{d}{dx}(5x^{\frac{1}{2}})$$
$$= 3\frac{d}{dx}(x^2) - 5\frac{d}{dx}(x^{\frac{1}{2}})$$
$$= 3(2x) - 5\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$
$$= 6x - \frac{5}{2}x^{-\frac{1}{2}}, \text{ or } 6x - \frac{5}{2\sqrt{x}}$$

b. We first expand $y = (3x + 2)^2$.

$$y = 9x^2 + 12x + 4$$
$$\frac{dy}{dx} = 9(2x) + 12(1) + 0$$
$$= 18x + 12$$

EXAMPLE 5 Selecting a strategy to determine the equation of a tangent

Determine the equation of the tangent to the graph of $f(x) = -x^3 + 3x^2 - 2$ at $x = 1$.

Solution A – Using the derivative

The slope of the tangent to the graph of f at any point is given by the derivative $f'(x)$.

For $f(x) = -x^3 + 3x^2 - 2$

$f'(x) = -3x^2 + 6x$

Now, $f'(1) = -3(1)^2 + 6(1)$

$= -3 + 6$

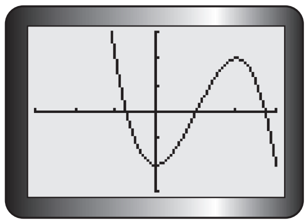
$= 3$

The slope of the tangent at $x = 1$ is 3 and the point of tangency is $(1, f(1)) = (1, 0)$.

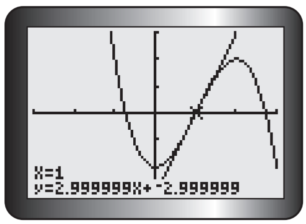
The equation of the tangent is $y - 0 = 3(x - 1)$ or $y = 3x - 3$.

Solution B – Using the graphing calculator

Draw the graph of the function using the graphing calculator.



Draw the tangent at the point on the function where $x = 1$. The calculator displays the equation of the tangent line.



The equation of the tangent line in this case is $y = 3x - 3$.

Tech Support

For help using the graphing calculator to graph functions and draw tangent lines See Technical Appendices p. 597 and p. 608.

EXAMPLE 6 Connecting the derivative to horizontal tangents

Determine points on the graph in Example 5 where the tangents are horizontal.

Solution

Horizontal lines have slope zero. We need to find the values of x that satisfy $f'(x) = 0$.

$-3x^2 + 6x = 0$
$-3x(x - 2) = 0$
$x = 0$ or $x = 2$

The graph of $f(x) = -x^3 + 3x^2 - 2$ has horizontal tangents at $(0, -2)$ and $(2, 2)$.

IN SUMMARY

Key Ideas

The following table summarizes the derivative rules in this section.

Rule	Function Notation	Leibniz Notation
Constant Function Rule	If $f(x) = k$, where k is a constant, then $f'(x) = 0$.	$\frac{d}{dx}(k) = 0$
Power Rule	If $f(x) = x^n$, where n is a real number, then $f'(x) = nx^{n-1}$.	$\frac{d}{dx}(x^n) = nx^{n-1}$
Constant Multiple Rule	If $f(x) = kg(x)$, then $f'(x) = kg'(x)$.	$\frac{d}{dx}(ky) = k\frac{dy}{dx}$
Sum Rule	If $f(x) = p(x) + q(x)$, then $f'(x) = p'(x) + q'(x)$.	$\frac{d}{dx}(f(x)) = \frac{d}{dx}(p(x)) + \frac{d}{dx}(q(x))$
Difference Rule	If $f(x) = p(x) - q(x)$, then $f'(x) = p'(x) - q'(x)$.	$\frac{d}{dx}(f(x)) = \frac{d}{dx}(p(x)) - \frac{d}{dx}(q(x))$

Need to Know

- To determine the derivative of a simple rational function, such as $f(x) = \frac{4}{x^6}$, express the function as a power, then use the power rule.
  If $f(x) = 4x^{-6}$, then $f'(x) = 4(-6)x^{(-6-1)} = -24x^{-7}$
- If you have a radical function such as $g(x) = \sqrt[3]{x^5}$, rewrite the function as $g(x) = x^{\frac{5}{3}}$, then use the power rule.
  If $g(x) = x^{\frac{5}{3}}$, then $g'(x) = \frac{5}{3}x^{\frac{2}{3}} = \frac{5}{3}\sqrt[3]{x^2}$

Exercise 2.2

PART A

1. What rules do you know for calculating derivatives? Give examples of each rule.

2. Determine $f'(x)$ for each of the following functions:

a. $f(x) = 4x - 7$
b. $f(x) = x^3 - x^2$
c. $f(x) = -x^2 + 5x + 8$
d. $f(x) = \sqrt[3]{x}$
e. $f(x) = \left(\frac{x}{2}\right)^4$
f. $f(x) = x^{-3}$

K 3. Differentiate each function. Use either Leibniz notation or prime notation, depending on which is appropriate.

a. $h(x) = (2x + 3)(x + 4)$
b. $f(x) = 2x^3 + 5x^2 - 4x - 3.75$
c. $s = t^2(t^2 - 2t)$
d. $y = \frac{1}{5}x^5 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + 1$
e. $g(x) = 5(x^2)^4$
f. $s(t) = \frac{t^5 - 3t^2}{2t}, t > 0$

4. Apply the differentiation rules you learned in this section to find the derivatives of the following functions:

a. $y = 3x^{\frac{5}{3}}$
b. $y = 4x^{-\frac{1}{2}} - \frac{6}{x}$
c. $y = \frac{6}{x^3} + \frac{2}{x^2} - 3$
d. $y = 9x^{-2} + 3\sqrt{x}$
e. $y = \sqrt{x} + 6\sqrt{x^3} + \sqrt{2}$
f. $y = \frac{1 + \sqrt{x}}{x}$

PART B

5. Let s represent the position of a moving object at time t. Find the velocity $v = \frac{ds}{dt}$ at time t.

a. $s = -2t^2 + 7t$
b. $s = 18 + 5t - \frac{1}{3}t^3$
c. $s = (t - 3)^2$

6. Determine $f'(a)$ for the given function $f(x)$ at the given value of a.

a. $f(x) = x^3 - \sqrt{x}, a = 4$
b. $f(x) = 7 - 6\sqrt{x} + 5x^{\frac{2}{3}}, a = 64$

7. Determine the slope of the tangent to each of the curves at the given point.

a. $y = 3x^4, (1, 3)$
b. $y = \frac{1}{x^{-5}}, (-1, -1)$
c. $y = \frac{2}{x}, (-2, -1)$
d. $y = \sqrt{16x^3}, (4, 32)$

8. Determine the slope of the tangent to the graph of each function at the point with the given x-coordinate.

a. $y = 2x^3 + 3x, x = 1$

b. $y = 2\sqrt{x} + 5, x = 4$

c. $y = \frac{16}{x^2}, x = -2$

d. $y = x^{-3}(x^{-1} + 1), x = 1$

9. Write an equation of the tangent to each curve at the given point.

a. $y = 2x - \frac{1}{x}, P(0.5, -1)$

b. $y = \frac{3}{x^2} - \frac{4}{x^3}, P(-1, 7)$

c. $y = \sqrt{3x^3}, P(3, 9)$

d. $y = \frac{1}{x}\left(x^2 + \frac{1}{x}\right), P(1, 2)$

e. $y = (\sqrt{x} - 2)(3\sqrt{x} + 8), P(4, 0)$

f. $y = \frac{\sqrt{x} - 2}{\sqrt[3]{x}}, P(1, -1)$

C 10. What is a normal to the graph of a function? Determine the equation of the normal to the graph of the function in question 9, part b., at the given point.

T 11. Determine the values of x so that the tangent to the function $y = \frac{3}{\sqrt[3]{x}}$ is parallel to the line $x + 16y + 3 = 0$.

12. Do the functions $y = \frac{1}{x}$ and $y = x^3$ ever have the same slope? If so, where?

13. Tangents are drawn to the parabola $y = x^2$ at $(2, 4)$ and $\left(-\frac{1}{8}, \frac{1}{64}\right)$. Prove that these lines are perpendicular. Illustrate with a sketch.

14. Determine the point on the parabola $y = -x^2 + 3x + 4$ where the slope of the tangent is 5. Illustrate your answer with a sketch.

15. Determine the coordinates of the points on the graph of $y = x^3 + 2$ at which the slope of the tangent is 12.

16. Show that there are two tangents to the curve $y = \frac{1}{5}x^5 - 10x$ that have a slope of 6.

17. Determine the equations of the tangents to the curve $y = 2x^2 + 3$ that pass through the following points:

a. point $(2, 3)$

b. point $(2, -7)$

18. Determine the value of a, given that the line $ax - 4y + 21 = 0$ is tangent to the graph of $y = \frac{a}{x^2}$ at $x = -2$.

A 19. It can be shown that, from a height of h metres, a person can see a distance of d kilometres to the horizon, where $d = 3.53\sqrt{h}$.

a. When the elevator of the CN Tower passes the 200 m height, how far can the passengers in the elevator see across Lake Ontario?

b. Find the rate of change of this distance with respect to height when the height of the elevator is 200 m.

20. An object drops from a cliff that is 150 m high. The distance, d, in metres, that the object has dropped at t seconds in modelled by $d(t) = 4.9t^2$.

    a. Find the average rate of change of distance with respect to time from 2 s to 5 s.

    b. Find the instantaneous rate of change of distance with respect to time at 4 s.

    c. Find the rate at which the object hits the ground to the nearest tenth.

21. A subway train travels from one station to the next in 2 min. Its distance, in kilometres, from the first station after t minutes is $s(t) = t^2 - \frac{1}{3}t^3$. At what times will the train have a velocity of 0.5 km/min?

22. While working on a high-rise building, a construction worker drops a bolt from 320 m above the ground. After t seconds, the bolt has fallen a distance of s metres, where $s(t) = 5t^2$, $0 \leq t \leq 8$. The function that gives the height of the bolt above ground at time t is $R(t) = 320 - 5t^2$. Use this function to determine the velocity of the bolt at $t = 2$.

23. Tangents are drawn from the point $(0, 3)$ to the parabola $y = -3x^2$. Find the coordinates of the points at which these tangents touch the curve. Illustrate your answer with a sketch.

24. The tangent to the cubic function that is defined by $y = x^3 - 6x^2 + 8x$ at point $A(3, -3)$ intersects the curve at another point, B. Find the coordinates of point B. Illustrate with a sketch.

25. a. Find the coordinates of the points, if any, where each function has a horizontal tangent line.

    i. $f(x) = 2x - 5x^2$

    ii. $f(x) = 4x^2 + 2x - 3$

    iii. $f(x) = x^3 - 8x^2 + 5x + 3$

    b. Suggest a graphical interpretation for each of these points.

PART C

26. Let $P(a, b)$ be a point on the curve $\sqrt{x} + \sqrt{y} = 1$. Show that the slope of the tangent at P is $-\sqrt{\frac{b}{a}}$.

27. For the power function $f(x) = x^n$, find the x-intercept of the tangent to its graph at point $(1, 1)$. What happens to the x-intercept as n increases without bound $(n \to +\infty)$? Explain the result geometrically.

28. For each function, sketch the graph of $y = f(x)$ and find an expression for $f'(x)$. Indicate any points at which $f'(x)$ does not exist.

    a. $f(x) = \begin{cases} x^2, x < 3 \\ x + 6, x \geq 3 \end{cases}$ b. $f(x) = |3x^2 - 6|$ c. $f(x) = ||x| - 1|$

Section 2.3—The Product Rule

In this section, we will develop a rule for differentiating the product of two functions, such as $f(x) = (3x^2 - 1)(x^3 + 8)$ and $g(x) = (x - 3)^3(x + 2)^2$, without first expanding the expressions.

You might suspect that the derivative of a product of two functions is simply the product of the separate derivatives. An example shows that this is not so.

EXAMPLE 1 **Reasoning about the derivative of a product of two functions**

Let $p(x) = f(x)g(x)$, where $f(x) = (x^2 + 2)$ and $g(x) = (x + 5)$.
Show that $p'(x) \neq f'(x)g'(x)$.

Solution

The expression $p(x)$ can be simplified.

$$p(x) = (x^2 + 2)(x + 5)$$
$$= x^3 + 5x^2 + 2x + 10$$
$$p'(x) = 3x^2 + 10x + 2$$
$$f'(x) = 2x \text{ and } g'(x) = 1, \text{ so } f'(x)g'(x) = (2x)(1) = 2x.$$

Since $2x$ is not the derivative of $p(x)$, we have shown that $p'(x) \neq f'(x)g'(x)$.

The correct method for differentiating a product of two functions uses the following rule.

The Product Rule

If $p(x) = f(x)g(x)$, then $p'(x) = f'(x)g(x) + f(x)g'(x)$.

If u and v are functions of x, $\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$.

In words, the product rule says, "the derivative of the product of two functions is equal to the derivative of the first function times the second function plus the first function times the derivative of the second function."

Proof:

$p(x) = f(x)g(x)$, so using the definition of the derivative,

$$p'(x) = \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x)g(x)}{h}.$$

To evaluate $p'(x)$, we subtract and add the same term in the numerator.

$$\begin{aligned}\text{Now, } p'(x) &= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}\\ &= \lim_{h\to 0}\left\{\left[\frac{f(x+h)-f(x)}{h}\right]g(x+h) + f(x)\left[\frac{g(x+h)-g(x)}{h}\right]\right\}\\ &= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]\lim_{h\to 0}g(x+h) + \lim_{h\to 0}f(x)\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]\\ &= f'(x)g(x) + f(x)g'(x)\end{aligned}$$

EXAMPLE 2

Applying the product rule

Differentiate $h(x) = (x^2 - 3x)(x^5 + 2)$ using the product rule.

Solution

$h(x) = (x^2 - 3x)(x^5 + 2)$

Using the product rule, we get

$$\begin{aligned}h'(x) &= \frac{d}{dx}[x^2 - 3x]\cdot(x^5 + 2) + (x^2 - 3x)\frac{d}{dx}[x^5 + 2]\\ &= (2x - 3)(x^5 + 2) + (x^2 - 3x)(5x^4)\\ &= 2x^6 - 3x^5 + 4x - 6 + 5x^6 - 15x^5\\ &= 7x^6 - 18x^5 + 4x - 6\end{aligned}$$

We can, of course, differentiate the function after we first expand. The product rule will be essential, however, when we work with products of polynomials such as $f(x) = (x^2 + 9)(x^3 + 5)^4$ or non-polynomial functions such as $f(x) = (x^2 + 9)\sqrt{x^3 + 5}$.

It is not necessary to simplify an expression when you are asked to calculate the derivative at a particular value of x. Because many expressions obtained using differentiation rules are cumbersome, it is easier to substitute, then evaluate the derivative expression.

The next example could be solved by finding the product of the two polynomials and then calculating the derivative of the resulting polynomial at $x = -1$. Instead, we will apply the product rule.

EXAMPLE 3 Selecting an efficient strategy to determine the value of the derivative

Find the value $h'(-1)$ for the function $h(x) = (5x^3 + 7x^2 + 3)(2x^2 + x + 6)$.

Solution

$h(x) = (5x^3 + 7x^2 + 3)(2x^2 + x + 6)$

Using the product rule, we get

$$\begin{aligned} h'(x) &= (15x^2 + 14x)(2x^2 + x + 6) + (5x^3 + 7x^2 + 3)(4x + 1) \\ h'(-1) &= [15(-1)^2 + 14(-1)][2(-1)^2 + (-1) + 6] \\ &\quad + [5(-1)^3 + 7(-1)^2 + 3][4(-1) + 1] \\ &= (1)(7) + (5)(-3) \\ &= -8 \end{aligned}$$

The following example illustrates the extension of the product rule to more than two functions.

EXAMPLE 4 Connecting the product rule to a more complex function

Find an expression for $p'(x)$ if $p(x) = f(x)g(x)h(x)$.

Solution

We temporarily regard $f(x)g(x)$ as a single function.

$$p(x) = [f(x)g(x)]h(x)$$

By the product rule,

$$p'(x) = [f(x)g(x)]'h(x) + [f(x)g(x)]h'(x)$$

A second application of the product rule yields

$$\begin{aligned} p'(x) &= [f'(x)g(x) + f(x)g'(x)]h(x) + f(x)g(x)h'(x) \\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x) \end{aligned}$$

This expression gives us the **extended product rule** for the derivative of a product of three functions. Its symmetrical form makes it easy to extend to a product of four or more functions.

The Power of a Function Rule for Positive Integers

Suppose that we now wish to differentiate functions such as $y = (x^2 - 3)^4$ or $y = (x^2 + 3x + 5)^6$.

These functions are of the form $y = u^n$, where n is a positive integer and $u = g(x)$ is a function whose derivative we can find. Using the product rule, we can develop an efficient method for differentiating such functions.

For $n = 2$,

$h(x) = [g(x)]^2$

$h(x) = g(x)g(x)$

Using the product rule,

$$\begin{aligned} h'(x) &= g'(x)g(x) + g(x)g'(x) \\ &= 2g'(x)g(x) \end{aligned}$$

Similarly, for $n = 3$, we can use the extended product rule.

$$\begin{aligned} \text{Thus, } h(x) &= [g(x)]^3 \\ &= g(x)g(x)g(x) \\ h'(x) &= g'(x)g(x)g(x) + g(x)g'(x)g(x) + g(x)g(x)g'(x) \\ &= 3[g(x)]^2 g'(x) \end{aligned}$$

These results suggest a generalization of the power rule.

The Power of a Function Rule for Integers

If u is a function of x, and n is an integer, then $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$.
In function notation, if $f(x) = [g(x)]^n$, then $f'(x) = n[g(x)]^{n-1}g'(x)$.

The power of a function rule is a *special case* of the chain rule, which we will discuss later in this chapter. We are now able to differentiate any polynomial, such as $h(x) = (x^2 + 3x + 5)^6$ or $h(x) = (1 - x^2)^4(2x + 6)^3$, without multiplying out the brackets. We can also differentiate rational functions, such as $f(x) = \frac{2x + 5}{3x - 1}$.

EXAMPLE 5 Applying the power of a function rule

For $h(x) = (x^2 + 3x + 5)^6$, find $h'(x)$.

Solution

Here $h(x)$ has the form $h(x) = [g(x)]^6$, where the "inner" function is $g(x) = x^2 + 3x + 5$.

By the power of a function rule, we get $h'(x) = 6(x^2 + 3x + 5)^5(2x + 3)$.

EXAMPLE 6 Selecting a strategy to determine the derivative of a rational function

Differentiate the rational function $f(x) = \frac{2x + 5}{3x - 1}$ by first expressing it as a product and then using the product rule.

Solution

$$\begin{aligned}
f(x) &= \frac{2x + 5}{3x - 1} \\
&= (2x + 5)(3x - 1)^{-1} && \text{(Express } f \text{ as a product)} \\
f'(x) &= \frac{d}{dx}[(2x + 5)](3x - 1)^{-1} + (2x + 5)\frac{d}{dx}\left[(3x - 1)^{-1}\right] && \text{(Product rule)} \\
&= 2(3x - 1)^{-1} + (2x + 5)(-1)(3x - 1)^{-2}\frac{d}{dx}(3x - 1) \\
& && \text{(Power of a function rule)} \\
&= 2(3x - 1)^{-1} - 3(2x + 5)(3x - 1)^{-2} \\
&= \frac{2}{(3x - 1)} - \frac{3(2x + 5)}{(3x - 1)^2} && \text{(Simplify)} \\
&= \frac{2(3x - 1)}{(3x - 1)^2} - \frac{6x + 15}{(3x - 1)^2} \\
&= \frac{6x - 2 - 6x - 15}{(3x - 1)^2} \\
&= \frac{-17}{(3x - 1)^2}
\end{aligned}$$

EXAMPLE 7 Using the derivative to solve a problem

The position s, in centimetres, of an object moving in a straight line is given by $s = t(6 - 3t)^4$, $t \geq 0$, where t is the time in seconds. Determine the object's velocity at $t = 2$.

Solution

The velocity of the object at any time $t \geq 0$ is $v = \frac{ds}{dt}$.

$$\begin{aligned}
v &= \frac{d}{dt}[t(6 - 3t)^4] \\
&= (1)(6 - 3t)^4 + (t)\frac{d}{dt}[(6 - 3t)^4] && \text{(Product rule)} \\
&= (6 - 3t)^4 + (t)[4(6 - 3t)^3(-3)] && \text{(Power of a function rule)}
\end{aligned}$$

At $t = 2$, $v = 0 + (2)[4(0)(-3)]$

$= 0$

We conclude that the object is at rest at $t = 2$ s.

IN SUMMARY

Key Ideas

- The derivative of a product of differentiable functions is not the product of their derivatives.
- The **product rule** for differentiation:
  If $h(x) = f(x)g(x)$, then $h'(x) = f'(x)g(x) + f(x)g'(x)$.
- The **power of a function rule** for integers:
  If $f(x) = [g(x)]^n$, then $f'(x) = n[g(x)]^{n-1}g'(x)$.

Need to Know

- In some cases, it is easier to expand and simplify the product before differentiating, rather than using the product rule.
  $$\begin{aligned} \text{If } f(x) &= 3x^4(5x^3 - 7) \\ &= 15x^7 - 21x^4 \\ f'(x) &= 105x^6 - 84x^3 \end{aligned}$$
- If the derivative is needed at a particular value of the independent variable, it is not necessary to simplify before substituting.

Exercise 2.3

PART A

1. Use the product rule to differentiate each function. Simplify your answers.
   a. $h(x) = x(x - 4)$
   b. $h(x) = x^2(2x - 1)$
   c. $h(x) = (3x + 2)(2x - 7)$
   d. $h(x) = (5x^7 + 1)(x^2 - 2x)$
   e. $s(t) = (t^2 + 1)(3 - 2t^2)$
   f. $f(x) = \dfrac{x - 3}{x + 3}$

K 2. Use the product rule and the power of a function rule to differentiate the following functions. Do not simplify.
   a. $y = (5x + 1)^3(x - 4)$
   b. $y = (3x^2 + 4)(3 + x^3)^5$
   c. $y = (1 - x^2)^4(2x + 6)^3$
   d. $y = (x^2 - 9)^4(2x - 1)^3$

3. When is it not appropriate to use the product rule? Give examples.

4. Let $F(x) = [b(x)][c(x)]$. Express $F'(x)$ in terms of $b(x)$ and $c(x)$.

PART B

5. Determine the value of $\frac{dy}{dx}$ for the given value of x.
   a. $y = (2 + 7x)(x - 3), x = 2$
   b. $y = (1 - 2x)(1 + 2x), x = \frac{1}{2}$
   c. $y = (3 - 2x - x^2)(x^2 + x - 2), x = -2$
   d. $y = x^3(3x + 7)^2, x = -2$
   e. $y = (2x + 1)^5(3x + 2)^4, x = -1$
   f. $y = x(5x - 2)(5x + 2), x = 3$
6. Determine the equation of the tangent to the curve $y = (x^3 - 5x + 2)(3x^2 - 2x)$ at the point $(1, -2)$.
7. Determine the point(s) where the tangent to the curve is horizontal.
   a. $y = 2(x - 29)(x + 1)$
   b. $y = (x^2 + 2x + 1)(x^2 + 2x + 1)$
8. Use the extended product rule to differentiate the following functions. Do not simplify.
   a. $y = (x + 1)^3(x + 4)(x - 3)^2$
   b. $y = x^2(3x^2 + 4)^2(3 - x^3)^4$

A

9. A 75 L gas tank has a leak. After t hours, the remaining volume, V, in litres is $V(t) = 75\left(1 - \frac{t}{24}\right)^2, 0 \le t \le 24$. Use the product rule to determine how quickly the gas is leaking from the tank when the tank is 60% full of gas.

C

10. Determine the slope of the tangent to $h(x) = 2x(x + 1)^3(x^2 + 2x + 1)^2$ at $x = -2$. Explain how to find the equation of the normal at $x = -2$.

PART C

T

11. a. Determine an expression for $f'(x)$ if $f(x) = g_1(x)g_2(x)g_3(x) \ldots g_{n-1}(x)g_n(x)$.
    b. If $f(x) = (1 + x)(1 + 2x)(1 + 3x) \ldots (1 + nx)$, find $f'(0)$.
12. Determine a quadratic function $f(x) = ax^2 + bx + c$ if its graph passes through the point $(2, 19)$ and it has a horizontal tangent at $(-1, -8)$.
13. Sketch the graph of $f(x) = |x^2 - 1|$.
    a. For what values of x is f not differentiable?
    b. Find a formula for f', and sketch the graph of f'.
    c. Find $f'(x)$ at $x = -2$, 0, and 3.
14. Show that the line $4x - y + 11 = 0$ is tangent to the curve $y = \frac{16}{x^2} - 1$.

Mid-Chapter Review

1. a. Sketch the graph of $f(x) = x^2 - 5x$.
   b. Calculate the slopes of the tangents to $f(x) = x^2 - 5x$ at points with x-coordinates 0, 1, 2, ..., 5.
   c. Sketch the graph of the derivative function $f'(x)$.
   d. Compare the graphs of $f(x)$ and $f'(x)$.
2. Use the definition of the derivative to find $f'(x)$ for each function.
   a. $f(x) = 6x + 15$
   b. $f(x) = 2x^2 - 4$
   c. $f(x) = \dfrac{5}{x + 5}$
   d. $f(x) = \sqrt{x - 2}$
3. a. Determine the equation of the tangent to the curve $y = x^2 - 4x + 3$ at $x = 1$.
   b. Sketch the graph of the function and the tangent.
4. Differentiate each of the following functions:
   a. $y = 6x^4$
   b. $y = 10x^{\frac{1}{2}}$
   c. $g(x) = \dfrac{2}{x^3}$
   d. $y = 5x + \dfrac{3}{x^2}$
   e. $y = (11t + 1)^2$
   f. $y = \dfrac{x - 1}{x}$
5. Determine the equation of the tangent to the graph of $f(x) = 2x^4$ that has slope 1.
6. Determine $f'(x)$ for each of the following functions:
   a. $f(x) = 4x^2 - 7x + 8$
   b. $f(x) = -2x^3 + 4x^2 + 5x - 6$
   c. $f(x) = \dfrac{5}{x^2} - \dfrac{3}{x^3}$
   d. $f(x) = \sqrt{x} + \sqrt[3]{x}$
   e. $f(x) = 7x^{-2} - 3\sqrt{x}$
   f. $f(x) = -4x^{-1} + 5x - 1$
7. Determine the equation of the tangent to the graph of each function.
   a. $y = -3x^2 + 6x + 4$ when $x = 1$
   b. $y = 3 - 2\sqrt{x}$ when $x = 9$
   c. $f(x) = -2x^4 + 4x^3 - 2x^2 - 8x + 9$ when $x = 3$
8. Determine the derivative using the product rule.
   a. $f(x) = (4x^2 - 9x)(3x^2 + 5)$
   b. $f(t) = (-3t^2 - 7t + 8)(4t - 1)$
   c. $y = (3x^2 + 4x - 6)(2x^2 - 9)$
   d. $y = (3 - 2x^3)^3$

9. Determine the equation of the tangent to
$y = (5x^2 + 9x - 2)(-x^2 + 2x + 3)$ at $(1, 48)$.

10. Determine the point(s) where the tangent to the curve $y = 2(x - 1)(5 - x)$ is horizontal.

11. If $y = 5x^2 - 8x + 4$, determine $\frac{dy}{dx}$ from first principles.

12. A tank holds 500 L of liquid, which takes 90 min to drain from a hole in the bottom of the tank. The volume, V, remaining in the tank after t minutes is

$$V(t) = 500\left(1 - \frac{t}{90}\right)^2, \text{ where } 0 \leq t \leq 90$$

a. How much liquid remains in the tank at 1 h?

b. What is the average rate of change of volume with respect to time from 0 min to 60 min?

c. How fast is the liquid draining at 30 min?

13. The volume of a sphere is given by $V(r) = \frac{4}{3}\pi r^3$.

a. Determine the average rate of change of volume with respect to radius as the radius changes from 10 cm to 15 cm.

b. Determine the rate of change of volume when the radius is 8 cm.

14. A classmate says, "The derivative of a cubic polynomial function is a quadratic polynomial function." Is the statement always true, sometimes true, or never true? Defend your choice in words, and provide two examples to support your argument.

15. Show that $\frac{dy}{dx} = (a + 4b)x^{a+4b-1}$ if $y = \frac{x^{2a+3b}}{x^{a-b}}$ and a and b are integers.

16. a. Determine $f'(3)$, where $f(x) = -6x^3 + 4x - 5x^2 + 10$.

b. Give two interpretations of the meaning of $f'(3)$.

17. The population, P, of a bacteria colony at t hours can be modelled by

$$P(t) = 100 + 120t + 10t^2 + 2t^3$$

a. What is the initial population of the bacteria colony?

b. What is the population of the colony at 5 h?

c. What is the growth rate of the colony at 5 h?

18. The relative percent of carbon dioxide, C, in a carbonated soft drink at t minutes can be modelled by $C(t) = \frac{100}{t}$, where $t > 2$. Determine $C'(t)$ and interpret the results at 5 min, 50 min, and 100 min. Explain what is happening.

Section 2.4—The Quotient Rule

In the previous section, we found that the derivative of the product of two functions is not the product of their derivatives. The quotient rule gives the derivative of a function that is the quotient of two functions. It is derived from the product rule.

The Quotient Rule

If $h(x) = \frac{f(x)}{g(x)}$, then $h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$, $g(x) \neq 0$.

In Leibniz notation, $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$.

Proof:

We want to find $h'(x)$, given that $\qquad h(x) = \frac{f(x)}{g(x)}, \; g(x) \neq 0.$

We rewrite this as a product: $\qquad h(x)g(x) = f(x).$

Using the product rule, $h'(x)g(x) + h(x)g'(x) = f'(x).$

Solving for $h'(x)$, we get
$$h'(x) = \frac{f'(x) - h(x)g'(x)}{g(x)}$$
$$= \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)}$$
$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

The quotient rule provides us with an alternative approach to differentiate rational functions, in addition to what we learned last section.

Memory Aid for the Product and Quotient Rules

It is worth noting that the quotient rule is similar to the product rule in that both have $f'(x)g(x)$ and $f(x)g'(x)$. For the product rule, we put an addition sign between the terms. For the quotient rule, we put a subtraction sign between the terms and then divide the result by the square of the original denominator.

Take note that in the quotient rule the $f'(x)g(x)$ term must come first. This isn't the case with the product rule.

EXAMPLE 1

Applying the quotient rule

Determine the derivative of $h(x) = \frac{3x - 4}{x^2 + 5}$.

Solution

Since $h(x) = \frac{f(x)}{g(x)}$, where $f(x) = 3x - 4$ and $g(x) = x^2 + 5$, use the quotient rule to find $h'(x)$.

Using the quotient rule, we get
$$\begin{aligned} h'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \\ &= \frac{(3)(x^2 + 5) - (3x - 4)(2x)}{(x^2 + 5)^2} \\ &= \frac{3x^2 + 15 - 6x^2 + 8x}{(x^2 + 5)^2} \\ &= \frac{-3x^2 + 8x + 15}{(x^2 + 5)^2} \end{aligned}$$

EXAMPLE 2

Selecting a strategy to determine the equation of a line tangent to a rational function

Determine the equation of the tangent to $y = \dfrac{2x}{x^2 + 1}$ at $x = 0$.

Solution A – Using the derivative

The slope of the tangent to the graph of f at any point is given by the derivative $\frac{dy}{dx}$.

By the quotient rule,
$$\frac{dy}{dx} = \frac{(2)(x^2 + 1) - (2x)(2x)}{(x^2 + 1)^2}$$

At $x = 0$,
$$\frac{dy}{dx} = \frac{(2)(0 + 1) - (0)(0)}{(0 + 1)^2} = 2$$

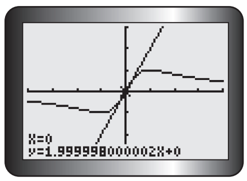
The slope of the tangent at $x = 0$ is 2 and the point of tangency is $(0, 0)$. The equation of the tangent is $y = 2x$.

Tech *Support*

For help using the graphing calculator to graph functions and draw tangent lines see Technical Appendices p. 597 and p. 608.

Solution B – Using the graphing calculator

Draw the graph of the function using the graphing calculator.



Draw the tangent at the point on the function where $x = 0$. The calculator displays the equation of the tangent line.

The equation of the tangent line in this case is $y = 2x$.

EXAMPLE 3 Using the quotient rule to solve a problem

Determine the coordinates of each point on the graph of $f(x) = \dfrac{2x + 8}{\sqrt{x}}$ where the tangent is horizontal.

Solution

The slope of the tangent at any point on the graph is given by $f'(x)$. Using the quotient rule,

$$f'(x) = \frac{(2)(\sqrt{x}) - (2x + 8)\left(\frac{1}{2}x^{-\frac{1}{2}}\right)}{(\sqrt{x})^2}$$

$$= \frac{2\sqrt{x} - \dfrac{2x + 8}{2\sqrt{x}}}{x}$$

$$= \frac{\dfrac{2x}{\sqrt{x}} - \dfrac{x + 4}{\sqrt{x}}}{x}$$

$$= \frac{\dfrac{2x - x - 4}{\sqrt{x}}}{x}$$

$$= \frac{x - 4}{x\sqrt{x}}$$

The tangent will be horizontal when $f'(x) = 0$; that is, when $x = 4$. The point on the graph where the tangent is horizontal is $(4, 8)$.

IN SUMMARY

Key Ideas

- The derivative of a quotient of two differentiable functions is not the quotient of their derivatives.
- The **quotient rule** for differentiation:

  If $h(x) = \dfrac{f(x)}{g(x)}$, then $h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$, $g(x) \neq 0$.

Need to Know

- To find the derivative of a rational function, you can use two methods:

Leave the function in fraction form, and use the quotient rule.	OR	Express the function as a product, and use the product and power of a function rules.
$f(x) = \dfrac{x - 2}{1 + x}$		$f(x) = (x - 2)(1 + x)^{-1}$

Exercise 2.4

PART A

1. What are the exponent rules? Give examples of each rule.
2. Copy the table, and complete it *without* using the quotient rule.

Function	Rewrite	Differentiate and Simplify, if Necessary
$f(x) = \frac{x^2 + 3x}{x}, x \neq 0$		
$g(x) = \frac{3x^{\frac{5}{3}}}{x}, x \neq 0$		
$h(x) = \frac{1}{10x^5}, x \neq 0$		
$y = \frac{8x^3 + 6x}{2x}, x \neq 0$		
$s = \frac{t^2 - 9}{t - 3}, t \neq 3$		

C 3. What are the different ways to find the derivative of a rational function? Give examples.

PART B

K 4. Use the quotient rule to differentiate each function. Simplify your answers.

a. $h(x) = \frac{x}{x + 1}$
b. $h(t) = \frac{2t - 3}{t + 5}$
c. $h(x) = \frac{x^3}{2x^2 - 1}$
d. $h(x) = \frac{1}{x^2 + 3}$
e. $y = \frac{x(3x + 5)}{1 - x^2}$
f. $y = \frac{x^2 - x + 1}{x^2 + 3}$

5. Determine $\frac{dy}{dx}$ at the given value of x.

a. $y = \frac{3x + 2}{x + 5}, x = -3$
b. $y = \frac{x^3}{x^2 + 9}, x = 1$
c. $y = \frac{x^2 - 25}{x^2 + 25}, x = 2$
d. $y = \frac{(x + 1)(x + 2)}{(x - 1)(x - 2)}, x = 4$

6. Determine the slope of the tangent to the curve $y = \frac{x^3}{x^2 - 6}$ at point $(3, 9)$.
7. Determine the points on the graph of $y = \frac{3x}{x - 4}$ where the slope of the tangent is $-\frac{12}{25}$.

T 8. Show that there are no tangents to the graph of $f(x) = \frac{5x + 2}{x + 2}$ that have a negative slope.

9. Find the point(s) at which the tangent to the curve is horizontal.

a. $y = \dfrac{2x^2}{x - 4}$

b. $y = \dfrac{x^2 - 1}{x^2 + x - 2}$

A 10. An initial population, p, of 1000 bacteria grows in number according to the equation $p(t) = 1000\left(1 + \dfrac{4t}{t^2 + 50}\right)$, where t is in hours. Find the rate at which the population is growing after 1 h and after 2 h.

11. Determine the equation of the tangent to the curve $y = \dfrac{x^2 - 1}{3x}$ at $x = 2$.

12. A motorboat coasts toward a dock with its engine off. Its distance s, in metres, from the dock t seconds after the engine is turned off is $s(t) = \dfrac{10(6 - t)}{t + 3}$ for $0 \leq t \leq 6$.

a. How far is the boat from the dock initially?

b. Find the velocity of the boat when it bumps into the dock.

13. a. The radius of a circular juice blot on a piece of paper towel t seconds after it was first seen is modelled by $r(t) = \dfrac{1 + 2t}{1 + t}$, where r is measured in centimetres. Calculate

i. the radius of the blot when it was first observed

ii. the time at which the radius of the blot was 1.5 cm

iii. the rate of increase of the area of the blot when the radius was 1.5 cm

b. According to this model, will the radius of the blot ever reach 2 cm? Explain your answer.

14. The graph of $f(x) = \dfrac{ax + b}{(x - 1)(x - 4)}$ has a horizontal tangent line at $(2, -1)$. Find a and b. Check using a graphing calculator.

15. The concentration, c, of a drug in the blood t hours after the drug is taken orally is given by $c(t) = \dfrac{5t}{2t^2 + 7}$. When does the concentration reach its maximum value?

16. The position from its starting point, s, of an object that moves in a straight line at time t seconds is given by $s(t) = \dfrac{t}{t^2 + 8}$. Determine when the object changes direction.

PART C

17. Consider the function $f(x) = \dfrac{ax + b}{cx + d}$, $x \neq -\dfrac{d}{c}$, where a, b, c, and d are nonzero constants. What condition on a, b, c, and d ensures that each tangent to the graph of f has a positive slope?

Section 2.5—The Derivatives of Composite Functions

Recall that one way of combining functions is through a process called **composition**. We start with a number x in the domain of g, find its image $g(x)$, and then take the value of f at $g(x)$, provided that $g(x)$ is in the domain of f. The result is the new function $h(x) = f(g(x))$, which is called the **composite function** of f and g, and is denoted $(f \circ g)$.

Definition of a composite function

Given two functions f and g, the **composite function** $(f \circ g)$ is defined by $(f \circ g)(x) = f(g(x))$.

EXAMPLE 1 **Reflecting on the process of composition**

If $f(x) = \sqrt{x}$ and $g(x) = x + 5$, find each of the following values:

a. $f(g(4))$ b. $g(f(4))$ c. $f(g(x))$ d. $g(f(x))$

Solution

a. Since $g(4) = 9$, we have $f(g(4)) = f(9) = 3$.

b. Since $f(4) = 2$, we have $g(f(4)) = g(2) = 7$. *Note:* $f(g(4)) \neq g(f(4))$.

c. $f(g(x)) = f(x + 5) = \sqrt{x + 5}$

d. $g(f(x)) = g(\sqrt{x}) = \sqrt{x} + 5$ *Note:* $f(g(x)) \neq g(f(x))$.

The chain rule states how to compute the derivative of the composite function $h(x) = f(g(x))$ in terms of the derivatives of f and g.

The Chain Rule

If f and g are functions that have derivatives, then the composite function $h(x) = f(g(x))$ has a derivative given by $h'(x) = f'(g(x))g'(x)$.

In words, the chain rule says, "the derivative of a composite function is the product of the derivative of the outer function evaluated at the inner function and the derivative of the inner function."

Proof:

By the definition of the derivative, $[f(g(x))]' = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$.

Assuming that $g(x+h) - g(x) \neq 0$, we can write

$$[f(g(x))]' = \lim_{h \to 0} \left[\left(\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \right) \left(\frac{g(x+h) - g(x)}{h} \right) \right]$$

$$= \lim_{h \to 0} \left[\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \right] \lim_{h \to 0} \left[\frac{g(x+h) - g(x)}{h} \right]$$

(Property of limits)

Since $\lim_{h \to 0}[g(x+h) - g(x)] = 0$, let $g(x+h) - g(x) = k$ and $k \to 0$ as $h \to 0$. We obtain

$$[f(g(x))]' = \lim_{k \to 0} \left[\frac{f(g(x) + k) - f(g(x))}{k} \right] \lim_{h \to 0} \left[\frac{g(x+h) - g(x)}{h} \right]$$

Therefore, $[f(g(x))]' = f'(g(x))g'(x)$.

This proof is not valid for all circumstances. When dividing by $g(x+h) - g(x)$, we assume that $g(x+h) - g(x) \neq 0$. A proof that covers all cases can be found in advanced calculus textbooks.

EXAMPLE 2 Applying the chain rule

Differentiate $h(x) = (x^2 + x)^{\frac{3}{2}}$.

Solution

The inner function is $g(x) = x^2 + x$, and the outer function is $f(x) = x^{\frac{3}{2}}$.

The derivative of the inner function is $g'(x) = 2x + 1$.

The derivative of the outer function is $f'(x) = \frac{3}{2}x^{\frac{1}{2}}$.

The derivative of the outer function evaluated at the inner function $g(x)$ is $f'(x^2 + x) = \frac{3}{2}(x^2 + x)^{\frac{1}{2}}$.

By the chain rule, $h'(x) = \frac{3}{2}(x^2 + x)^{\frac{1}{2}}(2x + 1)$.

The Chain Rule in Leibniz Notation

If y is a function of u and u is a function of x (so that y is a composite function), then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$, provided that $\frac{dy}{du}$ and $\frac{du}{dx}$ exist.

If we interpret derivatives as rates of change, the chain rule states that if y is a function of x through the intermediate variable u, then the rate of change of y

with respect to x is equal to the product of the rate of change of y with respect to u and the rate of change of u with respect to x.

EXAMPLE 3

Applying the chain rule using Leibniz notation

If $y = u^3 - 2u + 1$, where $u = 2\sqrt{x}$, find $\frac{dy}{dx}$ at $x = 4$.

Solution

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (3u^2 - 2)\left[2\left(\frac{1}{2}x^{-\frac{1}{2}}\right)\right]$$

$$= (3u^2 - 2)\left(\frac{1}{\sqrt{x}}\right)$$

It is not necessary to write the derivative entirely in terms of x.

When $x = 4$, $u = 2\sqrt{4} = 4$ and $\frac{dy}{dx} = [3(4)^2 - 2]\left(\frac{1}{\sqrt{4}}\right) = (46)\left(\frac{1}{2}\right) = 23$.

EXAMPLE 4

Selecting a strategy involving the chain rule to solve a problem

An environmental study of a certain suburban community suggests that the average daily level of carbon monoxide in the air can be modelled by the function $C(p) = \sqrt{0.5p^2 + 17}$, where $C(p)$ is in parts per million and population p is expressed in thousands. It is estimated that t years from now, the population of the community will be $p(t) = 3.1 + 0.1t^2$ thousand. At what rate will the carbon monoxide level be changing with respect to time three years from now?

Solution

We are asked to find the value of $\frac{dC}{dt}$, when $t = 3$.

We can find the rate of change by using the chain rule.

Therefore, $\frac{dC}{dt} = \frac{dC}{dp}\frac{dp}{dt}$

$$= \frac{d(0.5p^2 + 17)^{\frac{1}{2}}}{dp}\frac{d(3.1 + 0.1t^2)}{dt}$$

$$= \left[\frac{1}{2}(0.5p^2 + 17)^{-\frac{1}{2}}(0.5)(2p)\right](0.2t)$$

When $t = 3$, $p(3) = 3.1 + 0.1(3)^2 = 4$.

$$\text{So, } \frac{dC}{dt} = \left[\frac{1}{2}(0.5(4)^2 + 17)^{-\frac{1}{2}}(0.5)(2(4))\right](0.2(3))$$

$$= 0.24$$

Since the sign of $\frac{dC}{dt}$ is positive, the carbon monoxide level will be increasing at the rate of 0.24 parts per million per year three years from now.

EXAMPLE 5 Using the chain rule to differentiate a power of a function

If $y = (x^2 - 5)^7$, find $\frac{dy}{dx}$.

Solution

The inner function is $g(x) = x^2 - 5$, and the outer function is $f(x) = x^7$.
By the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= 7(x^2 - 5)^6(2x)\\ &= 14x(x^2 - 5)^6\end{aligned}$$

Example 5 is a special case of the chain rule in which the outer function is a power function of the form $y = [g(x)]^n$. This leads to a generalization of the power rule seen earlier.

Power of a Function Rule

If n is a real number and $u = g(x)$, then $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$,

or $\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)$.

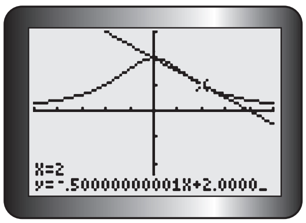
EXAMPLE 6 Connecting the derivative to the slope of a tangent

Using a graphing calculator, sketch the graph of the function $f(x) = \frac{8}{x^2 + 4}$.

Find the equation of the tangent at the point $(2, 1)$ on the graph.

Solution

Using a graphing calculator, the graph is



Tech Support

For help using the graphing calculator to graph functions and draw tangent lines see Technical Appendices p. 597 and p. 608.

The slope of the tangent at point $(2, 1)$ is given by $f'(2)$.
We first write the function as $f(x) = 8(x^2 + 4)^{-1}$.

By the power of a function rule, $f'(x) = -8(x^2 + 4)^{-2}(2x)$.

The slope at $(2, 1)$ is

$$\begin{aligned}f'(2) &= -8(4 + 4)^{-2}(4)\\ &= -\frac{32}{(8)^2}\\ &= -0.5\end{aligned}$$

The equation of the tangent is $y - 1 = -\frac{1}{2}(x - 2)$, or $x + 2y - 4 = 0$.

EXAMPLE 7 Combining derivative rules to differentiate a complex product

Differentiate $h(x) = (x^2 + 3)^4(4x - 5)^3$. Express your answer in a simplified factored form.

Solution

Here we use the product rule and the chain rule.

$$
\begin{aligned}
h'(x) &= \frac{d}{dx}[(x^2+3)^4] \cdot (4x-5)^3 + \frac{d}{dx}[(4x-5)^3] \cdot (x^2+3)^4 && \text{(Product rule)}\\
&= [4(x^2+3)^3(2x)] \cdot (4x-5)^3 + [3(4x-5)^2(4)] \cdot (x^2+3)^4 && \text{(Chain rule)}\\
&= 8x(x^2+3)^3(4x-5)^3 + 12(4x-5)^2(x^2+3)^4 && \text{(Simplify)}\\
&= 4(x^2+3)^3(4x-5)^2[2x(4x-5) + 3(x^2+3)] && \text{(Factor)}\\
&= 4(x^2+3)^3(4x-5)^2(11x^2 - 10x + 9)
\end{aligned}
$$

EXAMPLE 8 Combining derivative rules to differentiate a complex quotient

Determine the derivative of $g(x) = \left(\frac{1+x^2}{1-x^2}\right)^{10}$.

Solution A – Using the product and chain rule

There are several approaches to this problem. You could keep the function as it is and use the chain rule and the quotient rule. You could also decompose the function and express it as $g(x) = \frac{(1+x^2)^{10}}{(1-x^2)^{10}}$, and then apply the quotient rule and the chain rule. Here we will express the function as the product $g(x) = (1+x^2)^{10}(1-x^2)^{-10}$ and apply the product rule and the chain rule.

$$
\begin{aligned}
g'(x) &= \frac{d}{dx}\left[(1+x^2)^{10}\right](1-x^2)^{-10} + (1+x^2)^{10}\frac{d}{dx}\left[(1-x^2)^{-10}\right]\\
&= 10(1+x^2)^9(2x)(1-x^2)^{-10} + (1+x^2)^{10}(-10)(1-x^2)^{-11}(-2x)\\
&= 20x(1+x^2)^9(1-x^2)^{-10} + (20x)(1+x^2)^{10}(1-x^2)^{-11} && \text{(Simplify)}\\
&= 20x(1+x^2)^9(1-x^2)^{-11}[(1-x^2) + (1+x^2)] && \text{(Factor)}\\
&= 20x(1+x^2)^9(1-x^2)^{-11}(2)\\
&= \frac{40x(1+x^2)^9}{(1-x^2)^{11}} && \text{(Rewrite using positive exponents)}
\end{aligned}
$$

Solution B – Using the chain and quotient rule

In this solution, we will use the chain rule and the quotient rule, where $u = \frac{1+x^2}{1-x^2}$ is the inner function and u^{10} is the outer function.

$$g'(x) = \frac{dg}{du}\frac{du}{dx}$$

$$g'(x) = \frac{d\left[\left(\frac{1+x^2}{1-x^2}\right)^{10}\right]}{d\left(\frac{1+x^2}{1-x^2}\right)} \frac{d}{dx}\left(\frac{1+x^2}{1-x^2}\right)$$ (Chain rule and quotient rule)

$$= 10\left(\frac{1+x^2}{1-x^2}\right)^9 \frac{d}{dx}\left(\frac{1+x^2}{1-x^2}\right)$$

$$= 10\left(\frac{1+x^2}{1-x^2}\right)^9 \left[\frac{2x(1-x^2) - (-2x)(1+x^2)}{(1-x^2)^2}\right]$$ (Expand)

$$= 10\left(\frac{1+x^2}{1-x^2}\right)^9 \left[\frac{2x - 2x^3 + 2x + 2x^3}{(1-x^2)^2}\right]$$ (Simplify)

$$= 10\left(\frac{1+x^2}{1-x^2}\right)^9 \left[\frac{4x}{(1-x^2)^2}\right]$$

$$= \frac{10(1+x^2)^9}{(1-x^2)^9} \frac{4x}{(1-x^2)^2}$$

$$= \frac{40x(1+x^2)^9}{(1-x^2)^{11}}$$

IN SUMMARY

Key Idea

- The **chain rule**:
  If y is a function of u, and u is a function of x (i.e., y is a composite function), then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$, provided that $\frac{dy}{du}$ and $\frac{du}{dx}$ exist.
  Therefore, if $h(x) = (f \circ g)(x)$, then
  $$h'(x) = f'(g(x)) \cdot g'(x)$$ (Function notation)
  or
  $$\frac{d[h(x)]}{dx} = \frac{d[f(g(x))]}{d[g(x)]} \cdot \frac{d[g(x)]}{dx}$$ (Leibniz notation)

Need to Know

- When the outer function is a power function of the form $y = [g(x)]^n$, we have a special case of the chain rule, called the **power of a function rule**:
  $$\frac{d}{dx}[g(x)]^n = \frac{d[g(x)]^n}{d[g(x)]} \cdot \frac{d[g(x)]}{dx} = n[g(x)]^{n-1} \cdot g'(x)$$

Exercise 2.5

PART A

1. Given $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$, find the following value:
   a. $f(g(1))$
   b. $g(f(1))$
   c. $g(f(0))$
   d. $f(g(-4))$
   e. $f(g(x))$
   f. $g(f(x))$

2. For each of the following pairs of functions, find the composite functions $(f \circ g)$ and $(g \circ f)$. What is the domain of each composite function? Are the composite functions equal?
   a. $f(x) = x^2$
      $g(x) = \sqrt{x}$
   b. $f(x) = \frac{1}{x}$
      $g(x) = x^2 + 1$
   c. $f(x) = \frac{1}{x}$
      $g(x) = \sqrt{x + 2}$

C

3. What is the rule for calculating the derivative of the composition of two differentiable functions? Give examples, and show how the derivative is determined.

4. Differentiate each function. Do not expand any expression before differentiating.
   a. $f(x) = (2x + 3)^4$
   b. $g(x) = (x^2 - 4)^3$
   c. $h(x) = (2x^2 + 3x - 5)^4$
   d. $f(x) = (\pi^2 - x^2)^3$
   e. $y = \sqrt{x^2 - 3}$
   f. $f(x) = \frac{1}{(x^2 - 16)^5}$

PART B

K

5. Rewrite each of the following in the form $y = u^n$ or $y = ku^n$, and then differentiate.
   a. $y = -\frac{2}{x^3}$
   b. $y = \frac{1}{x + 1}$
   c. $y = \frac{1}{x^2 - 4}$
   d. $y = \frac{3}{9 - x^2}$
   e. $y = \frac{1}{5x^2 + x}$
   f. $y = \frac{1}{(x^2 + x + 1)^4}$

6. Given $h = g \circ f$, where f and g are continuous functions, use the information in the table to evaluate $h(-1)$ and $h'(-1)$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
−1	1	18	−5	−15
0	−2	5	−1	−11
1	−1	−4	3	−7
2	4	−9	7	−3
3	13	−10	11	1

7. Given $f(x) = (x - 3)^2$, $g(x) = \frac{1}{x}$, and $h(x) = f(g(x))$, determine $h'(x)$.

8. Differentiate each function. Express your answer in a simplified factored form.
   a. $f(x) = (x + 4)^3(x - 3)^6$
   b. $y = (x^2 + 3)^3(x^3 + 3)^2$
   c. $y = \dfrac{3x^2 + 2x}{x^2 + 1}$
   d. $h(x) = x^3(3x - 5)^2$
   e. $y = x^4(1 - 4x^2)^3$
   f. $y = \left(\dfrac{x^2 - 3}{x^2 + 3}\right)^4$
9. Find the rate of change of each function at the given value of t. Leave your answers as rational numbers, or in terms of roots and the number π.
   a. $s(t) = t^{\frac{1}{3}}(4t - 5)^{\frac{2}{3}}, t = 8$
   b. $s(t) = \left(\dfrac{t - \pi}{t - 6\pi}\right)^{\frac{1}{3}}, t = 2\pi$
10. For what values of x do the curves $y = (1 + x^3)^2$ and $y = 2x^6$ have the same slope?
11. Find the slope of the tangent to the curve $y = (3x - x^2)^{-2}$ at $\left(2, \frac{1}{4}\right)$.
12. Find the equation of the tangent to the curve $y = (x^3 - 7)^5$ at $x = 2$.
13. Use the chain rule, in Leibniz notation, to find $\frac{dy}{dx}$ at the given value of x.
    a. $y = 3u^2 - 5u + 2, u = x^2 - 1, x = 2$
    b. $y = 2u^3 + 3u^2, u = x + x^{\frac{1}{2}}, x = 1$
    c. $y = u(u^2 + 3)^3, u = (x + 3)^2, x = -2$
    d. $y = u^3 - 5(u^3 - 7u)^2, u = \sqrt{x}, x = 4$
14. Find $h'(2)$, given $h(x) = f(g(x)), f(u) = u^2 - 1, g(2) = 3$, and $g'(2) = -1$.

A

15. A 50 000 L tank can be drained in 30 min. The volume of water remaining in the tank after t minutes is $V(t) = 50\,000\left(1 - \frac{t}{30}\right)^2, 0 \le t \le 30$. At what rate, to the nearest whole number, is the water flowing out of the tank when $t = 10$?
16. The function $s(t) = (t^3 + t^2)^{\frac{1}{2}}, t \ge 0$, represents the displacement s, in metres, of a particle moving along a straight line after t seconds. Determine the velocity when $t = 3$.

PART C

17. a. Write an expression for $h'(x)$ if $h(x) = p(x)q(x)r(x)$.
    b. If $h(x) = x(2x + 7)^4(x - 1)^2$, find $h'(-3)$.

T

18. Show that the tangent to the curve $y = (x^2 + x - 2)^3 + 3$ at the point $(1, 3)$ is also the tangent to the curve at another point.
19. Differentiate $y = \dfrac{x^2(1 - x^3)}{(1 + x)^3}$.

Technology Extension: Derivatives on Graphing Calculators

Numerical derivatives can be approximated on a TI-83/84 Plus using **nDeriv(**.

To approximate $f'(0)$ for $f(x) = \frac{2x}{x^2 + 1}$ follow these steps:

Press MATH, and scroll down to **8:nDeriv(** under the MATH menu.

Press ENTER, and the display on the screen will be **nDeriv(**.

To find the derivative, key in the *expression*, the *variable*, the *value* at which we want the derivative, and a value for ε.

For this example, the display will be **nDeriv** $(2X/(X^2 + 1), X, 0, 0.01)$.

Press ENTER, and the value **1.99980002** will be returned.

Therefore, $f'(0)$is approximately 1.999 800 02.

A better approximation can be found by using a smaller value for ε, such as $\varepsilon = 0.0001$. The default value for ε is 0.001.

Try These:

a. Use the **nDeriv(** function on a graphing calculator to determine the value of the derivative of each of the following functions at the given point.

i. $f(x) = x^3, x = -1$

ii. $f(x) = x^4, x = 2$

iii. $f(x) = x^3 - 6x, x = -2$

iv. $f(x) = (x^2 + 1)(2x - 1)^4, x = 0$

v. $f(x) = x^2 + \frac{16}{x} - 4\sqrt{x}, x = 4$

vi. $f(x) = \frac{x^2 - 1}{x^2 + x - 2}, x = -1$

b. Determine the actual value of each derivative at the given point using the rules of differentiation.

The TI-89, TI-92, an TI-Nspire can find exact symbolic and numerical derivatives.

If you have access to either model, try some of the functions above and compare your answers with those found using a TI-83/84 Plus. For example, on the TI-89 press DIFFERENTIATE under the CALCULATE menu, key $d(2x/(x^2 + 1), x)|x = 0$ and press ENTER.

CAREER LINK WRAP-UP | Investigate and Apply

CHAPTER 2: THE ELASTICITY OF DEMAND

An electronics retailing chain has established the monthly price (p)–demand (n) relationship for an electronic game as

$$n(p) = 1000 - 10\,\frac{(p-1)^{\frac{4}{3}}}{\sqrt[3]{p}}$$

They are trying to set a price level that will provide maximum revenue (R). They know that when demand is *elastic* ($E > 1$), a drop in price will result in higher overall revenues ($R = np$), and that when demand is *inelastic* ($E < 1$), an increase in price will result in higher overall revenues. To complete the questions in this task, you will have to use the elasticity definition

$$E = -\left[\left(\frac{\Delta n}{n}\right) \div \left(\frac{\Delta p}{p}\right)\right]$$

converted into differential $\left(\frac{\Delta n}{\Delta p} = \frac{dn}{dp}\right)$ notation.

a. Determine the elasticity of demand at \$20 and \$80, classifying these price points as having elastic or inelastic demand. What does this say about where the optimum price is in terms of generating the maximum revenue? Explain. Also calculate the revenue at the \$20 and \$80 price points.

b. Approximate the demand curve as a linear function (tangent) at a price point of \$50. Plot the demand function and its linear approximation on the graphing calculator. What do you notice? Explain this by looking at the demand function.

c. Use your linear approximation to determine the price point that will generate the maximum revenue. (*Hint:* Think about the specific value of E where you will not want to increase or decrease the price to generate higher revenues.) What revenue is generated at this price point?

d. A second game has a price–demand relationship of

$$n(p) = \frac{12\,500}{p - 25}$$

The price is currently set at \$50. Should the company increase or decrease the price? Explain.

Key Concepts Review

Now that you have completed your study of derivatives in Chapter 2, you should be familiar with such concepts as derivatives of polynomial functions, the product rule, the quotient rule, the power rule for rational exponents, and the chain rule. Consider the following summary to confirm your understanding of the key concepts.

- The derivative of f at a is given by $f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$ or, alternatively, by $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.
- The derivative function of $f(x)$ with respect to x is $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$.
- The derivative of a function at a point $(a, f(a))$ can be interpreted as
  - the slope of the tangent line at this point
  - the instantaneous rate of change at this point

Summary of Differentiation Techniques

Rule	Function Notation	Leibniz Notation
Constant	$f(x) = k,\ f'(x) = 0$	$\frac{d}{dx}(k) = 0$
Linear	$f(x) = x,\ f'(x) = 1$	$\frac{d}{dx}(x) = 1$
Power	$f(x) = x^n,\ f'(x) = nx^{n-1}$	$\frac{d}{dx}(x^n) = nx^{n-1}$
Constant Multiple	$f(x) = kg(x),\ f'(x) = kg'(x)$	$\frac{d}{dx}(ky) = k\frac{dy}{dx}$
Sum or Difference	$f(x) = p(x) \pm q(x)$, $f'(x) = p'(x) \pm q'(x)$	$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$
Product	$h(x) = f(x)g(x)$ $h'(x) = f'(x)g(x) + f(x)g'(x)$	$\frac{d}{dx}[f(x)g(x)] = \left[\frac{d}{dx}f(x)\right]g(x) + f(x)\left[\frac{d}{dx}g(x)\right]$
Quotient	$h(x) = \frac{f(x)}{g(x)}$ $h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\left[\frac{d}{dx}f(x)\right]g(x) - f(x)\left[\frac{d}{dx}g(x)\right]}{[g(x)]^2}$
Chain	$h(x) = f(g(x)),\ h'(x) = f'(g(x))g'(x)$	$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$, where u is a function of x
Power of a Function	$f(x) = [g(x)]^n,\ f'(x) = n[g(x)]^{n-1}g'(x)$	$y = u^n,\ \frac{dy}{dx} = nu^{n-1}\frac{du}{dx}$, where u is a function of x

Review Exercise

1. Describe the process of finding a derivative using the definition of $f'(x)$.

2. Use the definition of the derivative to find $f'(x)$ for each of the following functions:

   a. $y = 2x^2 - 5x$
   b. $y = \sqrt{x - 6}$
   c. $y = \dfrac{x}{4 - x}$

3. Differentiate each of the following functions:

   a. $y = x^2 - 5x + 4$
   b. $f(x) = x^{\frac{3}{4}}$
   c. $y = \dfrac{7}{3x^4}$
   d. $y = \dfrac{1}{x^2 + 5}$
   e. $y = \dfrac{3}{(3 - x^2)^2}$
   f. $y = \sqrt{7x^2 + 4x + 1}$

4. Determine the derivative of the given function. In some cases, it will save time if you rearrange the function before differentiating.

   a. $f(x) = \dfrac{2x^3 - 1}{x^2}$
   b. $g(x) = \sqrt{x}(x^3 - x)$
   c. $y = \dfrac{x}{3x - 5}$
   d. $y = \sqrt{x - 1}(x + 1)$
   e. $f(x) = (\sqrt{x} + 2)^{-\frac{2}{3}}$
   f. $y = \dfrac{x^2 + 5x + 4}{x + 4}$

5. Determine the derivative, and give your answer in a simplified form.

   a. $y = x^4(2x - 5)^6$
   b. $y = x\sqrt{x^2 + 1}$
   c. $y = \dfrac{(2x - 5)^4}{(x + 1)^3}$
   d. $y = \left(\dfrac{10x - 1}{3x + 5}\right)^6$
   e. $y = (x - 2)^3(x^2 + 9)^4$
   f. $y = (1 - x^2)^3(6 + 2x)^{-3}$

6. If f is a differentiable function, find an expression for the derivative of each of the following functions:

   a. $g(x) = f(x^2)$
   b. $h(x) = 2xf(x)$

7. a. If $y = 5u^2 + 3u - 1$ and $u = \dfrac{18}{x^2 + 5}$, find $\dfrac{dy}{dx}$ when $x = 2$.

   b. If $y = \dfrac{u + 4}{u - 4}$ and $u = \dfrac{\sqrt{x} + x}{10}$, find $\dfrac{dy}{dx}$ when $x = 4$.

   c. If $y = f(\sqrt{x^2 + 9})$ and $f'(5) = -2$, find $\dfrac{dy}{dx}$ when $x = 4$.

8. Determine the slope of the tangent at point $(1, 4)$ on the graph of $f(x) = (9 - x^2)^{\frac{2}{3}}$.

9. For what values of x does the curve $y = -x^3 + 6x^2$ have a slope of -12? For what values of x does the curve $y = -x^3 + 6x^2$ have a slope of -15? Use a graphing calculator to graph the function and confirm your results.

10. a. Determine the values of x where the graph of each function has a horizontal tangent.

    i. $y = (x^2 - 4)^5$ ii. $y = (x^3 - x)^2$

    b. Use a graphing calculator to graph each function and its tangent at the values you found from part a. to confirm your result.

11. Determine the equation of the tangent to each function at the given point.

    a. $y = (x^2 + 5x + 2)^4$, $(0, 16)$ b. $y = (3x^{-2} - 2x^3)^5$, $(1, 1)$

12. A tangent to the parabola $y = 3x^2 - 7x + 5$ is perpendicular to $x + 5y - 10 = 0$. Determine the equation of the tangent.

13. The line $y = 8x + b$ is tangent to the curve $y = 2x^2$. Determine the point of tangency and the value of b.

14. a. Using a graphing calculator, graph the function $f(x) = \frac{x^3}{x^2 - 6}$.

    b. Using the draw function or an equivalent function on your calculator or graphing software, find the equations of the tangents where the slope is zero.

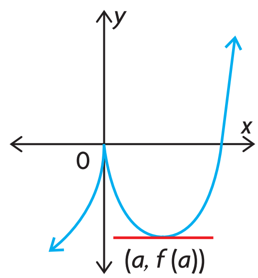
    c. Setting $f'(x) = 0$, find the coordinates of the points where the slope is zero.

    d. Determine the slope of the tangent to the graph at $(2, -4)$. Use the graph to verify that your answer is reasonable.

15. Consider the function $f(x) = 2x^{\frac{5}{3}} - 5x^{\frac{2}{3}}$.

    a. Determine the slope of the tangent at the point where the graph crosses the x-axis.

    b. Determine the value of a shown in the graph of $f(x)$ given below.



16. A rested student is able to memorize M words after t minutes, where $M = 0.1t^2 - 0.001t^3, 0 \leq t \leq \frac{200}{3}$.

    a. How many words are memorized in the first 10 min? How many words are memorized in the first 15 min?

    b. What is the memory rate at $t = 10$? What is the memory rate at $t = 15$?

17. A grocery store determines that, after t hours on the job, a new cashier can scan $N(t) = 20 - \frac{30}{\sqrt{9 + t^2}}$ items per minute.

    a. Find $N'(t)$, the rate at which the cashier's productivity is changing.

    b. According to this model, does the cashier ever stop improving? Why?

18. An athletic-equipment supplier experiences weekly costs of $C(x) = \frac{1}{3}x^3 + 40x + 700$ in producing x baseball gloves per week.

    a. Find the marginal cost, $C'(x)$.

    b. Find the production level x at which the marginal cost is \$76 per glove.

19. A manufacturer of kitchen appliances experiences revenue of $R(x) = 750x - \frac{x^2}{6} - \frac{2}{3}x^3$ dollars from the sale of x refrigerators per month.

    a. Find the marginal revenue, $R'(x)$.

    b. Find the marginal revenue when 10 refrigerators per month are sold.

20. An economist has found that the demand function for a particular new product is given by $D(p) = \frac{20}{\sqrt{p - 1}}$, $p > 1$. Find the slope of the demand curve at the point $(5, 10)$.

21. Kathy has diabetes. Her blood sugar level, B, one hour after an insulin injection, depends on the amount of insulin, x, in milligrams injected.

    $$B(x) = -0.2x^2 + 500, 0 \leq x \leq 40$$

    a. Find $B(0)$ and $B(30)$.

    b. Find $B'(0)$ and $B'(30)$.

    c. Interpret your results.

    d. Consider the values of $B'(50)$ and $B(50)$. Comment on the significance of these values. Why are restrictions given for the original function?

22. Determine which functions are differentiable at $x = 1$. Give reasons for your choices.

    a. $f(x) = \dfrac{3x}{1 - x^2}$

    b. $g(x) = \dfrac{x - 1}{x^2 + 5x - 6}$

    c. $h(x) = \sqrt[3]{(x - 2)^2}$

    d. $m(x) = |3x - 3| - 1$

23. At what x-values is each function *not* differentiable? Explain.

    a. $f(x) = \dfrac{3}{4x^2 - x}$

    b. $f(x) = \dfrac{x^2 - x - 6}{x^2 - 9}$

    c. $f(x) = \sqrt{x^2 - 7x + 6}$

24. At a manufacturing plant, productivity is measured by the number of items, p, produced per employee per day over the previous 10 years. Productivity is modelled by $p(t) = \frac{25t}{t + 1}$, where t is the number of years measured from 10 years ago. Determine the rate of change of p with respect to t.

25. Choose a simple polynomial function in the form $f(x) = ax + b$. Use the quotient rule to find the derivative of the reciprocal function $\frac{1}{ax + b}$. Repeat for other polynomial functions, and devise a rule for finding the derivative of $\frac{1}{f(x)}$. Confirm your rule using first principles.

26. Given $f(x) = \frac{(2x - 3)^2 + 5}{2x - 3}$,
    a. Express f as the composition of two simpler functions.
    b. Use this composition to determine $f'(x)$.

27. Given $g(x) = \sqrt{2x - 3} + 5(2x - 3)$,
    a. Express g as the composition of two simpler functions.
    b. Use this composition to determine $g'(x)$.

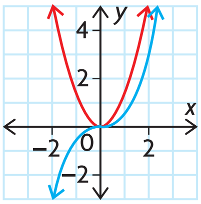
28. Determine the derivative of each function.
    a. $f(x) = (2x - 5)^3(3x^2 + 4)^5$
    b. $g(x) = (8x^3)(4x^2 + 2x - 3)^5$
    c. $y = (5 + x)^2(4 - 7x^3)^6$
    d. $h(x) = \dfrac{6x - 1}{(3x + 5)^4}$
    e. $y = \dfrac{(2x^2 - 5)^3}{(x + 8)^2}$
    f. $f(x) = \dfrac{-3x^4}{\sqrt{4x - 8}}$
    g. $g(x) = \left(\dfrac{2x + 5}{6 - x^2}\right)^4$
    h. $y = \left[\dfrac{1}{(4x + x^2)^3}\right]^3$

29. Find numbers a, b, and c so that the graph of $f(x) = ax^2 + bx + c$ has x-intercepts at $(0, 0)$ and $(8, 0)$, and a tangent with slope 16 where $x = 2$.

30. An ant colony was treated with an insecticide and the number of survivors, A, in hundreds at t hours is $A(t) = -t^3 + 5t + 750$.
    a. Find $A'(t)$.
    b. Find the rate of change of the number of living ants in the colony at 5 h.
    c. How many ants were in the colony before it was treated with the insecticide?
    d. How many hours after the insecticide was applied were no ants remaining in the colony?

Chapter 2 Test

1. Explain when you need to use the chain rule.



2. The graphs of a function and its derivative are shown at the left. Label the graphs f and f', and write a short paragraph stating the criteria you used to make your selection.
3. Use the definition of the derivative to find $\frac{d}{dx}(x - x^2)$.
4. Determine $\frac{dy}{dx}$ for each of the following functions:
   a. $y = \frac{1}{3}x^3 - 3x^{-5} + 4\pi$
   b. $y = 6(2x - 9)^5$
   c. $y = \frac{2}{\sqrt{x}} + \frac{x}{\sqrt{3}} + 6\sqrt[3]{x}$
   d. $y = \left(\frac{x^2 + 6}{3x + 4}\right)^5$ (Leave your answer in a simplified factored form.)
   e. $y = x^2\sqrt[3]{6x^2 - 7}$ (Simplify your answer.)
   f. $y = \frac{4x^5 - 5x^4 + 6x - 2}{x^4}$ (Simplify your answer.)
5. Determine the slope of the tangent to the graph of $y = (x^2 + 3x - 2)(7 - 3x)$ at $(1, 8)$.
6. Determine $\frac{dy}{dx}$ at $x = -2$ for $y = 3u^2 + 2u$ and $u = \sqrt{x^2 + 5}$.
7. Determine the equation of the tangent to $y = (3x^{-2} - 2x^3)^5$ at $(1, 1)$.
8. The amount of pollution in a certain lake is $P(t) = (t^{\frac{1}{4}} + 3)^3$, where t is measured in years and P is measured in parts per million (ppm). At what rate is the amount of pollution changing after 16 years?
9. At what point on the curve $y = x^4$ does the normal have a slope of 16?
10. Determine the points on the curve $y = x^3 - x^2 - x + 1$ where the tangent is horizontal.
11. For what values of a and b will the parabola $y = x^2 + ax + b$ be tangent to the curve $y = x^3$ at point $(1, 1)$?

Chapter 3

DERIVATIVES AND THEIR APPLICATIONS

We live in a world that is always in flux. Sir Isaac Newton's name for calculus was "the method of fluxions." He recognized in the seventeenth century, as you probably recognize today, that understanding change is important. Newton was what we might call a "mathematical physicist." He developed his method of fluxions to gain a better understanding of the natural world, including motion and gravity. But change is not limited to the natural world, and, since Newton's time, the use of calculus has spread to include applications in the social sciences. Psychology, business, and economics are just a few of the areas in which calculus continues to be an effective problem-solving tool. As we shall see in this chapter, anywhere functions can be used as models, the derivative is certain to be meaningful and useful.

CHAPTER EXPECTATIONS

In this chapter, you will

- make connections between the concept of motion and the concept of derivatives, **Section 3.1**
- solve problems involving rates of change, **Section 3.1**
- determine second derivatives, **Section 3.1**
- determine the extreme values of a function, **Section 3.2**
- solve problems by applying mathematical models and their derivatives to determine, interpret, and communicate the mathematical results, **Section 3.3**
- solve problems by determining the maximum and minimum values of a mathematical model, **Career Link, Sections 3.3, 3.4**

Review of Prerequisite Skills

In Chapter 2, we developed an understanding of derivatives and differentiation. In this chapter, we will consider a variety of applications of derivatives. The following skills will be helpful:

- graphing polynomial and simple rational functions
- working with circles in standard position
- solving polynomial equations
- finding the equations of tangents and normals
- using the following formulas:
  circle: circumference: $C = 2\pi r$, area: $A = \pi r^2$
  right circular cylinder: surface area: $SA = 2\pi rh + 2\pi r^2$, volume: $V = \pi r^2 h$

Exercise

1. Sketch the graph of each function.

a. $2x + 3y - 6 = 0$

b. $3x - 4y = 12$

c. $y = \sqrt{x}$

d. $y = \sqrt{x - 2}$

e. $y = x^2 - 4$

f. $y = -x^2 + 9$

2. Solve each equation, where $x, t \in \mathbf{R}$.

a. $3(x - 2) + 2(x - 1) - 6 = 0$

b. $\frac{1}{3}(x - 2) + \frac{2}{5}(x + 3) = \frac{x - 5}{2}$

c. $t^2 - 4t + 3 = 0$

d. $2t^2 - 5t - 3 = 0$

e. $\frac{6}{t} + \frac{t}{2} = 4$

f. $x^3 + 2x^2 - 3x = 0$

g. $x^3 - 8x^2 + 16x = 0$

h. $4t^3 + 12t^2 - t - 3 = 0$

i. $4t^4 - 13t^2 + 9 = 0$

3. Solve each inequality, where $x \in \mathbf{R}$.

a. $3x - 2 > 7$

b. $x(x - 3) > 0$

c. $-x^2 + 4x > 0$

4. Determine the area of each figure. Leave your answers in terms of π, where applicable.

a. square: perimeter 20 cm

b. rectangle: length 8 cm, width 6 cm

c. circle: radius 7 cm

d. circle: circumference 12π cm

5. Two measures of each right circular cylinder are given. Calculate the two remaining measures.

	Radius, r	**Height, h**	**Surface Area, $S = 2\pi rh + 2\pi r^2$**	**Volume, $V = \pi r^2 h$**
a.	4 cm	3 cm		
b.	4 cm			96π cm^3
c.		6 cm		216π cm^3
d.	5 cm		120π cm^2	

6. Calculate total surface area and volume for cubes with the following side lengths:

a. 3 cm

b. $\sqrt{5}$ cm

c. $2\sqrt{3}$ cm

d. $2k$ cm

7. Express each set of numbers using interval notation.

a. $\{x \in \mathbf{R} \mid x > 3\}$

b. $\{x \in \mathbf{R} \mid x \le -2\}$

c. $\{x \in \mathbf{R} \mid x < 0\}$

d. $\{x \in \mathbf{R} \mid x \ge -5\}$

e. $\{x \in \mathbf{R} \mid -2 < x \le 8\}$

f. $\{x \in \mathbf{R} \mid -4 < x < 4\}$

8. Express each interval using set notation, where $x \in \mathbf{R}$.

a. $(5, \infty)$

b. $(-\infty, 1]$

c. $(-\infty, \infty)$

d. $[-10, 12]$

e. $(-1, 3)$

f. $[2, 20)$

9. Use graphing technology to graph each function and determine its maximum and/or minimum values.

a. $f(x) = x^2 - 5$

b. $f(x) = -x^2 - 10x$

c. $f(x) = 3x^2 - 30x + 82$

d. $f(x) = |x| - 1$

e. $f(x) = 3\sin x + 2$

f. $f(x) = -2\cos 2x - 5$

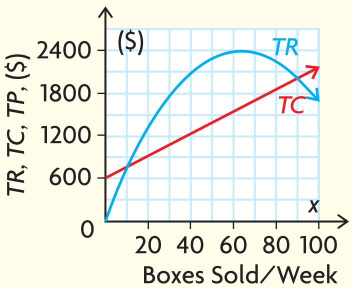
CHAPTER 3: MAXIMIZING PROFITS

We live in a world that demands we determine the best, the worst, the maximum, and the minimum. Through mathematical modelling, calculus can be used to establish optimum operating conditions for processes that seem to have competing variables. For example, minimizing transportation costs for a delivery vehicle would seem to require the driver to travel as fast as possible to reduce hourly wages. Higher rates of speed, however, increase gas consumption. With calculus, an optimal speed can be established to minimize the total cost of driving the delivery vehicle, considering both gas consumption and hourly wages. In this chapter, calculus tools will be used in realistic contexts to solve optimization problems—from business applications (such as minimizing cost) to psychology (such as maximizing learning).

Case Study—Entrepreneurship

In the last 10 years, the Canadian economy has seen a dramatic increase in the number of small businesses. An ability to use graphs to interpret the marginal profit (a calculus concept) will help an entrepreneur make good business decisions.

A person with an old family recipe for gourmet chocolates decides to open her own business. Her weekly total revenue (*TR*) and total cost (*TC*) curves are plotted on the set of axes shown.



DISCUSSION QUESTIONS

Make a rough sketch of the graph in your notes, and answer the following questions:

1. What sales interval would keep the company profitable? What do we call these values?
2. Superimpose the total profit (*TP*) curve over the *TR* and *TC* curves. What would the sales level have to be to obtain maximum profits? Estimate the slopes on the *TR* and *TC* curves at this level of sales. Should they be the same? Why or why not?
3. On a separate set of axes, sketch the marginal profit (the extra profit earned by selling one more box of chocolates), $MP = \frac{dTP}{dx}$. What can you say about the marginal profit as the level of sales progresses from just less than the maximum to the maximum, and then to just above the maximum? Does this make sense? Explain.

Section 3.1—Higher-Order Derivatives, Velocity, and Acceleration

Derivatives arise in the study of motion. The velocity of a car is the rate of change of displacement at a specific point in time. We have already developed the rules of differentiation and learned how to interpret the derivative at a point on a curve. We can now extend the applications of differentiation to higher-order derivatives. This will allow us to discuss the applications of the first and second derivatives to rates of change as an object moves in a straight line, either vertically or horizontally, such as a space shuttle taking off into space or a car moving along a straight section of road.

Higher-Order Derivatives

The function $y = f(x)$ has a first derivative $y = f'(x)$. The **second derivative** of $y = f(x)$ is the derivative of $y = f'(x)$.

The derivative of $f(x) = 10x^4$ with respect to x is $f'(x) = 40x^3$. If we differentiate $f'(x) = 40x^3$, we obtain $f''(x) = 120x^2$. This new function is called the second derivative of $f(x) = 10x^4$, and is denoted $f''(x)$.

For $y = 2x^3 - 5x^2$, the first derivative is $\frac{dy}{dx} = 6x^2 - 10x$ and the second derivative is $\frac{d^2y}{dx^2} = 12x - 10$.

Note the appearance of the superscripts in the second derivative. The reason for this choice of notation is that the second derivative is the derivative of the first derivative. That is, we write $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$.

Other notations that are used to represent first and second derivatives of $y = f(x)$ are $\frac{dy}{dx} = f'(x) = y'$ and $\frac{d^2y}{dx^2} = f''(x) = y''$.

EXAMPLE 1 **Selecting a strategy to determine the second derivative of a rational function**

Determine the second derivative of $f(x) = \frac{x}{1 + x}$ when $x = 1$.

Solution

Write $f(x) = \frac{x}{1 + x}$ as a product, and differentiate.

$$f(x) = x(x + 1)^{-1}$$

$$f'(x) = (1)(x + 1)^{-1} + (x)(-1)(x + 1)^{-2}(1)$$

(Product and power of a function rule)

$$= \frac{1}{x + 1} - \frac{x}{(x + 1)^2}$$

$$= \frac{1(x+1)}{(x+1)^2} - \frac{x}{(x+1)^2}$$ (Simplify)

$$= \frac{x+1-x}{(x+1)^2}$$

$$= \frac{1}{(1+x)^2}$$ (Rewrite as a power function)

$$= (1+x)^{-2}$$

Differentiating again to determine the second derivative,

$$f''(x) = -2(1+x)^{-3}(1)$$ (Power of a function rule)

$$= \frac{-2}{(1+x)^3}$$

When $x = 1$, $f''(1) = \frac{-2}{(1+1)^3}$ (Evaluate)

$$= \frac{-2}{8}$$

$$= -\frac{1}{4}$$

Velocity and Acceleration—Motion on a Straight Line

One reason for introducing the derivative is the need to calculate rates of change. Consider the motion of an object along a straight line. Examples are a car moving along a straight section of road, a ball dropped from the top of a building, and a rocket in the early stages of flight.

When studying motion along a line, we assume that the object is moving along a number line, which gives us an origin of reference, as well as positive and negative directions. The position of the object on the line relative to the origin is a function of time, t, and is commonly denoted by $s(t)$.

The rate of change of $s(t)$ with respect to time is the object's **velocity**, $v(t)$, and the rate of change of the velocity with respect to time is its **acceleration**, $a(t)$. The absolute value of the velocity is called **speed**.

Motion on a Straight Line

An object that moves along a straight line with its position determined by a function of time, $s(t)$, has a velocity of $\boldsymbol{v(t) = s'(t)}$ and an acceleration of $\boldsymbol{a(t) = v'(t) = s''(t)}$ at time t.
In Leibniz notation,

$$\boldsymbol{v = \frac{ds}{dt}} \text{ and } \boldsymbol{a = \frac{dv}{dt} = \frac{d^2s}{dt^2}}.$$

The speed of the object is $|\boldsymbol{v(t)}|$.

The units of velocity are displacement divided by time. The most common units are m/s.

The units of acceleration are displacement divided by $(\text{time})^2$. The most common units are metres per second per second, or metres per second squared, or m/s^2.

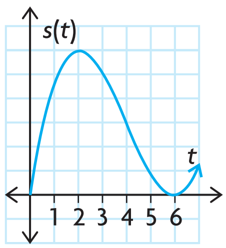
Since we are assuming that the motion is along the number line, it follows that when the object is moving to the right at time t, $v(t) > 0$ and when the object is moving to the left at time t, $v(t) < 0$. If $v(t) = 0$, the object is stationary at time t.

The object is accelerating when $a(t)$ and $v(t)$ are both positive or both negative. That is, the product of $a(t)$ and $v(t)$ is positive.

The object is decelerating when $a(t)$ is positive and $v(t)$ is negative, *or* when $a(t)$ is negative and $v(t)$ is positive. This happens when the product of $a(t)$ and $v(t)$ is negative.

EXAMPLE 2

Reasoning about the motion of an object along a straight line



An object is moving along a straight line. Its position, $s(t)$, to the right of a fixed point is given by the graph shown. When is the object moving to the right, when is it moving to the left, and when is it at rest?

Solution

The object is moving to the right whenever $s(t)$ is increasing, or $v(t) > 0$.

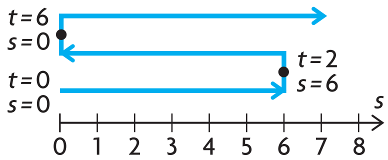
From the graph, $s(t)$ is increasing for $0 < t < 2$ and for $t > 6$.

For $2 < t < 6$, the value of $s(t)$ is decreasing, or $v(t) < 0$, so the object is moving to the left.

At $t = 2$, the direction of motion of the object changes from right to left, $v(t) = 0$, so the object is stationary at $t = 2$.

At $t = 6$, the direction of motion of the object changes from left to right, $v(t) = 0$, so the object is stationary at $t = 6$.

The motion of the object can be illustrated by the following position diagram.



EXAMPLE 3

Connecting motion to displacement, velocity, and acceleration

The position of an object moving on a line is given by $s(t) = 6t^2 - t^3$, $t \geq 0$, where s is in metres and t is in seconds.

a. Determine the velocity and acceleration of the object at $t = 2$.
b. At what time(s) is the object at rest?
c. In which direction is the object moving at $t = 5$?
d. When is the object moving in a positive direction?
e. When does the object return to its initial position?

Solution

a. The velocity at time t is $v(t) = s'(t) = 12t - 3t^2$.

At $t = 2$, $v(2) = 12(2) - 3(2)^2 = 12$.

The acceleration at time t is $a(t) = v'(t) = s''(t) = 12 - 6t$.

At $t = 2$, $a(2) = 12 - 6(2) = 0$.

At $t = 2$, the velocity is 12 m/s and the acceleration is 0 m/s^2.

We note that at $t = 2$, the object is moving at a constant velocity, since the acceleration is 0 m/s^2. The object is neither speeding up nor slowing down.

b. The object is at rest when the velocity is 0—that is, when $v(t) = 0$.

$12t - 3t^2 = 0$

$3t(4 - t) = 0$

$t = 0$ or $t = 4$

The object is at rest at $t = 0$ s and at $t = 4$ s.

c. To determine the direction of motion, we use the velocity at time $t = 5$.

$$\begin{aligned} v(5) &= 12(5) - 3(5)^2 \\ &= -15 \end{aligned}$$

The object is moving in a negative direction at $t = 5$.

d. The object moves in a positive direction when $v(t) > 0$.

$12t - 3t^2 > 0$ (Divide by -3)

$t^2 - 4t < 0$ (Factor)

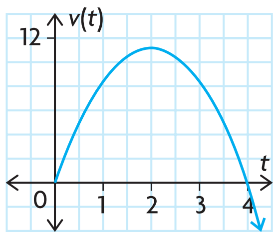
$t(t - 4) < 0$

There are two cases to consider since a product is negative when the first factor is positive and the second is negative, and vice versa.

Case 1	Case 2
$t > 0$ and $t - 4 < 0$	$t < 0$ and $t - 4 > 0$
so $t > 0$ and $t < 4$	so $t < 0$ and $t > 4$
$0 < t < 4$	no solution

Therefore, $0 < t < 4$.

The graph of the velocity function is a parabola opening downward, as shown.



From the graph and the algebraic solution above, we conclude that $v(t) > 0$ for $0 < t < 4$.

The object is moving to the right during the interval $0 < t < 4$.

e. At $t = 0$, $s(0) = 0$. Therefore, the object's initial position is at 0.

To find other times when the object is at this point, we solve $s(t) = 0$.

$6t^2 - t^3 = 0$ (Factor)

$t^2(6 - t) = 0$ (Solve)

$t = 0$ or $t = 6$

The object returns to its initial position after 6 s.

EXAMPLE 4 Analyzing motion along a horizontal line

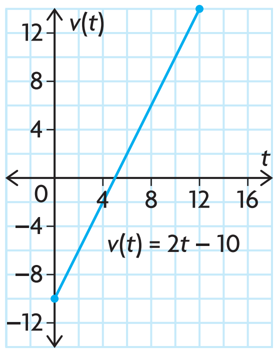
Discuss the motion of an object moving on a horizontal line if its position is given by $s(t) = t^2 - 10t$, $0 \le t \le 12$, where s is in metres and t is in seconds. Include the initial velocity, final velocity, and any acceleration in your discussion.

Solution

The initial position of the object occurs at time $t = 0$. Since $s(0) = 0$, the object starts at the origin.

The velocity at time t is $v(t) = s'(t) = 2t - 10 = 2(t - 5)$.

The object is at rest when $v(t) = 0$.



$$2(t - 5) = 0$$
$$t = 5$$

So the object is at rest after $t = 5$ s.

$v(t) > 0$ for $5 < t \le 12$, therefore the object is moving to the right during this time interval.

$v(t) < 0$ for $0 \le t < 5$, therefore the object is moving to the left during this time interval.

The initial velocity is $v(0) = -10$. So initially, the object is moving 10 m/s to the left.

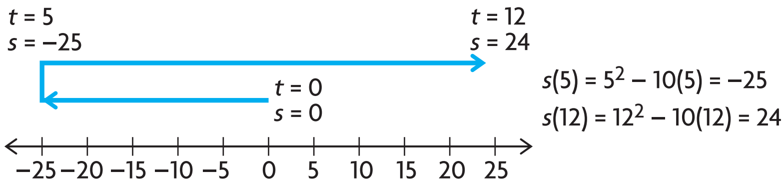
At $t = 12$, $v(12) = 2(12) - 10 = 14$.

So the final velocity is 14 m/s to the right. The velocity graph is shown.

The acceleration at time t is $a(t) = v'(t) = s''(t) = 2$. The acceleration is always 2 m/s^2. This means that the object is constantly increasing its velocity at a rate of 2 metres per second per second.

In conclusion, the object moves to the left for $0 \le t < 5$ and to the right for $5 < t \le 12$. The initial velocity is -10 m/s and the final velocity is 14 m/s.

To draw a diagram of the motion, determine the object's position at $t = 5$ and $t = 12$. (The actual path of the object is back and forth on a line.)



EXAMPLE 5 Analyzing motion under gravity near the surface of Earth

A baseball is hit vertically upward. The position function $s(t)$, in metres, of the ball above the ground is $s(t) = -5t^2 + 30t + 1$, where t is in seconds.

a. Determine the maximum height reached by the ball.
b. Determine the velocity of the ball when it is caught 1 m above the ground.

Solution

a. The maximum height occurs when the velocity of the ball is zero—that is, when the slope of the tangent to the graph is zero.

The velocity function is $v(t) = s'(t) = -10t + 30$.

Solving $v(t) = 0$, we obtain $t = 3$. This is the moment when the ball changes direction from up to down.

$$\begin{aligned} s(3) &= -5(3)^2 + 30(3) + 1 \\ &= 46 \end{aligned}$$

Therefore, the maximum height reached by the ball is 46 m.

b. When the ball is caught, $s(t) = 1$. To find the time at which this occurs, solve

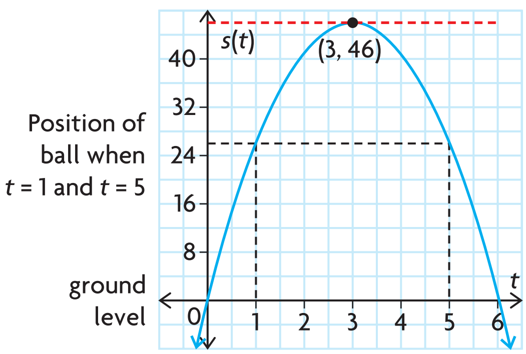
$$1 = -5t^2 + 30t + 1$$
$$0 = -5t(t - 6)$$
$$t = 0 \text{ or } t = 6$$

Since $t = 0$ is the time at which the ball leaves the bat, the time at which the ball is caught is $t = 6$.

The velocity of the ball when it is caught is $v(6) = -10(6) + 30 = -30$ m/s.

This negative value is reasonable, since the ball is falling (moving in a negative direction) when it is caught.

Note, however, that the graph of $s(t)$ does not represent the path of the ball. We think of the ball as moving in a straight line along a vertical s-axis, with the direction of motion reversing when $s = 46$.



To see this, note that the ball is at the same height at time $t = 1$, when $s(1) = 26$, and at time $t = 5$, when $s(5) = 26$.

IN SUMMARY

Key Ideas

- The derivative of the derivative function is called the second derivative.
- If the position of an object, $s(t)$, is a function of time, t, then the first derivative of this function represents the velocity of the object at time t.
  $v(t) = s'(t) = \frac{ds}{dt}$
- Acceleration, $a(t)$, is the instantaneous rate of change of velocity with respect to time. Acceleration is the first derivative of the velocity function and the second derivative of the position function.
  $a(t) = v'(t) = s''(t)$, or $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

Need to Know

- Negative velocity, $v(t) < 0$ or $s'(t) < 0$, indicates that an object is moving in a negative direction (left or down) at time t.
- Positive velocity, $v(t) > 0$ or $s'(t) > 0$, indicates that an object is moving in a positive direction (right or up) at time t.
- Zero velocity, $v(t) = 0$ or $s'(t) = 0$, indicates that an object is stationary and that a possible change in direction may occur at time t.
- Notations for the second derivative are $f''(x)$, $\frac{d^2y}{dx^2}$, $\frac{d^2}{dx^2}[f(x)]$, or y'' of a function $y = f(x)''$.
- Negative acceleration, $a(t) < 0$ or $v'(t) < 0$, indicates that the velocity is decreasing.
- Positive acceleration, $a(t) > 0$ or $v'(t) > 0$, indicates that the velocity is increasing.
- Zero acceleration, $a(t) = 0$ or $v'(t) = 0$, indicates that the velocity is constant and the object is neither accelerating nor decelerating.
- An object is accelerating (speeding up) when its velocity and acceleration have the same signs.
- An object is decelerating (slowing down) when its velocity and acceleration have opposite signs.
- The speed of an object is the magnitude of its velocity at time t.
  speed = $|v(t)| = |s'(t)|$.

Exercise 3.1

PART A

C 1. Explain and discuss the difference in velocity at times $t = 1$ and $t = 5$ for $v(t) = 2t - t^2$.

2. Determine the second derivative of each of the following:

a. $y = x^{10} + 3x^6$

b. $f(x) = \sqrt{x}$

c. $y = (1 - x)^2$

d. $h(x) = 3x^4 - 4x^3 - 3x^2 - 5$

e. $y = 4x^{\frac{3}{2}} - x^{-2}$

f. $f(x) = \dfrac{2x}{x + 1}$

g. $y = x^2 + \dfrac{1}{x^2}$

h. $g(x) = \sqrt{3x - 6}$

i. $y = (2x + 4)^3$

j. $h(x) = \sqrt[3]{x^5}$

K 3. Each of the following position functions describes the motion of an object along a straight line. Find the velocity and acceleration as functions of t, $t \geq 0$.

a. $s(t) = 5t^2 - 3t + 15$

b. $s(t) = 2t^3 + 36t - 10$

c. $s(t) = t - 8 + \dfrac{6}{t}$

d. $s(t) = (t - 3)^2$

e. $s(t) = \sqrt{t + 1}$

f. $s(t) = \dfrac{9t}{t + 3}$

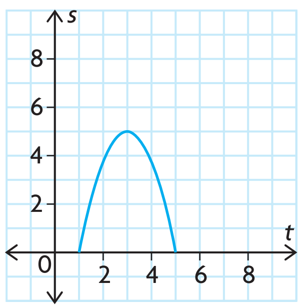
4. Answer the following questions for each position versus time graph below:

i. When is the velocity zero?

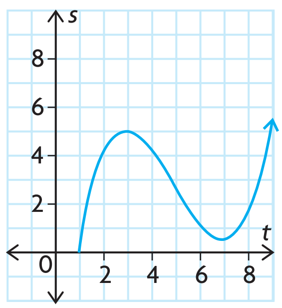
ii. When is the object moving in a positive direction?

iii. When is the object moving in a negative direction?

a.



b.



5. A particle moves along a straight line with the equation of motion $s = \frac{1}{3}t^3 - 2t^2 + 3t$, $t \geq 0$.
   a. Determine the particle's velocity and acceleration at any time t.
   b. When does the motion of the particle change direction?
   c. When does the particle return to its initial position?

PART B

6. Each function describes the position of an object that moves along a straight line. Determine whether the object is moving in a positive or negative direction at time $t = 1$ and at time $t = 4$.
   a. $s(t) = -\frac{1}{3}t^2 + t + 4$
   b. $s(t) = t(t - 3)^2$
   c. $s(t) = t^3 - 7t^2 + 10t$
7. Starting at $t = 0$, a particle moves along a line so that its position after t seconds is $s(t) = t^2 - 6t + 8$, where s is in metres.
   a. What is its velocity at time t?
   b. When is its velocity zero?
8. When an object is launched vertically from ground level with an initial velocity of 40 m/s, its position after t seconds is $s(t) = 40t - 5t^2$ metres above ground level.
   a. When does the object stop rising?
   b. What is its maximum height?
9. An object moves in a straight line, and its position, s, in metres after t seconds is $s(t) = 8 - 7t + t^2$.
   a. Determine the velocity when $t = 5$
   b. Determine the acceleration when $t = 5$.

A 10. The position function of a moving object is $s(t) = t^{\frac{5}{2}}(7 - t)$, $t \geq 0$, in metres, at time t, in seconds.
   a. Calculate the object's velocity and acceleration at any time t.
   b. After how many seconds does the object stop?
   c. When does the motion of the object change direction?
   d. When is its acceleration positive?
   e. When does the object return to its original position?

11. A ball is thrown upward, and its height, h, in metres above the ground after t seconds is given by $h(t) = -5t^2 + 25t$, $t \geq 0$.
   a. Calculate the ball's initial velocity.
   b. Calculate its maximum height.
   c. When does the ball strike the ground, and what is its velocity at this time?

12. A dragster races down a 400 m strip in 8 s. Its distance, in metres, from the starting line after t seconds is $s(t) = 6t^2 + 2t$.

   a. Determine the dragster's velocity and acceleration as it crosses the finish line.

   b. How fast was it moving 60 m down the strip?

13. For each of the following position functions, discuss the motion of an object moving on a horizontal line, where s is in metres and t is in seconds. Make a graph similar to that in Example 4, showing the motion for $t \geq 0$. Find the velocity and acceleration, and determine the extreme positions (farthest left and right) for $t \geq 0$.

   a. $s(t) = 10 + 6t - t^2$

   b. $s(t) = t^3 - 12t - 9$

14. If the position function of an object is $s(t) = t^5 - 10t^2$, at what time, t, in seconds, will the acceleration be zero? Is the object moving toward or away from the origin at this instant?

T 15. The position–time relationship for a moving object is given by $s(t) = kt^2 + (6k^2 - 10k)t + 2k$, where k is a non-zero constant.

   a. Show that the acceleration is constant.

   b. Find the time at which the velocity is zero, and determine the position of the object when this occurs.

PART C

16. An elevator is designed to start from a resting position without a jerk. It can do this if the acceleration function is continuous.

   a. Show that the acceleration is continuous at $t = 0$ for the following position function

$$s(t) = \begin{cases} 0, \text{ if } t < 0 \\ \dfrac{t^3}{t^2 + 1}, \text{ if } t \geq 0 \end{cases}$$

   b. What happens to the velocity and acceleration for very large values of t?

17. An object moves so that its velocity, v, is related to its position, s, according to $v = \sqrt{b^2 + 2gs}$, where b and g are constants. Show that the acceleration of the object is constant.

18. Newton's law of motion for a particle of mass m moving in a straight line says that $F = ma$, where F is the force acting on the particle and a is the acceleration of the particle. In relativistic mechanics, this law is replaced by $F = \dfrac{m_0 \frac{d}{dt} v}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$, where m_0 is the mass of the particle measured at rest, v is the velocity of the particle and c is the speed of light. Show that

$$F = \frac{m_0 a}{\left(1 - \left(\frac{v}{c}\right)^2\right)^{\frac{3}{2}}}.$$

Section 3.2—Maximum and Minimum on an Interval (Extreme Values)

INVESTIGATION The purpose of this investigation is to determine how the derivative can be used to determine the maximum (largest) value or the minimum (smallest) value of a function on a given interval. Together, these are called the **absolute extrema** on an interval.

A. For each of the following functions, determine, by completing the square, the value of x that produces a maximum or minimum function value on the given interval.

i. $f(x) = -x^2 + 6x - 3, 0 \leq x \leq 5$

ii. $f(x) = -x^2 - 2x + 11, -3 \leq x \leq 4$

iii. $f(x) = 4x^2 - 12x + 7, -1 \leq x \leq 4$

B. For each function in part A, determine the value of c such that $f'(c) = 0$.

C. Compare the values obtained in parts A and B for each function. Why does it make sense to say that the pattern you discovered is not merely a coincidence?

D. Using a graphing calculator, graph each of the following functions and determine all the values of x that produce a maximum or minimum value on the given interval.

i. $f(x) = x^3 - 3x^2 - 8x + 10, -2 \leq x \leq 4$

ii. $f(x) = x^3 - 12x + 5, -3 \leq x \leq 3$

iii. $f(x) = 3x^3 - 15x^2 + 9x + 23, 0 \leq x \leq 4$

iv. $f(x) = -2x^3 + 12x + 7, -2 \leq x \leq 2$

v. $f(x) = -x^3 - 2x^2 + 15x + 23, -4 \leq x \leq 3$

E. For each function in part D, determine all the values of c such that $f'(c) = 0$.

F. Compare the values obtained in parts D and E for each function. What do you notice?

G. From your comparisons in parts C and F, state a method for using the derivative of a function to determine values of the variable that give maximum or minimum values of the function.

H. Repeat part D for the following functions, using the given intervals.

i. $f(x) = -x^2 + 6x - 3, 4 \le x \le 8$

ii. $f(x) = 4x^2 - 12x + 7, 2 \le x \le 6$

iii. $f(x) = x^3 - 3x^2 - 9x + 10, -2 \le x \le 6$

iv. $f(x) = x^3 - 12x + 5, 0 \le x \le 5$

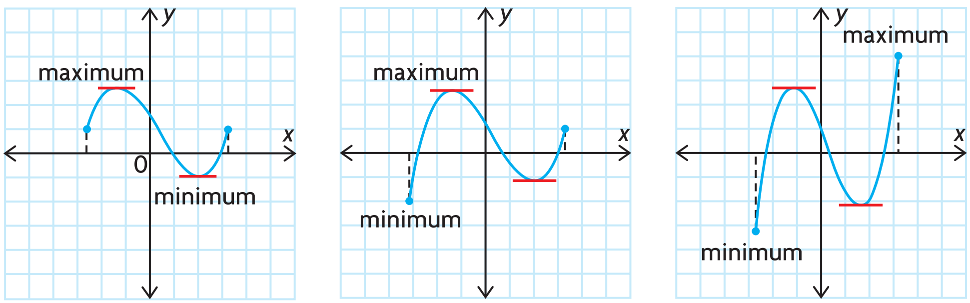
v. $f(x) = x^3 - 5x^2 + 3x + 7, -2 \le x \le 5$

I. In parts C and F, you saw that a maximum or minimum can occur at points $(c, f(c))$, where $f'(c) = 0$. From your observations in part H, state other values of the variable that can produce a maximum or minimum in a given interval.

Checkpoint: Check Your Understanding

The maximum value of a function that has a derivative at all points in an interval occurs at a "peak" $(f'(c) = 0)$ or at an endpoint of the interval. The minimum value occurs at a "valley" $(f'(c) = 0)$ or at an endpoint. This is true no matter how many peaks and valleys the graph has in the interval.

In the following three graphs, the derivative equals zero at two points:



Algorithm for Finding Maximum or Minimum (Extreme) Values

If a function $f(x)$ has a derivative at every point in the interval $a \le x \le b$, calculate $f(x)$ at

- all points in the interval $a \le x \le b$, where $f'(x) = 0$
- the endpoints $x = a$ and $x = b$

The maximum value of $f(x)$ on the interval $a \le x \le b$ is the largest of these values, and the minimum value of $f(x)$ on the interval is the smallest of these values.

When using the algorithm above it is important to consider the function $f(x)$ on a finite interval—that is, an interval that includes its endpoints. Otherwise, the function may not attain a maximum or minimum value.

EXAMPLE 1 Selecting a strategy to determine absolute extrema

Find the extreme values of the function $f(x) = -2x^3 + 9x^2 + 4$ on the interval $x \in [-1, 5]$.

Solution

The derivative is $f'(x) = -6x^2 + 18x$.

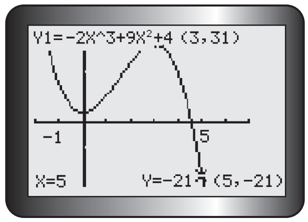
If we set $f'(x) = 0$, we obtain $-6x(x - 3) = 0$, so $x = 0$ or $x = 3$.

Both values lie in the given interval, $[-1, 5]$.

We can then evaluate $f(x)$ for these values and at the endpoints $x = -1$ and $x = 5$ to obtain

$$\begin{aligned} f(-1) &= 15 \\ f(0) &= 4 \\ f(3) &= 31 \\ f(5) &= -21 \end{aligned}$$

Therefore, the maximum value of $f(x)$ on the interval $-1 \leq x \leq 5$ is $f(3) = 31$, and the minimum value is $f(5) = -21$.



Graphing the function on this interval verifies our analysis.

EXAMPLE 2 Solving a problem involving absolute extrema

The amount of current, in amperes (A), in an electrical system is given by the function $C(t) = -t^3 + t^2 + 21t$, where t is the time in seconds and $0 \leq t \leq 5$. Determine the times at which the current is at its maximum and minimum, and determine the amount of current in the system at these times.

Solution

The derivative is $\frac{dC}{dt} = -3t^2 + 2t + 21$.

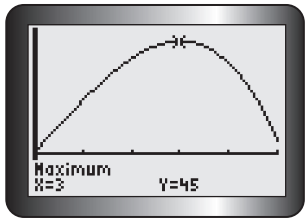
If we set $\frac{dC}{dt} = 0$, we obtain

$$-3t^2 + 2t + 21 = 0 \qquad \text{(Multiply by } -1\text{)}$$
$$3t^2 - 2t - 21 = 0 \qquad \text{(Factor)}$$
$$(3t + 7)(t - 3) = 0 \qquad \text{(Solve)}$$

Therefore, $t = -\frac{7}{3}$ or $t = 3$.

Only $t = 3$ is in the given interval, so we evaluate $C(t)$ at $t = 0$, $t = 3$, and $t = 5$ as follows:

$$C(0) = 0$$
$$C(3) = -3^3 + 3^2 + 21(3) = 45$$
$$C(5) = -5^3 + 5^2 + 21(5) = 5$$



The maximum is 45 A at time $t = 3$ s, and the minimum is 0 A at time $t = 0$ s.

Graphing the function on this interval verifies our analysis.

EXAMPLE 3 Selecting a strategy to determine the absolute minimum

The amount of light intensity on a point is given by the function $I(t) = \frac{t^2 + 2t + 16}{t + 2}$, where t is the time in seconds and $t \in [0, 14]$. Determine the time of minimal intensity.

Solution

Note that the function is not defined for $t = -2$. Since this value is not in the given interval, we need not worry about it.

The derivative is

$$I'(t) = \frac{(2t + 2)(t + 2) - (t^2 + 2t + 16)(1)}{(t + 2)^2} \quad \text{(Quotient rule)}$$
$$= \frac{2t^2 + 6t + 4 - t^2 - 2t - 16}{(t + 2)^2} \quad \text{(Expand and simplify)}$$
$$= \frac{t^2 + 4t - 12}{(t + 2)^2}$$

If we set $I'(t) = 0$, we only need to consider when the numerator is 0.

$$t^2 + 4t - 12 = 0 \quad \text{(Factor)}$$
$$(t + 6)(t - 2) = 0 \quad \text{(Solve)}$$
$$t = -6 \text{ or } t = 2$$

Only $t = 2$ is in the given interval, so we evaluate $I(t)$ for $t = 0, 2,$ and 14.

$$I(0) = 8$$

$$I(2) = \frac{4 + 4 + 16}{4} = 6$$

$$I(14) = \frac{14^2 + 2(14) + 16}{16} = 15$$

Note that the calculation can be simplified by rewriting the intensity function as shown.

$$I(t) = \frac{t^2 + 2t}{t + 2} + \frac{16}{t + 2}$$

$$= t + 16(t + 2)^{-1}$$

Then $I'(t) = 1 - 16(t + 2)^{-2}$

$$= 1 - \frac{16}{(t + 2)^2}$$

Setting $I'(t) = 0$ gives

$$1 = \frac{16}{(t + 2)^2}$$

$$t^2 + 4t + 4 = 16$$

$$t^2 + 4t - 12 = 0$$

As before, $t = -6$ or $t = 2$.
The evaluations are also simplified.

$$I(0) = 0 + \frac{16}{2} = 8$$

$$I(2) = 2 + \frac{16}{4} = 6$$

$$I(14) = 14 + \frac{16}{16} = 15$$

Either way, the minimum amount of light intensity occurs at $t = 2$ s on the given time interval.

IN SUMMARY

Key Ideas

- The maximum and minimum values of a function on an interval are also called extreme values, or absolute extrema.
- The maximum value of a function that has a derivative at all points in an interval occurs at a "peak" ($f'(c) = 0$) or at an endpoint of the interval, [a, b],
- The minimum value occurs at a "valley" ($f'(c) = 0$) or at an endpoint of the interval, [a, b].

Need to Know

- **Algorithm for Finding Extreme Values:**

  For a function $f(x)$ that has a derivative at every point in an interval [a, b], the maximum or minimum values can be found by using the following procedure:

  1. Determine $f'(x)$. Find all points in the interval $a \leq x \leq b$, where $f'(x) = 0$.
  2. Evaluate $f(x)$ at the endpoints a and b, and at points where $f'(x) = 0$.
  3. Compare all the values found in step 2.
     - The largest of these values is the maximum value of $f(x)$ on the interval $a \leq x \leq b$.
     - The smallest of these values is the minimum value of $f(x)$ on the interval $a \leq x \leq b$.

Exercise 3.2

PART A

C 1. State, with reasons, why the maximum/minimum algorithm can or cannot be used to determine the maximum and minimum values of the following functions:

a. $y = x^3 - 5x^2 + 10, -5 \leq x \leq 5$

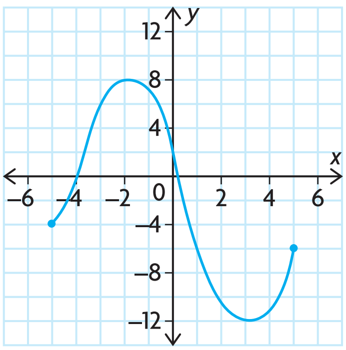
b. $y = \dfrac{3x}{x - 2}, -1 \leq x \leq 3$

c. $y = \dfrac{x}{x^2 - 4}, x \in [0, 5]$

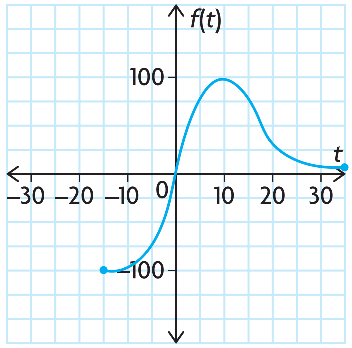
d. $y = \dfrac{x^2 - 1}{x + 3}, x \in [-2, 3]$

2. State the absolute maximum value and the absolute minimum value of each function, if the function is defined on the interval shown.

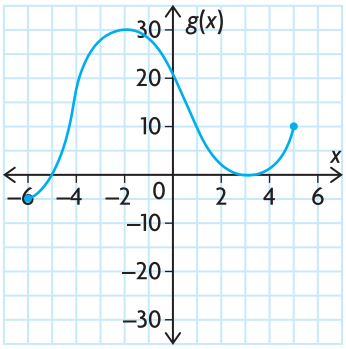
a.



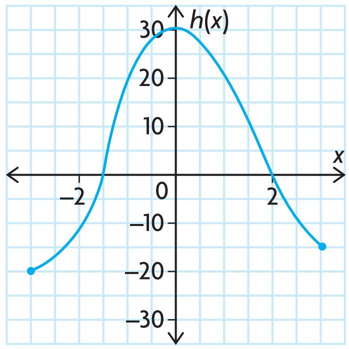
c.



b.



d.



K 3. Determine the absolute extrema of each function on the given interval. Illustrate your results by sketching the graph of each function.

a. $f(x) = x^2 - 4x + 3, 0 \leq x \leq 3$

b. $f(x) = (x - 2)^2, 0 \leq x \leq 2$

c. $f(x) = x^3 - 3x^2, -1 \leq x \leq 3$

d. $f(x) = x^3 - 3x^2, x \in [-2, 1]$

e. $f(x) = 2x^3 - 3x^2 - 12x + 1, x \in [-2, 0]$

f. $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x, x \in [0, 4]$

PART B

T 4. Using the algorithm for finding maximum or minimum values, determine the absolute extreme values of each function on the given interval.

a. $f(x) = x + \frac{4}{x}, 1 \leq x \leq 10$

b. $f(x) = 4\sqrt{x} - x, x \in [2, 9]$

c. $f(x) = \frac{1}{x^2 - 2x + 2}, 0 \leq x \leq 2$

d. $f(x) = 3x^4 - 4x^3 - 36x^2 + 20, x \in [-3, 4]$

e. $f(x) = \frac{4x}{x^2 + 1}, -2 \leq x \leq 4$

f. $f(x) = \frac{4x}{x^2 + 1}, x \in [2, 4]$

5. a. An object moves in a straight line. Its velocity, in m/s, at time t is $v(t) = \frac{4t^2}{4 + t^3}, t \geq 0$. Determine the maximum and minimum velocities over the time interval $1 \leq t \leq 4$.

b. Repeat part a., if $v(t) = \frac{4t^2}{1 + t^2}, t \geq 0$.

A 6. A swimming pool is treated periodically to control the growth of bacteria. Suppose that t days after a treatment, the number of bacteria per cubic centimetre is $N(t) = 30t^2 - 240t + 500$. Determine the lowest number of bacteria during the first week after the treatment.

7. The fuel efficiency, E, in litres per 100 kilometres, for a car driven at speed v, in km/h, is $E(v) = \frac{1600v}{v^2 + 6400}$.

a. If the speed limit is 100 km/h, determine the legal speed that will maximize the fuel efficiency.

b. Repeat part a., using a speed limit of 50 km/h.

c. Determine the speed intervals, within the legal speed limit of 0 km/h to 100 km/h, in which the fuel efficiency is increasing.

d. Determine the speed intervals, within the legal speed limit of 0 km/h to 100 km/h, in which the fuel efficiency is decreasing.

8. The concentration $C(t)$, in milligrams per cubic centimetre, of a certain medicine in a patient's bloodstream is given by $C(t) = \frac{0.1t}{(t + 3)^2}$, where t is the number of hours after the medicine is taken. Determine the maximum and minimum concentrations between the first and sixth hours after the medicine is taken.

9. Technicians working for the Ministry of Natural Resources found that the amount of a pollutant in a certain river can be represented by $P(t) = 2t + \frac{1}{(162t + 1)}$, $0 \le t \le 1$, where t is the time, in years, since a cleanup campaign started. At what time was the pollution at its lowest level?

10. A truck travelling at x km/h, where $30 \le x \le 120$, uses gasoline at the rate of $r(x)$ L/100 km, where $r(x) = \frac{1}{4}\left(\frac{4900}{x} + x\right)$. If fuel costs \$1.15/L, what speed will result in the lowest fuel cost for a trip of 200 km? What is the lowest total cost for the trip?

11. The polynomial function $f(x) = 0.001x^3 - 0.12x^2 + 3.6x + 10$, $0 \le x \le 75$, models the shape of a roller-coaster track, where f is the vertical displacement of the track and x is the horizontal displacement of the track. Both displacements are in metres. Determine the absolute maximum and minimum heights along this stretch of track.

12. a. Graph the cubic function with an absolute minimum at $(-2, -12)$, a local maximum at $(0, 3)$, a local minimum at $(2, -1)$, and an absolute maximum at $(4, 9)$. *Note*: local maximum and minimum values occur at peaks and valleys of a graph and do not have to be absolute extrema.
    b. What is the domain of this function?
    c. Where is the function increasing? Where is it decreasing?

13. What points on an interval must you consider to determine the absolute maximum or minimum value on the interval? Why?

PART C

T 14. In a certain manufacturing process, when the level of production is x units, the cost of production, in dollars, is $C(x) = 3000 + 9x + 0.05x^2$, $1 \le x \le 300$. What level of production, x, will minimize the unit cost, $U(x) = \frac{C(x)}{x}$? Keep in mind that the production level must be an integer.

15. Repeat question 14. If the cost of production is $C(x) = 6000 + 9x + 0.05x^2$, $1 \le x \le 300$.

Mid-Chapter Review

1. Determine the second derivative of each of the following functions:

   a. $h(x) = 3x^4 - 4x^3 - 3x^2 - 5$

   b. $f(x) = (2x - 5)^3$

   c. $y = \dfrac{15}{x + 3}$

   d. $g(x) = \sqrt{x^2 + 1}$

2. The displacement of an object in motion is described by $s(t) = t^3 - 21t^2 + 90t$, where the horizontal displacement, s, is measured in metres at t seconds.

   a. Calculate the displacement at 3 s.

   b. Calculate the velocity at 5 s.

   c. Calculate the acceleration at 4 s.

3. A ball is thrown upward. Its motion can be described by $h(t) = -4.9t^2 + 6t + 2$, where the height, h, is measured in metres at t seconds.

   a. Determine the initial velocity.

   b. When does the ball reach its maximum height?

   c. When does the ball hit the ground?

   d. What is the velocity of the ball when it hits the ground?

   e. What is the acceleration of the ball on the way up? What is its acceleration on the way down?

4. An object is moving horizontally. The object's displacement, s, in metres at t seconds is described by $s(t) = 4t - 7t^2 + 2t^3$.

   a. Determine the velocity and acceleration at $t = 2$.

   b. When is the object stationary? Describe the motion immediately before and after these times.

   c. At what time, to the nearest tenth of a second, is the acceleration equal to 0? Describe the motion at this time.

5. Determine the absolute extreme values of each function on the given interval, using the algorithm for finding maximum and minimum values.

   a. $f(x) = x^3 + 3x^2 + 1,\ -2 \le x \le 2$

   b. $f(x) = (x + 2)^2,\ -3 \le x \le 3$

   c. $f(x) = \dfrac{1}{x} - \dfrac{1}{x^3},\ x \in [1, 5]$

6. The volume, V, of 1 kg of H_2O at temperature t between 0 °C and 30 °C can be modelled by $V(t) = -0.000\,067t^3 + 0.008\,504\,3t^2 - 0.064\,26t + 999.87$. Volume is measured in cubic centimetres. Determine the temperature at which the volume of water is the greatest in the given interval.

7. Evaluate each of the following:
   a. $f'(3)$ if $f(x) = x^4 - 3x$
   b. $f'(-2)$ if $f(x) = 2x^3 + 4x^2 - 5x + 8$
   c. $f''(1)$ if $f(x) = -3x^2 - 5x + 7$
   d. $f''(-3)$ if $f(x) = 4x^3 - 3x^2 + 2x - 6$
   e. $f'(0)$ if $f(x) = 14x^2 + 3x - 6$
   f. $f''(4)$ if $f(x) = x^4 + x^5 - x^3$
   g. $f''\left(\frac{1}{3}\right)$ if $f(x) = -2x^5 + 2x - 6 - 3x^3$
   h. $f'\left(\frac{3}{4}\right)$ if $f(x) = -3x^3 - 7x^2 + 4x - 11$
8. On the surface of the Moon, an astronaut can jump higher because the force of gravity is less than it is on Earth. When a certain astronaut jumps, his height, in metres above the Moon's surface, can be modelled by $s(t) = t\left(-\frac{5}{6}t + 1\right)$, where t is measured in seconds. What is the acceleration due to gravity on the Moon?
9. The forward motion of a space shuttle, t seconds after touchdown, is described by $s(t) = 189t - t^{\frac{7}{3}}$, where s is measured in metres.
   a. What is the velocity of the shuttle at touchdown?
   b. How much time is required for the shuttle to stop completely?
   c. How far does the shuttle travel from touchdown to a complete stop?
   d. What is the deceleration 8 s after touchdown?
10. In a curling game, one team's skip slides a stone toward the rings at the opposite end of the ice. The stone's position, s, in metres at t seconds, can be modelled by $s(t) = 12t - 4t^{\frac{3}{2}}$. How far does the stone travel before it stops? How long is it moving?
11. After a football is punted, its height, h, in metres above the ground at t seconds, can be modelled by $h(t) = -4.9t^2 + 21t + 0.45$.
   a. Determine the restricted domain of this model.
   b. When does the ball reach its maximum height?
   c. What is the ball's maximum height?

Section 3.3—Optimization Problems

We frequently encounter situations in which we are asked to do the best we can. Such a request is vague unless we are given some conditions. Asking us to minimize the cost of making tables and chairs is not clear. Asking us to make the maximum number of tables and chairs possible, with a given amount of material, so that the costs of production are minimized allows us to construct a function that describes the situation. We can then determine the minimum (or maximum) of the function.

Such a procedure is called **optimization**. To optimize a situation is to realize the best possible outcome, subject to a set of restrictions. Because of these restrictions, the domain of the function is usually restricted. As you have seen earlier, in such situations, the maximum or minimum can be identified through the use of calculus, but might also occur at the ends of the restricted domain.

EXAMPLE 1

Solving a problem involving optimal area

A farmer has 800 m of fencing and wishes to enclose a rectangular field. One side of the field is against a country road that is already fenced, so the farmer needs to fence only the remaining three sides of the field. The farmer wants to enclose the maximum possible area and to use all the fencing. How does the farmer determine the dimensions to achieve this goal?

Solution

The farmer can achieve this goal by determining a function that describes the area, subject to the condition that the amount of fencing used is to be exactly 800 m, and by finding the maximum of the function. To do so, the farmer proceeds as follows:

Let the width of the enclosed area be x metres.

Then the length of the rectangular field is $(800 - 2x)$ m. The area of the field can be represented by the function $A(x)$, where

$$\begin{aligned} A(x) &= x(800 - 2x) \\ &= 800x - 2x^2 \end{aligned}$$

The domain of the function is $0 < x < 400$, since the amount of fencing is 800 m. To find the minimum and maximum values, determine $A'(x)$: $A'(x) = 800 - 4x$. Setting $A'(x) = 0$, we obtain $800 - 4x = 0$, so $x = 200$.

The minimum and maximum values can occur at $x = 200$ or at the ends of the domain, $x = 0$ and $x = 400$. Evaluating the area function at each of these gives

$$\begin{aligned} A(0) &= 0 \\ A(200) &= 200(800 - 400) \\ &= 80\,000 \\ A(400) &= 400(800 - 800) \\ &= 0 \end{aligned}$$

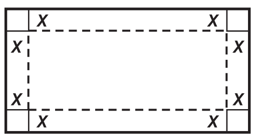
Sometimes, the ends of the domain produce results that are either not possible or unrealistic. In this case, $x = 200$ produces the maximum. The ends of the domain do not result in possible dimensions of a rectangle.

The maximum area that the farmer can enclose is 80 000 m^2, within a field 200 m by 400 m.

EXAMPLE 2

Solving a problem involving optimal volume

A piece of sheet metal, 60 cm by 30 cm, is to be used to make a rectangular box with an open top. Determine the dimensions that will give the box with the largest volume.



Solution

From the diagram, making the box requires the four corner squares to be cut out and discarded. Folding up the sides creates the box. Let each side of the squares be x centimetres.

$$\begin{aligned} \text{Therefore, height} &= x \\ \text{length} &= 60 - 2x \\ \text{width} &= 30 - 2x \end{aligned}$$

Since all dimensions are positive, $0 < x < 15$.

The volume of the box is the product of its dimensions and is given by the function $V(x)$, where

$$\begin{aligned} V(x) &= x(60 - 2x)(30 - 2x) \\ &= 4x^3 - 180x^2 + 1800x \end{aligned}$$

For extreme values, set $V'(x) = 0$.

$$V'(x) = 12x^2 - 360x + 1800$$
$$= 12(x^2 - 30x + 150)$$

Setting $V'(x) = 0$, we obtain $x^2 - 30x + 150 = 0$. Solving for x using the quadratic formula results in

$$x = \frac{30 \pm \sqrt{300}}{2}$$
$$= 15 \pm 5\sqrt{3}$$
$$x \doteq 23.7 \text{ or } x \doteq 6.3$$

Since $0 < x < 15$, $x = 15 - 5\sqrt{3} \doteq 6.3$. This is the only place within the interval where the derivative is 0.

To find the largest volume, substitute $x = 6.3$ in $V(x) = 4x^3 - 180x^2 + 1800x$.

$$V(6.3) = 4(6.3)^3 - 180(6.3)^2 + 1800(6.3)$$
$$= 5196$$

Notice that the endpoints of the domain did not have to be tested since it is impossible to make a box using the values $x = 0$ or $x = 15$.

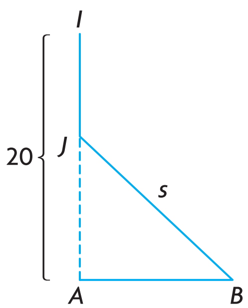
The maximum volume is obtained by cutting out corner squares of side length 6.3 cm. The length of the box is $60 - 2 \times 6.3 = 47.4$ cm, the width is about $30 - 2 \times 6.3 = 17.4$ cm, and the height is about 6.3 cm.

EXAMPLE 3

Solving a problem that minimizes distance

Ian and Ada are both training for a marathon. Ian's house is located 20 km north of Ada's house. At 9:00 a.m. one Saturday, Ian leaves his house and jogs south at 8 km/h. At the same time, Ada leaves her house and jogs east at 6 km/h. When are Ian and Ada closest together, given that they both run for 2.5 h?

Solution



If Ian starts at point I, he reaches point J after time t hours. Then $IJ = 8t$ km, and $JA = (20 - 8t)$ km.

If Ada starts at point A, she reaches point B after t hours, and $AB = 6t$ km. Now the distance they are apart is $s = JB$, and s can be expressed as a function of t by

$$s(t) = \sqrt{JA^2 + AB^2}$$
$$= \sqrt{(20 - 8t)^2 + (6t)^2}$$
$$= \sqrt{100t^2 - 320t + 400}$$
$$= (100t^2 - 320t + 400)^{\frac{1}{2}}$$

The domain for t is $0 \leq t \leq 2.5$.

$$s'(t) = \frac{1}{2}(100t^2 - 320t + 400)^{-\frac{1}{2}}(200t - 320)$$

$$= \frac{100t - 160}{\sqrt{100t^2 - 320t + 400}}$$

To obtain a minimum or maximum value, let $s'(t) = 0$.

$$\frac{100t - 160}{\sqrt{100t^2 - 320t + 400}} = 0$$

$$100t - 160 = 0$$

$$t = 1.6$$

Using the algorithm for finding extreme values,

$$s(0) = \sqrt{400} = 20$$

$$s(1.6) = \sqrt{100(1.6)^2 - 320(1.6) + 400} = 12$$

$$s(2.5) = \sqrt{225} = 15$$

Therefore, the minimum value of $s(t)$ is 12 km, which occurs at time 10:36 a.m.

IN SUMMARY

Key Ideas

- In an optimization problem, you must determine the maximum or minimum value of a quantity.
- An optimization problem can be solved using a mathematical model that is developed using information given in the problem. The numerical solution represents the extreme value of the model.

Need to Know

- **Algorithm for Solving Optimization Problems:**
  1. Understand the problem, and identify quantities that can vary. Determine a function in one variable that represents the quantity to be optimized.
  2. Whenever possible, draw a diagram, labelling the given and required quantities.
  3. Determine the domain of the function to be optimized, using the information given in the problem.
  4. Use the algorithm for extreme values to find the absolute maximum or minimum value in the domain.
  5. Use your result for step 4 to answer the original problem.

Exercise 3.3

PART A

1. A piece of wire, 100 cm long, needs to be bent to form a rectangle. Determine the dimensions of a rectangle with the maximum area.

C 2. Discuss the result of maximizing the area of a rectangle, given a fixed perimeter.

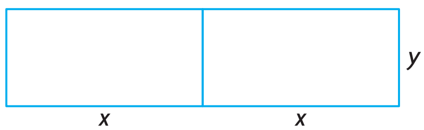
3. A farmer has 600 m of fence and wants to enclose a rectangular field beside a river. Determine the dimensions of the fenced field in which the maximum area is enclosed. (Fencing is required on only three sides: those that aren't next to the river.)

4. A rectangular piece of cardboard, 100 cm by 40 cm, is going to be used to make a rectangular box with an open top by cutting congruent squares from the corners. Calculate the dimensions (to one decimal place) for a box with the largest volume.

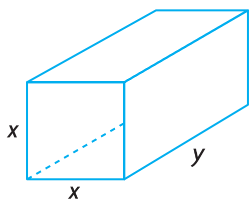
5. A rectangle has a perimeter of 440 cm. What dimensions will maximize the area of the rectangle?

6. What are the dimensions of a rectangle with an area of 64 m^2 and the smallest possible perimeter?

7. A rancher has 1000 m of fencing to enclose two rectangular corrals. The corrals have the same dimensions and one side in common. What dimensions will maximize the enclosed area?



8. A net enclosure for practising golf shots is open at one end, as shown. Find the dimensions that will minimize the amount of netting needed and give a volume of 144 m^2. (Netting is required only on the sides, the top, and the far end.)



PART B

9. The volume of a square-based rectangular cardboard box needs to be 1000 cm^3. Determine the dimensions that require the minimum amount of material to manufacture all six faces. Assume that there will be no waste material. The machinery available cannot fabricate material smaller than 2 cm in length.

10. Determine the area of the largest rectangle that can be inscribed inside a semicircle with a radius of 10 units. Place the length of the rectangle along the diameter.

A 11. A cylindrical-shaped tin can must have a capacity of 1000 cm^3.

   a. Determine the dimensions that require the minimum amount of tin for the can. (Assume no waste material.) According to the marketing department, the smallest can that the market will accept has a diameter of 6 cm and a height of 4 cm.

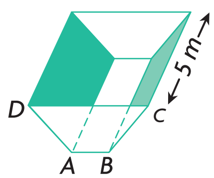
   b. Express your answer for part a. as a ratio of height to diameter. Does this ratio meet the requirements outlined by the marketing department?

12. a. Determine the area of the largest rectangle that can be inscribed in a right triangle if the legs adjacent to the right angle are 5 cm and 12 cm long. The two sides of the rectangle lie along the legs.

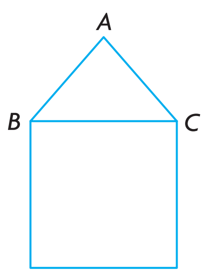
   b. Repeat part a. for a right triangle that has sides 8 cm and 15 cm.

   c. Hypothesize a conclusion for any right triangle.

T 13. a. An isosceles trapezoidal drainage gutter is to be made so that the angles at A and B in the cross-section $ABCD$ are each 120°. If the 5 m long sheet of metal that has to be bent to form the open-topped gutter and the width of the sheet of metal is 60 cm, then determine the dimensions so that the cross-sectional area will be a maximum.



   b. Calculate the maximum volume of water that can be held by this gutter.



14. The 6 segments of the window frame shown in the diagram are to be constructed from a piece of window framing material 6m in length. A carpenter wants to build a frame for a rural gothic style window, where $\triangle ABC$ is equilateral. The window must fit inside a space that is 1 m wide and 3 m high.

a. Determine the dimensions that should be used for the six pieces so that the maximum amount of light will be admitted. Assume no waste material for corner cuts and so on.

b. Would the carpenter get more light if the window was built in the shape of an equilateral triangle only? Explain.

15. A train leaves the station at 10:00 a.m. and travels due south at a speed of 60 km/h. Another train has been heading due west at 45 km/h and reaches the same station at 11:00 a.m. At what time were the two trains closest together?

16. A north–south highway intersects an east–west highway at point P. A vehicle crosses P at 1:00 p.m., travelling east at a constant speed of 60 km/h. At the same instant, another vehicle is 5 km north of P, travelling south at 80 km/h. Find the time when the two vehicles are closest to each other and the distance between them at this time.

PART C

17. In question 12, part c., you looked at two specific right triangles and observed that a rectangle with the maximum area that can be inscribed inside the triangle had dimensions equal to half the lengths of the sides adjacent to the rectangle. Prove that this is true for any right triangle.

18. Prove that any cylindrical can of volume k cubic units that is to be made using a minimum amount of material must have the height equal to the diameter.

19. A piece of wire, 100 cm long, is cut into two pieces. One piece is bent to form a square, and the other piece is bent to form a circle. Determine how the wire should be cut so that the total area enclosed is

    a. a maximum

    b. a minimum

20. Determine the minimal distance from point $(-3, 3)$ to the curve given by $y = (x - 3)^2$.

21. A chord joins any two points A and B on the parabola whose equation is $y^2 = 4x$. If C is the midpoint of AB, and CD is drawn parallel to the x-axis to meet the parabola at D, prove that the tangent at D is parallel to chord AB.

22. A rectangle lies in the first quadrant, with one vertex at the origin and two of the sides along the coordinate axes. If the fourth vertex lies on the line defined by $x + 2y - 10 = 0$, find the rectangle with the maximum area.

23. The base of a rectangle lies along the x-axis, and the upper two vertices are on the curve defined by $y = k^2 - x^2$. Determine the dimensions of the rectangle with the maximum area.

Section 3.4—Optimization Problems in Economics and Science

In the world of business, it is extremely important to manage costs effectively. Good control will allow for minimization of costs and maximization of profit. At the same time, there are human considerations. If your company is able to maximize profit but antagonizes customers or employees in the process, there may be problems in the future. For this reason, it may be important that, in addition to any mathematical constraints, you consider other more practical constraints on the domain when you construct a workable function.

The following examples will illustrate economic situations and domain constraints you may encounter.

EXAMPLE 1

Solving a problem to maximize revenue

A commuter train carries 2000 passengers daily from a suburb into a large city. The cost to ride the train is \$7.00 per person. Market research shows that 40 fewer people would ride the train for each \$0.10 increase in the fare, and 40 more people would ride the train for each \$0.10 decrease. If the capacity of the train is 2600 passengers, and carrying fewer than 1600 passengers means costs exceed revenue, what fare should the railway charge to get the largest possible revenue?

Solution

To maximize revenue, we require a revenue function. We know that revenue $=$ (number of passengers) $\times$ (fare per passenger).

To form a revenue function, the most straightforward choice for the independent variable comes from noticing that both the number of passengers and the fare per passenger change with each \$0.10 increase or decrease in the fare. If we let x represent the number of \$0.10 increases in the fare (for example, $x = 3$ represents a \$0.30 increase in the fare, whereas $x = -1$ represents a \$0.10 decrease in the fare), then we can write expressions for both the number of passengers and the fare per passenger in terms of x, as follows:

- the fare per passenger is $7 + 0.10x$
- the number of passengers is $2000 - 40x$

Since the number of passengers must be at least 1600, $2000 - 40x \geq 1600$, and $x \leq 10$. Since the number of passengers cannot exceed 2600, $2000 - 40x \leq 2600$, and $x \geq -15$.

The domain is $-15 \leq x \leq 10$.

The revenue function is

$$\begin{aligned} R(x) &= (7 + 0.10x)(2000 - 40x) \\ &= -4x^2 - 80x + 14\,000 \end{aligned}$$

From a practical point of view, we also require x to be an integer, so that the fare only varies by increments of \$0.10. We do not wish to consider fares that are not multiples of 10 cents.

Therefore, we need to find the absolute maximum value of the revenue function $R(x) = -4x^2 - 80x + 14\,000$ on the interval $-15 \le x \le 10$, where x must be an integer.

$R'(x) = -8x - 80$

$R'(x) = 0$ when $-8x - 80 = 0$ $x = -10$

$R'(x)$ is never undefined. Notice that $x = -10$, is in the domain. To determine the maximum revenue, we evaluate

$$\begin{aligned} R(-15) &= -4(-15)^2 - 80(-15) + 14\,000 \\ &= 14\,300 \\ R(-10) &= -4(-10)^2 - 80(-10) + 14\,000 \\ &= 14\,400 \\ R(10) &= -4(10)^2 - 80(10) + 14\,000 \\ &= 12\,800 \end{aligned}$$

Therefore, the maximum revenue occurs when there are -10 fare increases of \$0.10 each, or a fare decrease of $10(0.10) = \$1.00$. At a fare of \$6.00, the daily revenue is \$14 400, and the number of passengers is $2000 - 40(-10) = 2400$.

EXAMPLE 2

Solving a problem to minimize cost

A cylindrical chemical storage tank with a capacity of 1000 m^3 is going to be constructed in a warehouse that is 12 m by 15 m, with a height of 11 m. The specifications call for the base to be made of sheet steel that costs \$100/m^2, the top to be made of sheet steel that costs \$50/m^2, and the wall to be made of sheet steel that costs \$80/m^2.

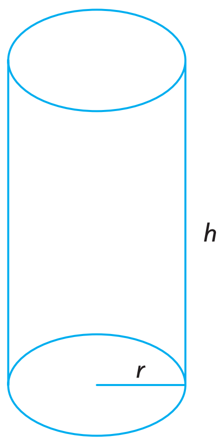
a. Determine whether it is possible for a tank of this capacity to fit in the warehouse. If it *is* possible, state the restrictions on the radius.

b. If fitting the tank in the warehouse is possible, determine the proportions that meet the conditions and that minimize the cost of the steel for construction.

All calculations should be accurate to two decimal places.

Solution

a. The radius of the tank cannot exceed 6 m, and the maximum height is 11 m. The volume, using $r = 6$ and $h = 11$, is $V = \pi r^2 h \doteq 1244 \text{ m}^3$.
It is possible to build a tank with a volume of 1000 m^3.
There are limits on the radius and the height. Clearly, $0 < r \leq 6$. Also, if $h = 11$, then $\pi r^2(11) \geq 1000$, so $r \geq 5.38$.
The tank can be constructed to fit in the warehouse. Its radius must be $5.38 \leq r \leq 6$.



b. If the height is h metres and the radius is r metres, then
- the cost of the base is $\$100(\pi r^2)$
- the cost of the top is $\$50(\pi r^2)$
- the cost of the wall is $\$80(2\pi rh)$

The cost of the tank is $C = 150\pi r^2 + 160\pi rh$.
Here we have two variable quantities, r and h.

However, since $V = \pi r^2 h = 1000$, $h = \frac{1000}{\pi r^2}$.

Substituting for h, we have a cost function in terms of r.

$$C(r) = 150\pi r^2 + 160\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$\text{or } C(r) = 150\pi r^2 + \frac{160\ 000}{r}$$

From part a., we know that the domain is $5.38 \leq r \leq 6$.
To find points where extreme values could occur, set $C'(r) = 0$.

$$300\pi r - \frac{160\ 000}{r^2} = 0$$

$$300\pi r = \frac{160\ 000}{r^2}$$

$$r^3 = \frac{1600}{3\pi}$$

$$r \doteq 5.54$$

This value is within the given domain, so we use the algorithm for finding maximum and minimum values.

$$C(5.38) = 150\pi(5.38)^2 + \frac{160\ 000}{5.38} \doteq 43\ 380$$

$$C(5.54) = 150\pi(5.54)^2 + \frac{160\ 000}{5.54} \doteq 43\ 344$$

$$C(6) = 150\pi(6)^2 + \frac{160\ 000}{6} \doteq 43\ 631$$

The minimal cost is approximately \$43 344, with a tank of radius 5.54 m and a height of $\frac{1000}{\pi(5.54)^2} = 10.37$ m.

When solving real-life optimization problems, there are often many factors that can affect the required functions and their domains. Such factors may not be obvious from the statement of the problem. We must do research and ask many questions to address all the factors. Solving an entire problem is a series of many steps, and optimization using calculus techniques is only one step in determining a solution.

IN SUMMARY

Key Ideas

- Profit, cost, and revenue are quantities whose rates of change are measured in terms of the number of units produced or sold.
- Economic situations usually involve minimizing costs or maximizing profits.

Need to Know

- To maximize revenue, we can use the revenue function.
  revenue = total revenue from the sale of x units = (price per unit) $\times$ x.
- Practical constraints, as well as mathematical constraints, must always be considered when constructing a model.
- Once the constraints on the model have been determined—that is the domain of the function—apply the extreme value algorithm to the function over the appropriately defined domain to determine the absolute extrema.

Exercise 3.4

PART A

K 1. The cost, in dollars, to produce x litres of maple syrup for the Elmira Maple Syrup Festival is $C(x) = 75(\sqrt{x} - 10)$, where $x \geq 400$.

a. What is the average cost of producing 625 L?

b. The marginal cost is $C'(x)$, and the marginal revenue is $R'(x)$. Marginal cost at x litres is the expected change in cost if we were to produce one additional litre of syrup. Similarly for marginal revenue. What is the marginal cost at 1225 L?

c. How much production is needed to achieve a marginal cost of \$0.50/L?

2. A sociologist determines that a foreign-language student has learned $N(t) = 20t - t^2$ vocabulary terms after t hours of uninterrupted study.
   a. How many terms are learned between times $t = 2$ and $t = 3$?
   b. What is the rate, in terms per hour, at which the student is learning at time $t = 2$?
   c. What is the maximum rate, in terms per hour, at which the student is learning?
3. A researcher found that the level of antacid in a person's stomach, t minutes after a certain brand of antacid tablet is taken, is $L(t) = \frac{6t}{t^2 + 2t + 1}$.
   a. Determine the value of t for which $L'(t) = 0$.
   b. Determine $L(t)$ for the value you found in part a.
   c. Using your graphing calculator, graph $L(t)$.
   d. From the graph, what can you predict about the level of antacid in a person's stomach after 1 min?
   e. What is happening to the level of antacid in a person's stomach from $2 \le t \le 8$?

PART B

A 4. The operating cost, C, in dollars per hour, for an airplane cruising at a height of h metres and an air speed of 200 km/h is given by $C = 4000 + \frac{h}{15} + \frac{15\,000\,000}{h}$ for the domain $1000 \le h \le 20\,000$. Determine the height at which the operating cost is at a minimum, and find the operating cost per hour at this height.

5. A rectangular piece of land is to be fenced using two kinds of fencing. Two opposite sides will be fenced using standard fencing that costs \$6/m, while the other two sides will require heavy-duty fencing that costs \$9/m. What are the dimensions of the rectangular lot of greatest area that can be fenced for a cost of \$9000?

6. A real estate office manages 50 apartments in a downtown building. When the rent is \$900 per month, all the units are occupied. For every \$25 increase in rent, one unit becomes vacant. On average, all units require \$75 in maintenance and repairs each month. How much rent should the real estate office charge to maximize profits?

7. A bus service carries 10 000 people daily between Ajax and Union Station, and the company has space to serve up to 15 000 people per day. The cost to ride the bus is \$20. Market research shows that if the fare increases by \$0.50, 200 fewer people will ride the bus. What fare should be charged to get the maximum revenue, given that the bus company must have at least \$130 000 in fares a day to cover operating costs?

T 8. The fuel cost per hour for running a ship is approximately one half the cube of the speed (measured in knots) plus additional fixed costs of \$216 per hour. Find the most economical speed to run the ship for a 500 M (nautical mile) trip. *Note:* Assume that there are no major disturbances, such as heavy tides or stormy seas.

9. A 20 000 m^3 rectangular cistern is to be made from reinforced concrete such that the interior length will be twice the height. If the cost is \$40/m^2 for the base, \$100/m^2 for the side walls, and \$200/m^2 for the roof, find the interior dimensions (to one decimal place) that will keep the cost to a minimum. To protect the water table, the building code specifies that no excavation can be more than 22 m deep. It also specifies that all cisterns must be at least 1 m deep.

C 10. The cost of producing an ordinary cylindrical tin can is determined by the materials used for the wall and the end pieces. If the end pieces are twice as expensive per square centimetre as the wall, find the dimensions (to the nearest millimetre) to make a 1000 cm^3 can at minimal cost.

11. Your neighbours operate a successful bake shop. One of their specialties is a very rich whipped-cream-covered cake. They buy the cakes from a supplier who charges \$6.00 per cake, and they sell 200 cakes weekly at \$10.00 each. Research shows that profit from the cake sales can be increased by increasing the price. Unfortunately, for every increase of \$0.50 cents, sales will drop by seven cakes.
    a. What is the optimal retail price for a cake to obtain a maximum weekly profit?
    b. The supplier, unhappy with reduced sales, informs the owners that if they purchase fewer than 165 cakes weekly, the cost per cake will increase to \$7.50. Now what is the optimal retail price per cake, and what is the bake shop's total weekly profit?
    c. Situations like this occur regularly in retail trade. Discuss the implications of reduced sales with increased total profit versus greater sales with smaller profits. For example, a drop in the number of customers could mean fewer sales of associated products.

12. Sandy is making a closed rectangular jewellery box with a square base from two different woods. The wood for the top and bottom costs \$20/m^2. The wood for the sides costs \$30/m^2. Find the dimensions that will minimize the cost of the wood for a volume of 4000 cm^3.

13. An electronics store is selling personal CD players. The regular price for each CD player is \$90. During a typical two weeks, the store sells 50 units. Past sales indicate that for every \$1 decrease in price, the store sells five more units during two weeks. Calculate the price that will maximize revenue.

14. A professional basketball team plays in an arena that holds 20 000 spectators. Average attendance at each game has been 14 000. The average ticket price is \$75. Market research shows that for each \$5 reduction in the ticket price, attendance increases by 800. Find the price that will maximize revenue.

15. Through market research, a computer manufacturer found that x thousand units of its new laptop will sell at a price of $2000 - 5x$ dollars per unit. The cost, C, in dollars to produce this many units is $C(x) = 15\,000\,000 + 1\,800\,000x + 75x^2$. Determine the level of sales that will maximize profit.

PART C

16. If the cost of producing x items is given by the function $C(x)$, and the total revenue when x items are sold is $R(x)$, then the profit function is $P(x) = R(x) - C(x)$. Show that the instantaneous rate of change in profit is 0 when the marginal revenue equals the marginal cost.

17. A fuel tank is being designed to contain 200 m^3 of gasoline, but the maximum length of a tank (measured from the tips of each hemisphere) that can be safely transported to clients is 16 m long. The design of the tank calls for a cylindrical part in the middle, with hemispheres at each end. If the hemispheres are twice as expensive per unit area as the cylindrical part, find the radius and height of the cylindrical part so the cost of manufacturing the tank will be minimal. Give your answers correct to the nearest centimetre.

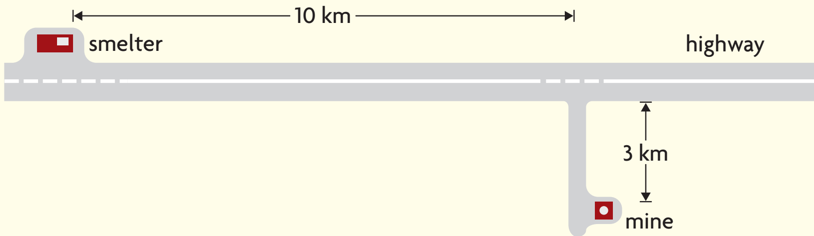
18. A truck crossing the prairies at a constant speed of 110 kilometres per hour gets gas mileage of 8 kilometre per litre. Gas costs \$1.15 per litre. The truck loses 0.10 kilometres per litre in fuel efficiency for each kilometre per hour increase in speed. The driver is paid \$35 per hour in wages and benefits. Fixed costs for running the truck are \$15.50 per hour. If a trip of 450 kilometres is planned, what speed will minimize operating expenses?

19. During a cough, the diameter of the trachea decreases. The velocity, v, of air in the trachea during a cough may be modelled by the formula $v(r) = Ar^2(r_0 - r)$, where A is a constant, r is the radius of the trachea during the cough, and r_0 is the radius of the trachea in a relaxed state. Find the radius of the trachea when the velocity is the greatest, and find the associated maximum velocity of air. Note that the domain for the problem is $0 \le r \le r_0$.

CAREER LINK WRAP-UP Investigate and Apply

CHAPTER 3: MAXIMIZING PROFITS

A construction company has been offered a contract for \$7.8 million to construct and operate a trucking route for five years to transport ore from a mine site to a smelter. The smelter is located on a major highway, and the mine is 3 km into a heavily forested area off the road.



Construction (capital) costs are estimated as follows:

- Repaving the highway will cost \$200 000/km.
- A new gravel road from the mine to the highway will cost \$500 000/km.

Operating conditions are as follows:

- There will be 100 round trips each day, for 300 days a year, for each of the five years the mine will be open.
- Operating costs on the gravel road will be \$65/h, and the speed limit will be 40 km/h.
- Operating costs on the highway will be \$50/h, and the speed limit will be 70 km/h.

Use calculus to determine if the company should accept the contract. Determine the average speeds of the trucks along the paved and gravel roads that produce optimum conditions (maximum profit). What is the maximum profit?

Key Concepts Review

In Chapter 3, you have considered a variety of applications of derivatives on an interval.

You should now be familiar with the following concepts:

- the position, velocity, and acceleration functions $s(t)$, $v(t)$, and $a(t)$, respectively, where $v(t) = s'(t)$ and $a(t) = v'(t) = s''(t)$
- the algorithm for finding absolute maximum and absolute minimum values
- derivatives that involve cost, revenue, and profit in the social sciences
- optimization problems (remember that you must first create a function to analyze, and that restrictions in the domain may be crucial)

Review Exercise

1. Determine f' and f'', if $f(x) = x^4 - \frac{1}{x^4}$.

2. For $y = x^9 - 7x^3 + 2$, find $\frac{d^2y}{dx^2}$.

3. Determine the velocity and acceleration of an object that moves along a straight line in such a way that its position is $s(t) = t^2 + (2t - 3)^{\frac{1}{2}}$.

4. Determine the velocity and acceleration as functions of time, t, for $s(t) = t - 7 + \frac{5}{t}, t \neq 0$.

5. A pellet is shot into the air. Its position above the ground at any time, t, is defined by $s(t) = 45t - 5t^2$ m. For what values of t, $t \geq 0$, is the upward velocity of the pellet positive? For what values of t is the upward velocity zero and negative? Draw a graph to represent the velocity of the pellet.

6. Determine the maximum and minimum of each function on the given interval.
   a. $f(x) = 2x^3 - 9x^2, -2 \leq x \leq 4$
   b. $f(x) = 12x - x^3, x \in [-3, 5]$
   c. $f(x) = 2x + \frac{18}{x}, 1 \leq x \leq 5$

7. A motorist starts braking for a stop sign. After t seconds, the distance, in metres, from the front of the car to the sign is $s(t) = 62 - 16t + t^2$.
   a. How far was the front of the car from the sign when the driver started braking?
   b. Does the car go beyond the stop sign before stopping?
   c. Explain why it is unlikely that the car would hit another vehicle that is travelling perpendicular to the motorist's road when the car first comes to a stop at the intersection.

8. The position function of an object that moves in a straight line is $s(t) = 1 + 2t - \frac{8}{t^2 + 1}, 0 \leq t \leq 2$. Calculate the maximum and minimum velocities of the object over the given time interval.

9. Suppose that the cost, in dollars, of manufacturing x items is approximated by $C(x) = 625 + 15x + 0.01x^2$, for $1 \leq x \leq 500$. The unit cost (the cost of manufacturing one item) would then be $U(x) = \frac{C(x)}{x}$. How many items should be manufactured to ensure that the unit cost is minimized?

10. For each of the following cost functions, determine

    i. the cost of producing 400 items

    ii. the average cost of each of the first 400 items produced

    iii. the marginal cost when $x = 400$, as well as the cost of producing the 401st item

    a. $C(x) = 3x + 1000$

    b. $C(x) = 0.004x^2 + 40x + 8000$

    c. $C(x) = \sqrt{x} + 5000$

    d. $C(x) = 100x^{-\frac{1}{2}} + 5x + 700$

11. Find the production level that minimizes the average cost per unit for the cost function $C(x) = 0.004x^2 + 40x + 16\,000$. Show that it is a minimum by using a graphing calculator to sketch the graph of the average cost function.

12. a. The position of an object moving along a straight line is described by the function $s(t) = 3t^2 - 10$ for $t \geq 0$. Is the object moving toward or away from its starting position when $t = 3$?

    b. Repeat the problem using $s(t) = -t^3 + 4t^2 - 10$ for $t \geq 0$.

13. A particle moving along a straight line will be s centimetres from a fixed point at time t seconds, where $t > 0$ and $s = 27t^3 + \frac{16}{t} + 10$.

    a. Determine when the velocity will be zero.

    b. Is the particle accelerating? Explain.

14. A box with a square base and no top must have a volume of 10 000 cm^3. If the smallest dimension is 5 cm, determine the dimensions of the box that minimize the amount of material used.

15. An animal breeder wishes to create five adjacent rectangular pens, each with an area of 2400 m^2. To ensure that the pens are large enough for grazing, the minimum for either dimension must be 10 m. Find the dimensions required for the pens to keep the amount of fencing used to a minimum.

16. You are given a piece of sheet metal that is twice as long as it is wide and has an area of 800 m^2. Find the dimensions of the rectangular box that would contain a maximum volume if it were constructed from this piece of metal by cutting out squares of equal area at all four corners and folding up the sides. The box will not have a lid. Give your answer correct to one decimal place.

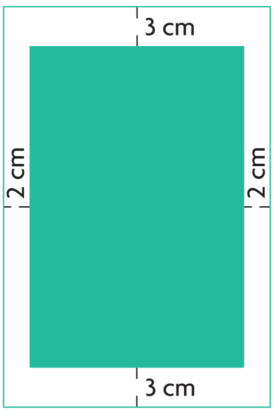
17. A cylindrical can needs to hold 500 cm^3 of apple juice. The height of the can must be between 6 cm and 15 cm, inclusive. How should the can be constructed so that a minimum amount of material will be used in the construction? (Assume that there will be no waste.)

18. In oil pipeline construction, the cost of pipe to go underwater is 60% more than the cost of pipe used in dry-land situations. A pipeline comes to a river that is 1 km wide at point A and must be extended to a refinery, R, on the other side, 8 km down the river. Find the best way to cross the river (assuming it is straight) so that the total cost of the pipe is kept to a minimum. (Give your answer correct to one decimal place.)

19. A train leaves the station at 10:00 p.m. and travels due north at a speed of 100 km/h. Another train has been heading due west at 120 km/h and reaches the same station at 11:00 p.m. At what time were the two trains closest together?

20. A store sells portable MP3 players for $100 each and, at this price, sells 120 MP3 players every month. The owner of the store wishes to increase his profit, and he estimates that, for every $2 increase in the price of MP3 players, one less MP3 player will be sold each month. If each MP3 player costs the store $70, at what price should the store sell the MP3 players to maximize profit?

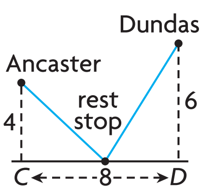
21. An offshore oil well, P, is located in the ocean 5 km from the nearest point on the shore, A. A pipeline is to be built to take oil from P to a refinery that is 20 km along the straight shoreline from A. If it costs $100 000 per kilometre to lay pipe underwater and only $75 000 per kilometre to lay pipe on land, what route from the well to the refinery will be the cheapest? (Give your answer correct to one decimal place.)



22. The printed area of a page in a book will be 81 cm^2. The margins at the top and bottom of the page will each be 3 cm deep. The margins at the sides of the page will each be 2 cm wide. What page dimensions will minimize the amount of paper?

23. A rectangular rose garden will be surrounded by a brick wall on three sides and by a fence on the fourth side. The area of the garden will be 1000 m^2. The cost of the brick wall is $192/m. The cost of the fencing is $48/m. Find the dimensions of the garden so that the cost of the materials will be as low as possible.

24. A boat leaves a dock at 2:00 p.m., heading west at 15 km/h. Another boat heads south at 12 km/h and reaches the same dock at 3:00 p.m. When were the boats closest to each other?



25. Two towns, Ancaster and Dundas, are 4 km and 6 km, respectively, from an old railroad line that has been made into a bike trail. Points C and D on the trail are the closest points to the two towns, respectively. These points are 8 km apart. Where should a rest stop be built to minimize the length of new trail that must be built from both towns to the rest stop?

26. Find the absolute maximum and minimum values.

a. $f(x) = x^2 - 2x + 6, -1 \leq x \leq 7$

b. $f(x) = x^3 + x^2, -3 \leq x \leq 3$

c. $f(x) = x^3 - 12x + 2, -5 \leq x \leq 5$

d. $f(x) = 3x^5 - 5x^3, -2 \leq x \leq 4$

27. Sam applies the brakes steadily to stop his car, which is travelling at 20 m/s. The position of the car, s, in metres at t seconds, is given by $s(t) = 20t - 0.3t^3$. Determine

a. the stopping distance b. the stopping time c. the deceleration at 2 s

28. Calculate each of the following:

a. $f''(2)$ if $f(x) = 5x^3 - x$

b. $f''(-1)$ if $f(x) = -2x^{-3} + x^2$

c. $f''(0)$ if $f(x) = (4x - 1)^4$

d. $f''(1)$ if $f(x) = \dfrac{2x}{x - 5}$

e. $f''(4)$ if $f(x) = \sqrt{x + 5}$

f. $f''(8)$ if $f(x) = \sqrt[3]{x^2}$

29. An object moves along a straight line. The object's position at time t is given by $s(t)$. Find the position, velocity, acceleration, and speed at the specified time.

a. $s(t) = \dfrac{2t}{t + 3}, t = 3$ b. $s(t) = t + \dfrac{5}{t + 2}, t = 1$

30. The function $s(t) = (t^2 + t)^{\frac{2}{3}}, t \geq 0$, represents the displacement, s, in metres, of a particle moving along a straight line after t seconds.

a. Determine $v(t)$ and $a(t)$.

b. Find the average velocity during the first 5 s.

c. Determine the velocity at exactly 5 s.

d. Find the average acceleration during the first 5 s.

e. Determine the acceleration at exactly 5 s.

Chapter 3 Test

1. Determine the second derivative of each of the following:
   a. $y = 7x^2 - 9x + 22$
   b. $f(x) = -9x^5 - 4x^3 + 6x - 12$
   c. $y = 5x^{-3} + 10x^3$
   d. $f(x) = (4x - 8)^3$
2. For each of the following displacement functions, calculate the velocity and acceleration at the indicated time:
   a. $s(t) = -3t^3 + 5t^2 - 6t,\ t = 3$
   b. $s(t) = (2t - 5)^3,\ t = 2$
3. The position function of an object moving horizontally along a straight line as a function of time is $s(t) = t^2 - 3t + 2$, $t \geq 0$, in metres, at time t, in seconds.
   a. Determine the velocity and acceleration of the object.
   b. Determine the position of the object when the velocity is 0.
   c. Determine the speed of the object when the position is 0.
   d. When does the object move to the left?
   e. Determine the average velocity from $t = 2$ to $t = 5$.
4. Determine the maximum and minimum of each function on the given interval.
   a. $f(x) = x^3 - 12x + 2,\ -5 \leq x \leq 5$
   b. $f(x) = x + \frac{9}{x},\ x \in [1, 6]$
5. After a football is punted, its height, h, in metres above the ground at t seconds, can be modelled by $h(t) = -4.9t^2 + 21t + 0.45$, $t \geq 0$.
   a. When does the football reach its maximum height?
   b. What is the football's maximum height?
6. A man purchased 2000 m of used wire fencing at an auction. He and his wife want to use the fencing to create three adjacent rectangular paddocks. Find the dimensions of the paddocks so that the fence encloses the largest possible area.
7. An engineer working on a new generation of computer called The Beaver is using compact VLSI circuits. The container design for the CPU is to be determined by marketing considerations and must be rectangular in shape. It must contain exactly 10 000 cm^3 of interior space, and the length must be twice the height. If the cost of the base is $\$0.02/cm^2$, the cost of the side walls is $\$0.05/cm^2$, and the cost of the upper face is $\$0.10/cm^2$, find the dimensions to the nearest millimetre that will keep the cost of the container to a minimum.
8. The landlord of a 50-unit apartment building is planning to increase the rent. Currently, residents pay $850 per month, and all the units are occupied. A real estate agency advises that every $100 increase in rent will result in 10 vacant units. What rent should the landlord charge to maximize revenue?

Chapter 4

CURVE SKETCHING

If you are having trouble figuring out a mathematical relationship, what do you do? Many people find that visualizing mathematical problems is the best way to understand them and to communicate them more meaningfully. Graphing calculators and computers are powerful tools for producing visual information about functions. Similarly, since the derivative of a function at a point is the slope of the tangent to the function at this point, the derivative is also a powerful tool for providing information about the graph of a function. It should come as no surprise, then, that the Cartesian coordinate system in which we graph functions and the calculus that we use to analyze functions were invented in close succession in the seventeenth century. In this chapter, you will see how to draw the graph of a function using the methods of calculus, including the first and second derivatives of the function.

CHAPTER EXPECTATIONS

In this chapter, you will

- determine properties of the graphs of polynomial and rational functions, **Sections 4.1, 4.3, 4.5**
- describe key features of a given graph of a function, **Sections 4.1, 4.2, 4.4**
- determine intercepts and positions of the asymptotes of a graph, **Section 4.3**
- determine the values of a function near its asymptotes, **Section 4.3**
- determine key features of the graph of a function, **Section 4.5, Career Link**
- sketch, by hand, the graph of the derivative of a given graph, **Section 4.2**
- determine, from the equation of a simple combination of polynomial or rational functions $\left(\text{such as } f(x) = x^2 + \frac{1}{x}\right)$, the key features of the graph of the function, using the techniques of differential calculus, and sketch the graph by hand, **Section 4.4**

Review of Prerequisite Skills

There are many features that we can analyze to help us sketch the graph of a function. For example, we can try to determine the x- and y-intercepts of the graph, we can test for horizontal and vertical asymptotes using limits, and we can use our knowledge of certain kinds of functions to help us determine domains, ranges, and possible symmetries.

In this chapter, we will use the derivatives of functions, in conjunction with the features mentioned above, to analyze functions and their graphs. Before you begin, you should

- be able to solve simple equations and inequalities
- know how to sketch graphs of parent functions and simple transformations of these graphs (including quadratic, cubic, and root functions)
- understand the intuitive concept of a limit of a function and be able to evaluate simple limits
- be able to determine the derivatives of functions using all known rules

Exercise

1. Solve each equation.

a. $2y^2 + y - 3 = 0$

b. $x^2 - 5x + 3 = 17$

c. $4x^2 + 20x + 25 = 0$

d. $y^3 + 4y^2 + y - 6 = 0$

2. Solve each inequality.

a. $3x + 9 < 2$

b. $5(3 - x) \geq 3x - 1$

c. $t^2 - 2t < 3$

d. $x^2 + 3x - 4 > 0$

3. Sketch the graph of each function.

a. $f(x) = (x + 1)^2 - 3$

b. $f(x) = x^2 - 5x - 6$

c. $f(x) = \dfrac{2x - 4}{x + 2}$

d. $f(x) = \sqrt{x - 2}$

4. Evaluate each limit.

a. $\lim\limits_{x \to 2^-} (x^2 - 4)$

b. $\lim\limits_{x \to 2} \dfrac{x^2 + 3x - 10}{x - 2}$

c. $\lim\limits_{x \to 3} \dfrac{x^3 - 27}{x - 3}$

d. $\lim\limits_{x \to 4^+} \sqrt{2x + 1}$

5. Determine the derivative of each function.

a. $f(x) = \frac{1}{4}x^4 + 2x^3 - \frac{1}{x}$

b. $f(x) = \frac{x + 1}{x^2 - 3}$

c. $f(x) = (3x^2 - 6x)^2$

d. $f(t) = \frac{2t}{\sqrt{t - 4}}$

6. Divide, and then write your answer in the form $ax + b + \frac{r}{q(x)}$. For example, $(x^2 + 4x - 5) \div (x - 2) = x + 6 + \frac{7}{x - 2}$.

a. $(x^2 - 5x + 4) \div (x + 3)$

b. $(x^2 + 6x - 9) \div (x - 1)$

7. Determine the points where the tangent is horizontal to $f(x) = x^3 + 0.5x^2 - 2x + 3$.

8. State each differentiation rule in your own words.

a. power rule

b. constant rule

c. product rule

d. quotient rule

e. chain rule

f. power of a function rule

9. Describe the end behaviour of each function as $x \to \infty$ and $x \to -\infty$.

a. $f(x) = 2x^2 - 3x + 4$

b. $f(x) = -2x^3 + 4x - 1$

c. $f(x) = -5x^4 + 2x^3 - 6x^2 + 7x - 1$

d. $f(x) = 6x^5 - 4x - 7$

10. For each function, determine the reciprocal, $y = \frac{1}{f(x)}$, and the equations of the vertical asymptotes of $y = \frac{1}{f(x)}$. Verify your results using graphing technology.

a. $f(x) = 2x$

b. $f(x) = -x + 3$

c. $f(x) = (x + 4)^2 + 1$

d. $f(x) = (x + 3)^2$

11. State the equation of the horizontal asymptote of each function.

a. $y = \frac{5}{x + 1}$

b. $y = \frac{4x}{x - 2}$

c. $y = \frac{3x - 5}{6x - 3}$

d. $y = \frac{10x - 4}{5x}$

12. For each function in question 11, determine the following:

a. the x- and y-intercepts

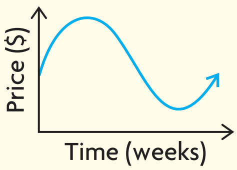
b. the domain and range

CAREER LINK Investigate

CHAPTER 4: PREDICTING STOCK VALUES

Stock-market analysts collect and interpret vast amounts of information and then predict trends in stock values. Stock analysts are classified into two main groups: the fundamentalists who predict stock values based on analysis of the companies' economic situations, and the technical analysts who predict stock values based on trends and patterns in the market. Technical analysts spend a significant amount of their time constructing and interpreting graphical models to find undervalued stocks that will give returns in excess of what the market predicts. In this chapter, your skills in producing and analyzing graphical models will be extended through the use of differential calculus.

Case Study: Technical Stock Analyst



To raise money for expansion, many privately owned companies give the public a chance to own part of their company through purchasing stock. Those who buy ownership expect to obtain a share in the future profits of the company. Some technical analysts believe that the greatest profits to be had in the stock market are through buying brand new stocks and selling them quickly. A technical analyst predicts that a stock's price over its first several weeks on the market will follow the pattern shown on the graph. The technical analyst is advising a person who purchased the stock the day it went on sale.

DISCUSSION QUESTIONS

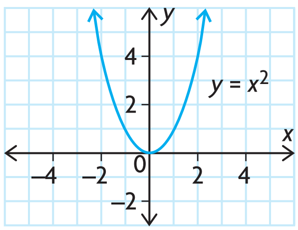
Make a rough sketch of the graph, and answer the following questions:

1. When would you recommend that the owner sell her shares? Label this point S on your graph. What do you notice about the slope, or instantaneous rate of change, of the graph at this point?
2. When would you recommend that the owner get back into the company and buy shares again? Label this point B on your graph. What do you notice about the slope, or instantaneous rate of change, of the graph at this point?
3. A concave-down section of a graph opens in a downward direction, and a concave-up section opens upward. On your graph, find the point where the concavity changes from concave down to concave up, and label this point C. Another analyst says that a change in concavity from concave down to concave up is a signal that a selling opportunity will soon occur. Do you agree with the analyst? Explain.

At the end of this chapter, you will have an opportunity to apply the tools of curve sketching to create, evaluate, and apply a model that could be used to advise clients on when to buy, sell, and hold new stocks.

Section 4.1—Increasing and Decreasing Functions

The graph of the quadratic function $f(x) = x^2$ is a parabola. If we imagine a particle moving along this parabola from left to right, we can see that, while the x-coordinates of the ordered pairs steadily increase, the y-coordinates of the ordered pairs along the particle's path first decrease and then increase. Determining the intervals in which a function increases and decreases is extremely useful for understanding the behaviour of the function. The following statements give a clear picture:



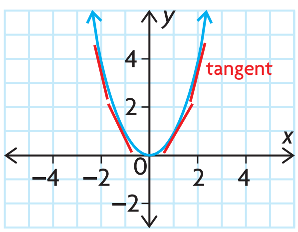
Intervals of Increase and Decrease

We say that a function f *is decreasing on an interval* if, for any value of $x_1 < x_2$ on the interval, $f(x_1) > f(x_2)$.

Similarly, we say that a function f *is increasing on an interval* if, for any value of $x_1 < x_2$ on the interval, $f(x_1) < f(x_2)$.

For the parabola with the equation $y = x^2$, the change from decreasing y-values to increasing y-values occurs at $(0, 0)$, the vertex of the parabola. The function $f(x) = x^2$ is decreasing on the interval $x < 0$ and is increasing on the interval $x > 0$.

If we examine tangents to the parabola anywhere on the interval where the y-values are decreasing (that is, on $x < 0$), we see that all of these tangents have negative slopes. Similarly, the slopes of tangents to the graph on the interval where the y-values are increasing are all positive.



For functions that are both continuous and differentiable, we can determine intervals of increasing and decreasing y-values using the derivative of the function. In the case of $y = x^2$, $\frac{dy}{dx} = 2x$. For $x < 0$, $\frac{dy}{dx} < 0$, and the slopes of the tangents are negative. The interval $x < 0$ corresponds to the decreasing portion of the graph of the parabola. For $x > 0$, $\frac{dy}{dx} > 0$, and the slopes of the tangents are positive on the interval where the graph is increasing.

We summarize this as follows: For a continuous and differentiable function, f, the function values (y-values) are increasing for all x-values where $f'(x) > 0$, and the function values (y-values) are decreasing for all x-values where $f'(x) < 0$.

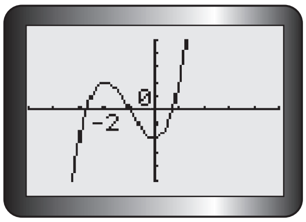
EXAMPLE 1 Using the derivative to reason about intervals of increase and decrease

Use your calculator to graph the following functions. Use the graph to estimate the values of x for which the function values (y-values) are increasing, and the values of x for which the y-values are decreasing. Verify your estimates with an algebraic solution.

a. $y = x^3 + 3x^2 - 2$ b. $y = \frac{x}{x^2 + 1}$

Solution

a. Using a calculator, we obtain the graph of $y = x^3 + 3x^2 - 2$. Using the TRACE key on the calculator, we estimate that the function values are increasing on $x < -2$, decreasing on $-2 < x < 0$, and increasing again on $x > 0$. To verify these estimates with an algebraic solution, we consider the slopes of the tangents.



The slope of a general tangent to the graph of $y = x^3 + 3x^2 - 2$ is given by $\frac{dy}{dx} = 3x^2 + 6x$. We first determine the values of x for which $\frac{dy}{dx} = 0$. These values tell us where the function has a **local maximum** or **local minimum** value. These are greatest and least values respectively of a function in relation to its neighbouring values.

Setting $\frac{dy}{dx} = 0$, we obtain $3x^2 + 6x = 0$

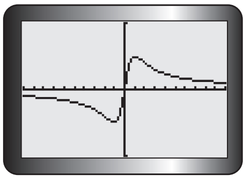
$$3x(x + 2) = 0$$
$$x = 0, x = -2$$

These values of x locate points on the graph where the slope of the tangent is zero (that is, where the tangent is horizontal).

Since this is a polynomial function it is continuous so $\frac{dy}{dx}$ is defined for all values of x. Because $\frac{dy}{dx} = 0$ only at $x = -2$ and $x = 0$, the derivative must be either positive or negative for all other values of x. We consider the intervals $x < -2$, $-2 < x < 0$, and $x > 0$.

Value of *x*	$x < -2$	$-2 < x < 0$	$x > 0$
Sign of $\frac{dy}{dx} = 3x(x + 2)$	$\frac{dy}{dx} > 0$	$\frac{dy}{dx} < 0$	$\frac{dy}{dx} > 0$
Slope of Tangents	positive	negative	positive
Values of *y* Increasing or Decreasing	increasing	decreasing	increasing

So $y = x^3 + 3x^2 - 2$ is increasing on the intervals $x < -2$ and $x > 0$ and is decreasing on the interval $-2 < x < 0$.



b. Using a calculator, we obtain the graph of $y = \frac{x}{x^2 + 1}$. Using the TRACE key on the calculator, we estimate that the function values (y-values) are decreasing on $x < -1$, increasing on $-1 < x < 1$, and decreasing again on $x > 1$.

We analyze the intervals of increasing/decreasing y-values for the function by determining where $\frac{dy}{dx}$ is positive and where it is negative.

$$y = \frac{x}{x^2 + 1}$$ (Express as a product)

$$= x(x^2 + 1)^{-1}$$

$$\frac{dy}{dx} = 1(x^2 + 1)^{-1} + x(-1)(x^2 + 1)^{-2}(2x)$$ (Product and chain rules)

$$= \frac{1}{x^2 + 1} - \frac{2x^2}{(x^2 + 1)^2}$$

$$= \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2}$$ (Simplify)

$$= \frac{-x^2 + 1}{(x^2 + 1)^2}$$

Setting $\frac{dy}{dx} = 0$, we obtain $\frac{-x^2 + 1}{(x^2 + 1)^2} = 0$ (Solve)

$$-x^2 + 1 = 0$$

$$x^2 = 1$$

$$x = 1 \text{ or } x = -1$$

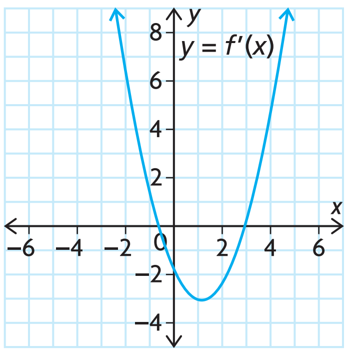
These values of x locate the points on the graph where the slope of the tangent is 0. Since the denominator of this rational function can never be 0, this function is continuous so $\frac{dy}{dx}$ is defined for all values of x. Because $\frac{dy}{dx} = 0$ at $x = -1$ and $x = 1$, we consider the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$.

Value of x	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Sign of $\frac{dy}{dx} = \frac{-x^2 + 1}{(x^2 + 1)^2}$	$\frac{dy}{dx} < 0$	$\frac{dy}{dx} > 0$	$\frac{dy}{dx} < 0$
Slope of Tangents	negative	positive	negative
Values of y Increasing or Decreasing	decreasing	increasing	decreasing

Then $y = \frac{x}{x^2 + 1}$ is increasing on the interval $(-1, 1)$ and is decreasing on the intervals $(-\infty, -1)$ and $(1, \infty)$.

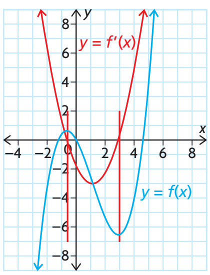
EXAMPLE 2 Graphing a function given the graph of the derivative

Consider the graph of $f'(x)$. Graph $f(x)$.



Solution

When the derivative, $f'(x)$, is positive, the graph of $f(x)$ is rising. When the derivative is negative, the graph is falling. In this example, the derivative changes sign from positive to negative at $x \doteq -0.6$. This indicates that the graph of $f(x)$ changes from increasing to decreasing, resulting in a local maximum for this value of x. The derivative changes sign from negative to positive at $x = 2.9$, indicating the graph of $f(x)$ changes from decreasing to increasing resulting in a local minimum for this value of x.

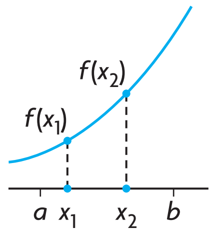


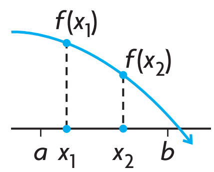
One possible graph of $f(x)$ is shown.

IN SUMMARY

Key Ideas

- A function f is **increasing** on an interval if, for any value of $x_1 < x_2$ in the interval, $f(x_1) < f(x_2)$.
- A function f is **decreasing** on an interval if, for any value of $x_1 < x_2$ in the interval, $f(x_1) > f(x_2)$.





- For a function f that is continuous and differentiable on an interval I
  - $f(x)$ is **increasing** on I if $f'(x) > 0$ for all values of x in I
  - $f(x)$ is **decreasing** on I if $f'(x) < 0$ for all values of x in I

Need to Know

- A function increases on an interval if the graph rises from left to right.
- A function decreases on an interval if the graph falls from left to right.
- The slope of the tangent at a point on a section of a curve that is increasing is always positive.
- The slope of the tangent at a point on a section of a curve that is decreasing is always negative.

Exercise 4.1

PART A

K 1. Determine the points at which $f'(x) = 0$ for each of the following functions:

a. $f(x) = x^3 + 6x^2 + 1$

b. $f(x) = \sqrt{x^2 + 4}$

c. $f(x) = (2x - 1)^2(x^2 - 9)$

d. $f(x) = \dfrac{5x}{x^2 + 1}$

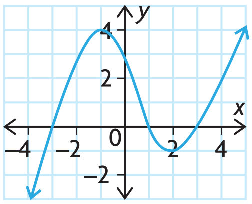
C 2. Explain how you would determine when a function is increasing or decreasing.

3. For each of the following graphs, state

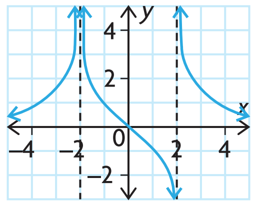
i. the intervals where the function is increasing

ii. the intervals where the function is decreasing

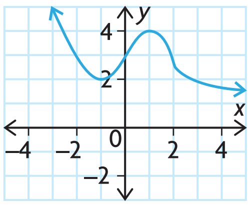
iii. the points where the tangent to the function is horizontal a.



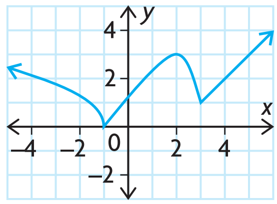
c.



b.



d.



4. Use a calculator to graph each of the following functions. Inspect the graph to estimate where the function is increasing and where it is decreasing. Verify your estimates with algebraic solutions.

a. $f(x) = x^3 + 3x^2 + 1$

b. $f(x) = x^5 - 5x^4 + 100$

c. $f(x) = x + \frac{1}{x}$

d. $f(x) = \frac{x - 1}{x^2 + 3}$

e. $f(x) = 3x^4 + 4x^3 - 12x^2$

f. $f(x) = x^4 + x^2 - 1$

PART B

5. Suppose that f is a differentiable function with the derivative $f'(x) = (x - 1)(x + 2)(x + 3)$. Determine the values of x for which the function f is increasing and the values of x for which the function is decreasing.

A 6. Sketch a graph of a function that is differentiable on the interval $-2 \le x \le 5$ and that satisfies the following conditions:

- The graph of f passes through the points $(-1, 0)$ and $(2, 5)$.
- The function f is decreasing on $-2 < x < -1$, increasing on $-1 < x < 2$, and decreasing again on $2 < x < 5$.

7. Find constants a, b, and c such that the graph of $f(x) = x^3 + ax^2 + bx + c$ will increase to the point $(-3, 18)$, decrease to the point $(1, -14)$, and then continue increasing.

8. Sketch a graph of a function f that is differentiable and that satisfies the following conditions:

- $f'(x) > 0$, when $x < -5$
- $f'(x) < 0$, when $-5 < x < 1$ and when $x > 1$
- $f'(-5) = 0$ and $f'(1) = 0$
- $f(-5) = 6$ and $f(1) = 2$

9. Each of the following graphs represents the derivative function $f'(x)$ of a function $f(x)$. Determine

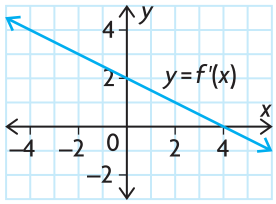
   i. the intervals where $f(x)$ is increasing

   ii. the intervals where $f(x)$ is decreasing

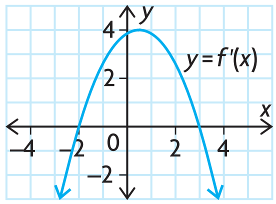
   iii. the x-coordinate for all local extrema of $f(x)$

   Assuming that $f(0) = 2$, make a rough sketch of the graph of each function.

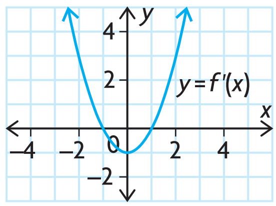
a.



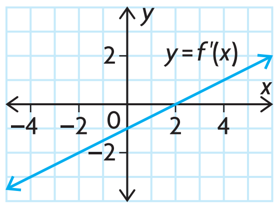
c.



b.



d.



10. Use the derivative to show that the graph of the quadratic function $f(x) = ax^2 + bx + c$, $a > 0$, is decreasing on the interval $x < -\frac{b}{2a}$ and increasing on the interval $x > -\frac{b}{2a}$.

11. For $f(x) = x^4 - 32x + 4$, find where $f'(x) = 0$, the intervals on which the function increases and decreases, and all the local extrema. Use graphing technology to verify your results.

12. Sketch a graph of the function g that is differentiable on the interval $-2 \leq x \leq 5$, decreases on $0 < x < 3$, and increases elsewhere on the domain. The absolute maximum of g is 7, and the absolute minimum is -3. The graph of g has local extrema at $(0, 4)$ and $(3, -1)$.

PART C

T

13. Let f and g be continuous and differentiable functions on the interval $a \leq x \leq b$. If f and g are both increasing on $a \leq x \leq b$, and if $f(x) > 0$ and $g(x) > 0$ on $a \leq x \leq b$, show that the product fg is also increasing on $a \leq x \leq b$.

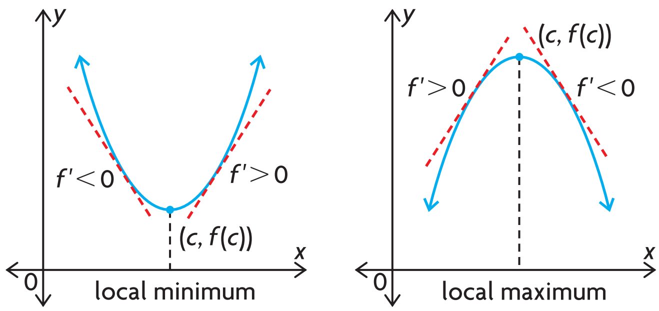
14. Let f and g be continuous and differentiable functions on the interval $a \leq x \leq b$. If f and g are both increasing on $a \leq x \leq b$, and if $f(x) < 0$ and $g(x) < 0$ on $a \leq x \leq b$, is the product fg increasing on $a \leq x \leq b$, decreasing, or neither?

Section 4.2—Critical Points, Local Maxima, and Local Minima

In Chapter 3, we learned that a maximum or minimum function value might occur at a point $(c, f(c))$ if $f'(c) = 0$. It is also possible that a maximum or minimum function value might occur at a point $(c, f(c))$ if $f'(c)$ is undefined. Since these points help to define the shape of the function's graph, they are called **critical points** and the values of c are called **critical numbers**. Combining this with the properties of increasing and decreasing functions, we have a **first derivative test** for local extrema.

The First Derivative Test

Test for local minimum and local maximum points. Let $f'(c) = 0$.



When moving left to right through x-values:

- if $f'(x)$ changes sign from negative to positive at $x = c$, then $f(x)$ has a local minimum at this point.
- if $f'(x)$ changes sign from positive to negative at $x = c$, then $f(x)$ has a local maximum at this point.

$f'(c) = 0$ may imply something other than the existence of a maximum or a minimum at $x = c$. There are also simple functions for which the derivative does not exist at certain points. In Chapter 2, we demonstrated three different ways that this could happen. For example, extrema could occur at points that correspond to cusps and corners on a function's graph and in these cases the derivative is undefined.

EXAMPLE 1 **Connecting the first derivative test to local extrema of a polynomial function**

For the function $y = x^4 - 8x^3 + 18x^2$, determine all the critical numbers. Determine whether each of these values of x gives a local maximum, a local minimum, or neither for the function.

Solution

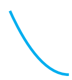
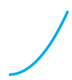
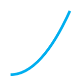
First determine $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 4x^3 - 24x^2 + 36x$$
$$= 4x(x^2 - 6x + 9)$$
$$= 4x(x - 3)^2$$

For critical numbers, let $\frac{dy}{dx} = 0$.

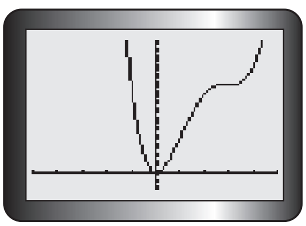
$$4x(x - 3)^2 = 0$$
$$x = 0 \text{ or } x = 3$$

Both values of x are in the domain of the function. There is a horizontal tangent at each of these values of x. To determine which of these values of x yield local maximum or minimum values of the function, we use a table to analyze the behaviour of $\frac{dy}{dx}$ and $y = x^4 - 8x^3 + 18x^2$.

Interval	$x < 0$	$0 < x < 3$	$x > 3$
$4x$	$-$	$+$	$+$
$(x - 3)^2$	$+$	$+$	$+$
$4x(x - 3)^2$	$(-)(+) = -$	$(+)(+) = +$	$(+)(+) = +$
$\frac{dy}{dx}$	< 0	> 0	> 0
$y = x^4 - 8x^3 + 18x^2$	decreasing	increasing	increasing
Shape of the Curve			

Using the information from the table, we see that there is a local minimum value of the function at $x = 0$, since the function values are decreasing before $x = 0$ and increasing after $x = 0$. We can also tell that there is neither a local maximum nor minimum value at $x = 3$, since the function values increase toward this point and increase away from it.

A calculator gives the following graph for $y = x^4 - 8x^3 + 18x^2$, which verifies our solution:



EXAMPLE 2 Reasoning about the significance of horizontal tangents

Determine whether or not the function $f(x) = x^3$ has a maximum or minimum at $(c, f(c))$, where $f'(c) = 0$.

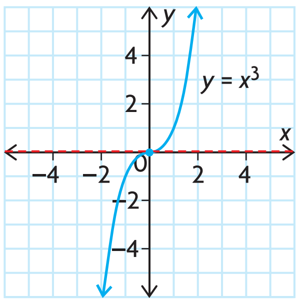
Solution

The derivative is $f'(x) = 3x^2$.

Setting $f'(x) = 0$ gives

$$3x^2 = 0$$
$$x = 0$$

$f(x)$ has a horizontal tangent at $(0, 0)$.



From the graph, it is clear that $(0, 0)$ is neither a maximum nor a minimum value since the values of this function are always increasing. Note that $f'(x) > 0$ for all values of x other than 0.

From this example, we can see that it is possible for a horizontal tangent to exist when $f'(c) = 0$, but that $(c, f(c))$ is neither a maximum nor a minimum. In the next example you will see that it is possible for a maximum or minimum to occur at a point at which the derivative does not exist.

EXAMPLE 3 Reasoning about the significance of a cusp

For the function $f(x) = (x + 2)^{\frac{2}{3}}$, determine the critical numbers. Use your calculator to sketch a graph of the function.

Solution

First determine $f'(x)$.

$$f'(x) = \frac{2}{3}(x + 2)^{-\frac{1}{3}}$$
$$= \frac{2}{3(x + 2)^{\frac{1}{3}}}$$

Note that there is no value of x for which $f'(x) = 0$ since the numerator is always positive. However, $f'(x)$ is undefined for $x = -2$.

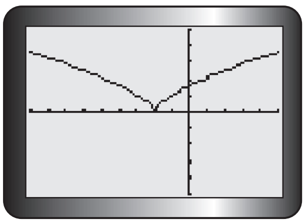
Since $f(-2) = (-2 + 2)^{\frac{2}{3}} = 0$, $x = -2$ is in the domain of $f(x) = (x + 2)^{\frac{2}{3}}$. We determine the slopes of tangents for x-values close to -2.

x	$f'(x) = \dfrac{2}{3(x+2)^{\frac{1}{3}}}$	x	$f'(x) = \dfrac{2}{3(x+2)^{\frac{1}{3}}}$
−2.1	−1.436 29	−1.9	1.436 29
−2.01	−3.094 39	−1.99	3.094 39
−2.001	−6.666 67	−1.999	6.666 67
−2.000 01	−30.943 9	−1.999 99	30.943 9

The slope of the tangent is undefined at the point $(-2, 0)$. Therefore, the function has one critical point, when $x = -2$.

In this example, the slopes of the tangents to the left of $x = -2$ are approaching $-\infty$, while the slopes to the right of $x = -2$ are approaching $+\infty$. Since the slopes on opposite sides of $x = -2$ are not approaching the same value, there is no tangent at $x = -2$ even though there is a point on the graph.

A calculator gives the following graph of $f(x) = (x + 2)^{\frac{2}{3}}$. There is a cusp at $(-2, 0)$.



If a value c is in the domain of a function $f(x)$, and if this value is such that $f'(c) = 0$ or $f'(c)$ is undefined, then $(c, f(c))$ is a critical point of the function f and c is called a critical number for f''.

In summary, critical points that occur when $\frac{dy}{dx} = 0$ give the locations of horizontal tangents on the graph of a function. Critical points that occur when $\frac{dy}{dx}$ is undefined give the locations of either vertical tangents or cusps (where we say that no tangent exists). Besides giving the location of interesting tangents (or lack thereof), critical points also determine other interesting features of the graph of a function.

Critical Numbers and Local Extrema

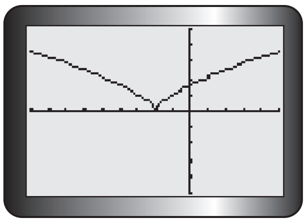
The critical number c determines the location of a local minimum value for a function f if $f(c) < f(x)$ for all values of x near c.

Similarly, the critical number c determines the location of a local maximum value for a function f if $f(c) > f(x)$ for all values of x near c.

Together, local maximum and minimum values of a function are called local extrema.

As mentioned earlier, a local minimum value of a function does not have to be the smallest value in the entire domain, just the smallest value in its neighbourhood. Similarly, a local maximum value of a function does not have to be the largest value in the entire domain, just the largest value in its neighbourhood. Local extrema occur graphically as peaks or valleys. The peaks and valleys can be either smooth or sharp.

To apply this reasoning, let's reconsider the graph of $f(x) = (x + 2)^{\frac{2}{3}}$.



The function $f(x) = (x + 2)^{\frac{2}{3}}$ has a local minimum value at $x = -2$, which also happens to be a critical value of the function.

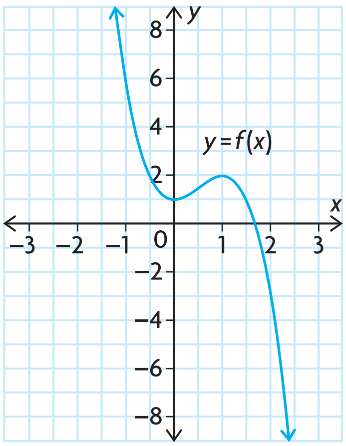
Every local maximum or minimum value of a function occurs at a critical point of the function.

In simple terms, peaks or valleys occur on the graph of a function at places where the tangent to the graph is horizontal, vertical, or does not exist.

How do we determine whether a critical point yields a local maximum or minimum value of a function without examining the graph of the function? We use the first derivative test to analyze whether the function is increasing or decreasing on either side of the critical point.

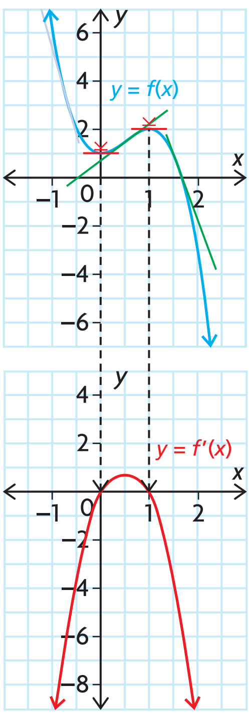
EXAMPLE 4 Graphing the derivative given the graph of a polynomial function

Given the graph of a polynomial function $y = f(x)$, graph $y = f'(x)$.



Solution

A polynomial function f is continuous for all values of x in the domain of f. The derivative of f, f', is also continuous for all values of x in the domain of f.



To graph $y = f'(x)$ using the graph of $y = f(x)$, first determine the slopes of the tangent lines, $f'(x_i)$, at certain x-values, x_i. These x-values include zeros, critical numbers, and numbers in each interval where f is increasing or decreasing. Then plot the corresponding ordered pairs on a graph. Draw a smooth curve through these points to complete the graph.

The given graph has a local minimum at (0, 1) and a local maximum at (1, 2). At these points, the tangents are horizontal. Therefore, $f'(0) = 0$ and $f'(1) = 0$.

At $x = \frac{1}{2}$, which is halfway between $x = 0$ and $x = 1$, the slope of the tangent is about $\frac{2}{3}$. So $f'\left(\frac{1}{2}\right) \doteq \frac{2}{3}$

The function $f(x)$ is decreasing when $f'(x) < 0$. The tangent lines show that $f'(x) < 0$ when $x < 0$ and when $x > 1$. Similarly, $f(x)$ is increasing when $f'(x) > 0$. The tangent lines show that $f'(x) > 0$ when $0 < x < 1$.

The shape of the graph of $f(x)$ suggests that $f(x)$ is a cubic polynomial with a negative leading coefficient. Assume that this is true. The derivative, $f'(x)$, may be a quadratic function with a negative leading coefficient. If it is, the graph of $f'(x)$ is a parabola that opens down.

Plot (0, 0), (1, 0), and $\left(\frac{1}{2}, \frac{2}{3}\right)$ on the graph of $f'(x)$. The graph of $f'(x)$ is a parabola that opens down and passes through these points.

IN SUMMARY

Key Idea

For a function $f(x)$, a **critical number** is a number, c, in the domain of $f(x)$ such that $f'(x) = 0$ or $f'(x)$ is undefined. As a result $(c, f(c))$ is called a critical point and usually corresponds to local or absolute extrema.

Need to Know

First Derivative Test

Let c be a critical number of a function f.

When moving through x-values from left to right:

- if $f'(x)$ changes from negative to positive at c, then $(c, f(c))$ is a **local minimum** of f.
- if $f'(x)$ changes from positive to negative at c, then $(c, f(c))$ is a **local maximum** of f.
- if $f'(x)$ does not change its sign at c, then $(c, f(c))$ is neither a local minimum or a local maximum.

Algorithm for Finding Local Maximum and Minimum Values of a Function *f*

1. Find critical numbers of the function (that is, determine where $f'(x) = 0$ and where $f'(x)$ is undefined) for all x-values in the domain of f.
2. Use the first derivative to analyze whether f is increasing or decreasing on either side of each critical number.
3. Based upon your findings in step 2., conclude whether each critical number locates a local maximum value of the function f, a local minimum value, or neither.

Exercise 4.2

PART A

C 1. Explain what it means to determine the critical points of the graph of a given function.

2. a. For the function $y = x^3 - 6x^2$, explain how you would find the critical points.

   b. Determine the critical points for $y = x^3 - 6x^2$, and then sketch the graph.

3. Find the critical points for each function. Use the first derivative test to determine whether the critical point is a local maximum, local minimum, or neither.

   a. $y = x^4 - 8x^2$ b. $f(x) = \dfrac{2x}{x^2 + 9}$ c. $y = x^3 + 3x^2 + 1$

4. Find the x- and y-intercepts of each function in question 3, and then sketch the curve.

5. Determine the critical points for each function. Determine whether the critical point is a local maximum or minimum, and whether or not the tangent is parallel to the horizontal axis.

   a. $h(x) = -6x^3 + 18x^2 + 3$
   b. $g(t) = t^5 + t^3$
   c. $y = (x - 5)^{\frac{1}{3}}$
   d. $f(x) = (x^2 - 1)^{\frac{1}{3}}$

6. Use graphing technology to graph the functions in question 5 and verify your results.

PART B

K 7. Determine the critical points for each of the following functions, and determine whether the function has a local maximum value, a local minimum value, or neither at the critical points. Sketch the graph of each function.

   a. $f(x) = -2x^2 + 8x + 13$
   b. $f(x) = \frac{1}{3}x^3 - 9x + 2$
   c. $f(x) = 2x^3 + 9x^2 + 12x$
   d. $f(x) = -3x^3 - 5x$
   e. $f(x) = \sqrt{x^2 - 2x + 2}$
   f. $f(x) = 3x^4 - 4x^3$

A 8. Suppose that f is a differentiable function with the derivative $f'(x) = (x + 1)(x - 2)(x + 6)$. Find all the critical numbers of f, and determine whether each corresponds to a local maximum, a local minimum, or neither.

9. Sketch a graph of a function f that is differentiable on the interval $-3 \leq x \leq 4$ and that satisfies the following conditions:

   - The function f is decreasing on $-1 < x < 3$ and increasing elsewhere on $-3 \leq x \leq 4$.
   - The largest value of f is 6, and the smallest value is 0.
   - The graph of f has local extrema at $(-1, 6)$ and $(3, 1)$.

10. Determine values of a, b, and c such that the graph of $y = ax^2 + bx + c$ has a relative maximum at $(3, 12)$ and crosses the y-axis at $(0, 1)$.

11. For $f(x) = x^2 + px + q$, find the values of p and q such that $f(1) = 5$ is an extremum of f on the interval $0 \leq x \leq 2$. Is this extremum a maximum value or a minimum value? Explain.

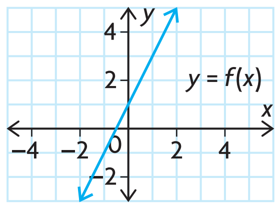
12. For $f(x) = x^3 - kx$, where $k \in \mathbf{R}$, find the values of k such that f has

    a. no critical numbers
    b. one critical number
    c. two critical numbers

T 13. Find values of a, b, c, and d such that $g(x) = ax^3 + bx^2 + cx + d$ has a local maximum at $(2, 4)$ and a local minimum at $(0, 0)$.

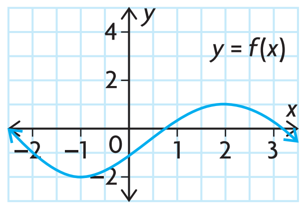
14. For each of the following graphs of the function $y = f(x)$, make a rough sketch of the derivative function $f'(x)$. By comparing the graphs of $f(x)$ and $f'(x)$, show that the intervals for which $f(x)$ is increasing correspond to the intervals where $f'(x)$ is positive. Also show that the intervals where $f(x)$ is decreasing correspond to the intervals for which $f'(x)$ is negative.

a.



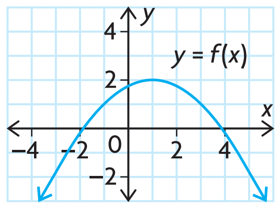
$f(x)$ is a linear function.

c.



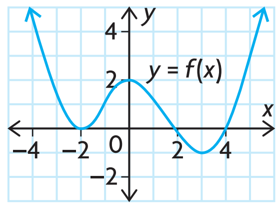
$f(x)$ is a cubic function.

b.



$f(x)$ is a quadratic function.

d.



$f(x)$ is a quartic function.

15. Consider the function $f(x) = 3x^4 + ax^3 + bx^2 + cx + d$.
    a. Find constants a, b, c, and d such that the graph of f will have horizontal tangents at $(-2, -73)$ and $(0, -9)$.
    b. There is a third point that has a horizontal tangent. Find this point.
    c. For all three points, determine whether each corresponds to a local maximum, a local minimum, or neither.

PART C

16. For each of the following polynomials, find the local extrema and the direction that the curve is opening for $x = 100$. Use this information to make a quick sketch of the curve.
    a. $y = 4 - 3x^2 - x^4$
    b. $y = 3x^5 - 5x^3 - 30x$

17. Suppose that $f(x)$ and $g(x)$ are positive functions (functions where $f(x) > 0$ and $g(x) > 0$) such that $f(x)$ has a local maximum and $g(x)$ has a local minimum at $x = c$. Show that the function $h(x) = \frac{f(x)}{g(x)}$ has a local maximum at $x = c$.

Section 4.3—Vertical and Horizontal Asymptotes

Adding, subtracting, or multiplying two polynomial functions yields another polynomial function. Dividing two polynomial functions results in a function that is not a polynomial. The quotient is a **rational function**. Asymptotes are among the special features of rational functions, and they play a significant role in curve sketching. In this section, we will consider vertical and horizontal asymptotes of rational functions.

INVESTIGATION

The purpose of this investigation is to examine the occurrence of vertical asymptotes for rational functions.

A. Use your graphing calculator to obtain the graph of $f(x) = \frac{1}{x - k}$ and the table of values for each of the following: $k = 3, 1, 0, -2, -4$, and -5.

B. Describe the behaviour of each graph as x approaches k from the right and from the left.

C. Repeat parts A and B for the function $f(x) = \frac{x + 3}{x - k}$ using the same values of k.

D. Repeat parts A and B for the function $f(x) = \frac{1}{x^2 + x - k}$ using the following values: $k = 2, 6$, and 12.

E. Make a general statement about the existence of a vertical asymptote for a rational function of the form $y = \frac{p(x)}{q(x)}$ if there is a value c such that $q(c) = 0$ and $p(c) \neq 0$.

Vertical Asymptotes and Rational Functions

Recall that the notation $x \to c^+$ means that x approaches c from the right. Similarly, $x \to c^-$ means that x approaches c from the left.

You can see from this investigation that as $x \to c$ from either side, the function values get increasingly large and either positive or negative depending on the value of $p(c)$. We say that the function values approach $+\infty$ (positive infinity) or $-\infty$ (negative infinity). These are not numbers. They are symbols that represent the behaviour of a function that increases or decreases without limit.

Because the symbol ∞ is not a number, the limits $\lim\limits_{x \to c^+} \frac{1}{x - c}$ and $\lim\limits_{x \to c^-} \frac{1}{x - c}$ *do not exist*. For convenience, however, we use the notation $\lim\limits_{x \to c^+} \frac{1}{x - c} = +\infty$ and $\lim\limits_{x \to c^-} \frac{1}{x - c} = -\infty$.

These limits form the basis for determining the asymptotes of simple functions.

Vertical Asymptotes of Rational Functions

A rational function of the form $f(x) = \frac{p(x)}{q(x)}$ has a vertical asymptote $x = c$ if $q(c) = 0$ and $p(c) \neq 0$.

EXAMPLE 1 Reasoning about the behaviour of a rational function near its vertical asymptotes

Determine any vertical asymptotes of the function $f(x) = \frac{x}{x^2 + x - 2}$, and describe the behaviour of the graph of the function for values of x near the asymptotes.

Solution

First, determine the values of x for which $f(x)$ is undefined by solving the following:

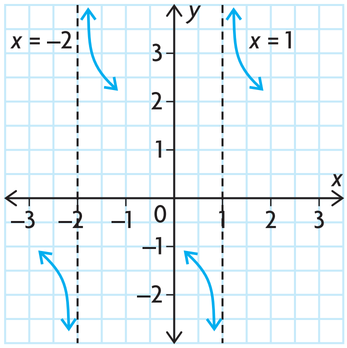
$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0$$
$$x = -2 \text{ or } x = 1$$

Neither of these values of x makes the numerator zero, so both of these values give vertical asymptotes. The equations of the asymptotes are $x = -2$ and $x = 1$.

To determine the behaviour of the graph near the asymptotes, it can be helpful to use a chart.

Values of x	x	$x + 2$	$x - 1$	$f(x) = \frac{x}{(x + 2)(x - 1)}$	$f(x) \to ?$
$x \to -2^-$	< 0	< 0	< 0	< 0	$-\infty$
$x \to -2^+$	< 0	> 0	< 0	> 0	$+\infty$
$x \to 1^-$	> 0	> 0	< 0	< 0	$-\infty$
$x \to 1^+$	> 0	> 0	> 0	> 0	$+\infty$

The behaviour of the graph can be illustrated as follows:



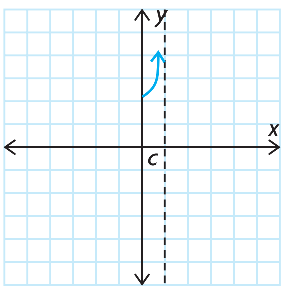
To proceed beyond this point, we require additional information.

Vertical Asymptotes and Infinite Limits

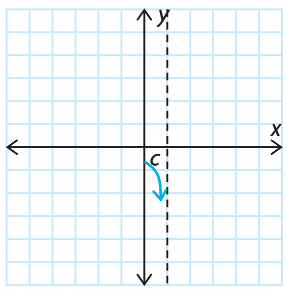
The graph of $f(x)$ has a vertical asymptote, $x = c$, if one of the following infinite limit statements is true:

$$\lim_{x \to c^-} f(x) = +\infty, \quad \lim_{x \to c^-} f(x) = -\infty, \quad \lim_{x \to c^+} f(x) = +\infty \quad \text{or} \quad \lim_{x \to c^+} f(x) = -\infty$$

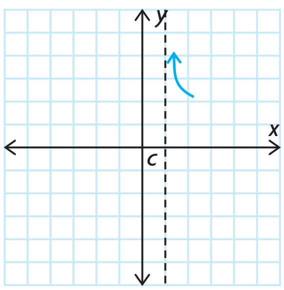
The following graphs correspond to each limit statement above:



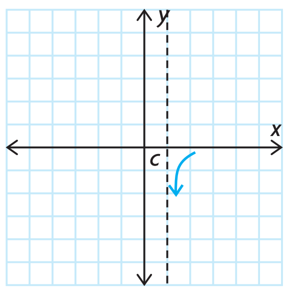
$\lim_{x \to c^-} f(x) = +\infty$



$\lim_{x \to c^-} f(x) = -\infty$



$\lim_{x \to c^+} f(x) = +\infty$



$\lim_{x \to c^+} f(x) = -\infty$

Horizontal Asymptotes and Rational Functions

Consider the behaviour of rational functions $f(x) = \frac{p(x)}{q(x)}$ as x increases without bound in both the positive and negative directions. The following notation is used to describe this behaviour:

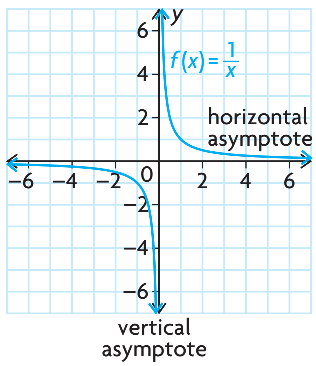
$$\lim_{x \to +\infty} f(x) \text{ and } \lim_{x \to -\infty} f(x)$$

The notation $x \to +\infty$ is read "x tends to positive infinity" and means that the values of x are positive and growing in magnitude without bound. Similarly, the notation $x \to -\infty$ is read "x tends to negative infinity" and means that the values of x are negative and growing in magnitude without bound.

The values of these limits can be determined by making two observations. The first observation is a list of simple limits, similar to those used for determining vertical asymptotes.

The Reciprocal Function and Limits at Infinity

$$\lim_{x \to +\infty} \frac{1}{x} = 0 \text{ and } \lim_{x \to -\infty} \frac{1}{x} = 0$$



The second observation is that a polynomial can always be written so the term of highest degree is a factor.

EXAMPLE 2

Expressing a polynomial function in an equivalent form

Write each function so the term of highest degree is a factor.

a. $p(x) = x^2 + 4x + 1$

b. $q(x) = 3x^2 - 4x + 5$

Solution

a. $p(x) = x^2 + 4x + 1$

$$= x^2\left(1 + \frac{4}{x} + \frac{1}{x^2}\right)$$

b. $q(x) = 3x^2 - 4x + 5$

$$= 3x^2\left(1 - \frac{4}{3x} + \frac{5}{3x^2}\right)$$

The value of writing a polynomial in this form is clear. It is easy to see that as x becomes large (either positive or negative), the value of the second factor always approaches 1.

We can now determine the limit of a rational function in which the degree of $p(x)$ is equal to or less than the degree of $q(x)$.

EXAMPLE 3

Selecting a strategy to evaluate limits at infinity

Determine the value of each of the following:

a. $\lim_{x \to +\infty} \frac{2x - 3}{x + 1}$

b. $\lim_{x \to -\infty} \frac{x}{x^2 + 1}$

c. $\lim_{x \to +\infty} \frac{2x^2 + 3}{3x^2 - x + 4}$

Solution

a. $$f(x) = \frac{2x - 3}{x + 1} = \frac{2x\left(1 - \frac{3}{2x}\right)}{x\left(1 + \frac{1}{x}\right)}$$ (Factor and simplify)

$$= \frac{2\left(1 - \frac{3}{2x}\right)}{1 + \frac{1}{x}}$$

$$\lim_{x \to +\infty} f(x) = \frac{2\left[\lim_{x \to +\infty}\left(1 - \frac{3}{2x}\right)\right]}{\lim_{x \to +\infty}\left(1 + \frac{1}{x}\right)}$$ (Apply limit properties)

$$= \frac{2(1 - 0)}{1 + 0}$$ (Evaluate)

$$= 2$$

b. $$g(x) = \frac{x}{x^2 + 1} \qquad \text{(Factor)}$$

$$= \frac{x(1)}{x^2\left(1 + \frac{1}{x^2}\right)} \qquad \text{(Simplify)}$$

$$= \frac{1}{x\left(1 + \frac{1}{x^2}\right)}$$

$$\lim_{x \to -\infty} g(x) = \frac{1}{\lim_{x \to -\infty} x \times \lim_{x \to -\infty} \left(1 + \frac{1}{x^2}\right)} \qquad \text{(Apply limit properties)}$$

$$= \frac{1}{\lim_{x \to -\infty} x \times (1)}$$

$$= \lim_{x \to -\infty} \frac{1}{x} \qquad \text{(Evaluate)}$$

$$= 0$$

c. To evaluate this limit, we can use the technique of dividing the numerator and denominator by the highest power of x in the denominator.

$$p(x) = \frac{2x^2 + 3}{3x^2 - x + 4} \qquad \text{(Divide by } x^2\text{)}$$

$$= \frac{(2x^2 + 3) \div x^2}{(3x^2 - x + 4) \div x^2} \qquad \text{(Simplify)}$$

$$= \frac{2 + \frac{3}{x^2}}{3 - \frac{1}{x} + \frac{4}{x^2}}$$

$$\lim_{x \to +\infty} p(x) = \frac{\lim_{x \to +\infty}\left(2 + \frac{3}{x^2}\right)}{\lim_{x \to +\infty}\left(3 - \frac{1}{x} + \frac{4}{x^2}\right)} \qquad \text{(Apply limit properties)}$$

$$= \frac{2 + 0}{3 - 0 + 0} \qquad \text{(Evaluate)}$$

$$= \frac{2}{3}$$

When $\lim_{x \to +\infty} f(x) = k$ or $\lim_{x \to -\infty} f(x) = k$, the graph of the function is approaching the line $y = k$. This line is a horizontal asymptote of the function. In Example 3, part a, $y = 2$ is a horizontal asymptote of $f(x) = \frac{2x - 3}{x + 1}$. Therefore, for large positive x-values, the y-values approach 2. This is also the case for large negative x-values.

To sketch the graph of the function, we need to know whether the curve approaches the horizontal asymptote from above or below. To find out, we need to consider $f(x) - k$, where k is the limit we just determined. This is illustrated in the following examples.

EXAMPLE 4 Reasoning about the end behaviours of a rational function

Determine the equations of any horizontal asymptotes of the function $f(x) = \frac{3x + 5}{2x - 1}$. State whether the graph approaches the asymptote from above or below.

Solution

$$f(x) = \frac{3x + 5}{2x - 1} = \frac{(3x + 5) \div x}{(2x - 1) \div x}$$

(Divide by x)

$$= \frac{3 + \frac{5}{x}}{2 - \frac{1}{x}}$$

$$\lim_{x \to +\infty} f(x) = \frac{\lim_{x \to +\infty}\left(3 + \frac{5}{x}\right)}{\lim_{x \to +\infty}\left(2 - \frac{1}{x}\right)}$$

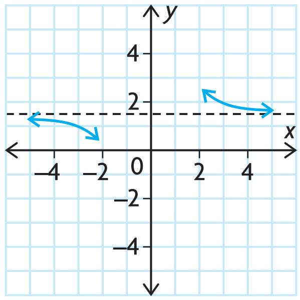
(Evaluate)

$$= \frac{3}{2}$$

Similarly, we can show that $\lim_{x \to -\infty} f(x) = \frac{3}{2}$. So, $y = \frac{3}{2}$ is a horizontal asymptote of the graph of $f(x)$ for both large positive and negative values of x. To determine whether the graph approaches the asymptote from above or below, we consider very large positive and negative values of x.

If x is large and positive (for example, if $x = 1000$), $f(x) = \frac{3005}{1999}$, which is greater than $\frac{3}{2}$. Therefore, the graph approaches the asymptote $y = \frac{3}{2}$ from above.

If x is large and negative (for example, if $x = -1000$), $f(x) = \frac{-2995}{-20\,001}$, which is less than $\frac{3}{2}$. This part of the graph approaches the asymptote $y = \frac{3}{2}$ from below, as illustrated in the diagram.



EXAMPLE 5 Selecting a limit strategy to analyze the behaviour of a rational function near its asymptotes

For the function $f(x) = \frac{3x}{x^2 - x - 6}$, determine the equations of all horizontal or vertical asymptotes. Illustrate the behaviour of the graph as it approaches the asymptotes.

Solution

For vertical asymptotes,

$$x^2 - x - 6 = 0$$

$$(x - 3)(x + 2) = 0$$

$$x = 3 \text{ or } x = -2$$

There are two vertical asymptotes, at $x = 3$ and $x = -2$.

Values of x	x	$x - 3$	$x + 2$	$f(x)$	$f(x) \rightarrow ?$
$x \rightarrow 3^-$	> 0	< 0	> 0	< 0	$-\infty$
$x \rightarrow 3^+$	> 0	> 0	> 0	> 0	$+\infty$
$x \rightarrow -2^-$	< 0	< 0	< 0	< 0	$-\infty$
$x \rightarrow -2^+$	< 0	< 0	> 0	> 0	$+\infty$

For horizontal asymptotes,

$$f(x) = \frac{3x}{x^2 - x - 6} \qquad \text{(Factor)}$$

$$= \frac{3x}{x^2\left(1 - \frac{1}{x} - \frac{6}{x^2}\right)} \qquad \text{(Simplify)}$$

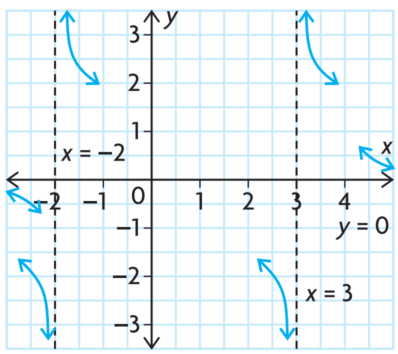
$$= \frac{3}{x\left(1 - \frac{1}{x} - \frac{6}{x^2}\right)}$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to\infty} \frac{3}{x} = 0$$

Similarly, we can show $\lim_{x\to-\infty} f(x) = 0$. Therefore, $y = 0$ is a horizontal asymptote of the graph of $f(x)$ for both large positive and negative values of x.

As x becomes large positively, $f(x) > 0$, so the graph is above the horizontal asymptote. As x becomes large negatively, $f(x) < 0$, so the graph is below the horizontal asymptote.

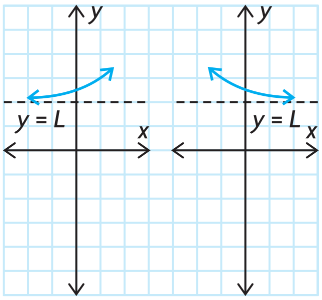
This diagram illustrates the behaviour of the graph as it nears the asymptotes:



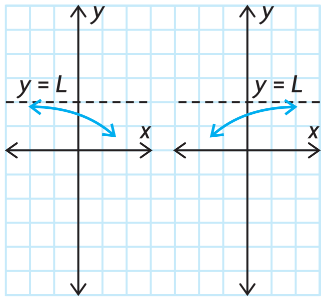
Horizontal Asymptotes and Limits at Infinity

If $\lim_{x\to+\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, we say that the line $y = L$ is a horizontal asymptote of the graph of $f(x)$.

The following graphs illustrate some typical situations:



$f(x) > L$, so the graph approaches from above.



$f(x) < L$, so the graph approaches from below.

In addition to vertical and horizontal asymptotes, it is possible for a graph to have **oblique asymptotes**. These are straight lines that are slanted and to which the curve becomes increasingly close. They occur with rational functions in which the degree of the numerator exceeds the degree of the denominator by exactly one. This is illustrated in the following example.

EXAMPLE 6

Reasoning about oblique asymptotes

Determine the equations of all asymptotes of the graph of $f(x) = \dfrac{2x^2 + 3x - 1}{x + 1}$.

Solution

Since $x + 1 = 0$ for $x = -1$, and $2x^2 + 3x - 1 \neq 0$ for $x = -1$, $x = -1$ is a vertical asymptote.

$$\text{Now } \lim_{x\to\infty} \frac{2x^2 + 3x - 1}{x + 1} = \lim_{x\to\infty} \frac{2x^2\left(1 + \frac{3}{2x} - \frac{1}{2x^2}\right)}{x\left(1 + \frac{1}{x}\right)}$$

$$= \lim_{x\to\infty} 2x$$

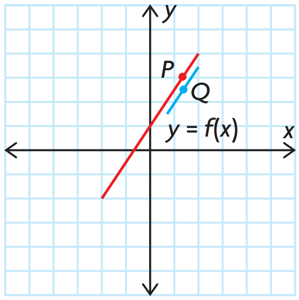
This limit does not exist, and, by a similar calculation, $\lim_{x\to-\infty} f(x)$ does not exist, so there is no horizontal asymptote.

Dividing the numerator by the denominator,

$$\begin{array}{r@{}l}
 & 2x + 1 \\
x + 1 \big) & \overline{2x^2 + 3x - 1} \\
 & \underline{2x^2 + 2x} \\
 & \qquad\;\; x - 1 \\
 & \qquad\;\; \underline{x + 1} \\
 & \qquad\quad\; -2
\end{array}$$

Thus, we can write $f(x)$ in the form $f(x) = 2x + 1 - \dfrac{2}{x + 1}$.

Now let's consider the straight line $y = 2x + 1$ and the graph of $y = f(x)$. For any value of x, we can determine point $P(x, 2x + 1)$ on the line and point $Q\left(x, 2x + 1 - \frac{2}{x + 1}\right)$ on the curve.



Then the vertical distance QP from the curve to the line is

$$QP = 2x + 1 - \left(2x + 1 - \frac{2}{x + 1}\right)$$
$$= \frac{2}{x + 1}$$

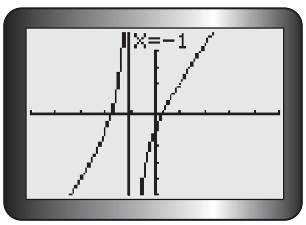
$$\lim_{x \to \infty} QP = \lim_{x \to \infty} \frac{2}{x + 1}$$
$$= 0$$

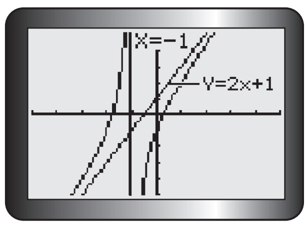
That is, as x gets very large, the curve approaches the line but never touches it. Therefore, the line $y = 2x + 1$ is an asymptote of the curve.

Since $\lim_{x \to -\infty} \frac{2}{x + 1} = 0$, the line is also an asymptote for large negative values of x.

In conclusion, there are two asymptotes of the graph of $f(x) = \frac{2x^2 + 3x - 1}{x + 1}$. They are $y = 2x + 1$ and $x = -1$.

Use a graphing calculator to obtain the graph of $f(x) = \frac{2x^2 + 3x - 1}{x + 1}$.





Note that the vertical asymptote $x = -1$ appears on the graph on the left, but the oblique asymptote $y = 2x + 1$ does not. Use the Y2 function to graph the oblique asymptote $y = 2x + 1$.

IN SUMMARY

Key Ideas

- The graph of $f(x)$ has a **vertical asymptote** $x = c$ if any of the following is true:
  $\lim_{x \to c^-} f(x) = +\infty$ $\quad\quad$ $\lim_{x \to c^-} f(x) = -\infty$
  $\lim_{x \to c^+} f(x) = +\infty$ $\quad\quad$ $\lim_{x \to c^+} f(x) = -\infty$
- The line $y = L$ is a **horizontal asymptote** of the graph of $f(x)$ if $\lim_{x \to +\infty} f(x) = L$ *or* $\lim_{x \to -\infty} f(x) = L$.
- In a rational function, an **oblique asymptote** occurs when the degree of the numerator is exactly one greater than the degree of the denominator**.**

Need to Know

The techniques for curve sketching developed to this point are described in the following algorithm. As we develop new ideas, the algorithm will be extended.

Algorithm for Curve Sketching (so far)

To sketch a curve, apply these steps in the order given.

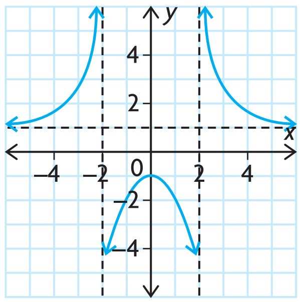
1. Check for any discontinuities in the domain. Determine if there are vertical asymptotes at these discontinuities, and determine the direction from which the curve approaches these asymptotes.
2. Find **both intercepts**.
3. Find any critical points.
4. Use the first derivative test to determine the type of critical points that may be present.
5. **Test end behaviour** by determining $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.
6. Construct an interval of increase/decrease table and identify all local or absolute extrema.
7. Sketch the curve.

Exercise 4.3

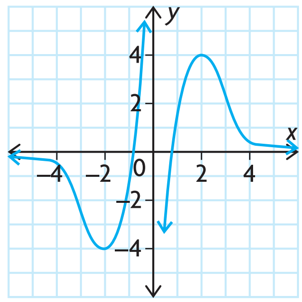
PART A

1. State the equations of the vertical and horizontal asymptotes of the curves shown.

a.



b.



C

2. Under what conditions does a rational function have vertical, horizontal, and oblique asymptotes?

3. Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, using the symbol "∞" when appropriate.

a. $f(x) = \dfrac{2x + 3}{x - 1}$

b. $f(x) = \dfrac{5x^2 - 3}{x^2 + 2}$

c. $f(x) = \dfrac{-5x^2 + 3x}{2x^2 - 5}$

d. $f(x) = \dfrac{2x^5 - 3x^2 + 5}{3x^4 + 5x - 4}$

4. For each of the following, check for discontinuities and state the equation of any vertical asymptotes. Conduct a limit test to determine the behaviour of the curve on either side of the asymptote.

a. $y = \dfrac{x}{x + 5}$

b. $f(x) = \dfrac{x + 2}{x - 2}$

c. $s = \dfrac{1}{(t - 3)^2}$

d. $y = \dfrac{x^2 - x - 6}{x - 3}$

e. $f(x) = \dfrac{6}{(x + 3)(x - 1)}$

f. $y = \dfrac{x^2}{x^2 - 1}$

5. For each of the following, determine the equations of any horizontal asymptotes. Then state whether the curve approaches the asymptote from above or below.

a. $y = \dfrac{x}{x + 4}$

b. $f(x) = \dfrac{2x}{x^2 - 1}$

c. $g(t) = \dfrac{3t^2 + 4}{t^2 - 1}$

d. $y = \dfrac{3x^2 - 8x - 7}{x - 4}$

PART B

K 6. For each of the following, check for discontinuities and then use at least two other tests to make a rough sketch of the curve. Verify using a calculator.

a. $y = \dfrac{x - 3}{x + 5}$

b. $f(x) = \dfrac{5}{(x + 2)^2}$

c. $g(t) = \dfrac{t^2 - 2t - 15}{t - 5}$

d. $y = \dfrac{(2 + x)(3 - 2x)}{(x^2 - 3x)}$

7. Determine the equation of the oblique asymptote for each of the following:

a. $f(x) = \dfrac{3x^2 - 2x - 17}{x - 3}$

b. $f(x) = \dfrac{2x^2 + 9x + 2}{2x + 3}$

c. $f(x) = \dfrac{x^3 - 1}{x^2 + 2x}$

d. $f(x) = \dfrac{x^3 - x^2 - 9x + 15}{x^2 - 4x + 3}$

8. a. For question 7, part a., determine whether the curve approaches the asymptote from above or below.

b. For question 7, part b., determine the direction from which the curve approaches the asymptote.

9. For each function, determine any vertical or horizontal asymptotes and describe its behaviour on each side of any vertical asymptote.

a. $f(x) = \dfrac{3x - 1}{x + 5}$

b. $g(x) = \dfrac{x^2 + 3x - 2}{(x - 1)^2}$

c. $h(x) = \dfrac{x^2 + x - 6}{x^2 - 4}$

d. $m(x) = \dfrac{5x^2 - 3x + 2}{x - 2}$

A 10. Use the algorithm for curve sketching to sketch the graph of each function.

a. $f(x) = \dfrac{3 - x}{2x + 5}$

b. $h(t) = 2t^3 - 15t^2 + 36t - 10$

c. $y = \dfrac{20}{x^2 + 4}$

d. $s(t) = t + \dfrac{1}{t}$

e. $g(x) = \dfrac{2x^2 + 5x + 2}{x + 3}$

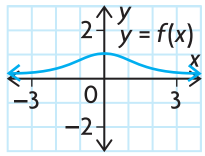
f. $s(t) = \dfrac{t^2 + 4t - 21}{t - 3}, t \geq -7$

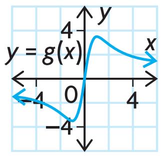
11. Consider the function $y = \frac{ax + b}{cx + d}$, where a, b, c, and d are constants, $a \neq 0, c \neq 0$.

a. Determine the horizontal asymptote of the graph.

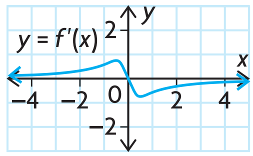
b. Determine the vertical asymptote of the graph.

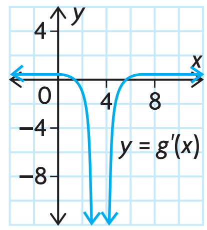
12. Use the features of each function's graph to sketch the graph of its first derivative.

a. 

b. 

13. A function’s derivative is shown in each graph. Use the graph to sketch a possible graph for the original function.

a. 

b. 

14. Let $f(x) = \frac{-x - 3}{x^2 - 5x - 14}$, $g(x) = \frac{x - x^3}{x - 3}$, $h(x) = \frac{x^3 - 1}{x^2 + 4}$, and $r(x) = \frac{x^2 + x - 6}{x^2 - 16}$. How can you can tell from its equation which of these functions has

a. a horizontal asymptote?

b. an oblique asymptote?

c. no vertical asymptote?

Explain. Determine the equations of all asymptote(s) for each function. Describe the behaviour of each function close to its asymptotes.

PART C

T 15. Find constants a and b such that the graph of the function defined by $f(x) = \frac{ax + 5}{3 - bx}$ will have a vertical asymptote at $x = 5$ and a horizontal asymptote at $y = -3$.

16. To understand why we cannot work with the symbol ∞ as though it were a real number, consider the functions $f(x) = \frac{x^2 + 1}{x + 1}$ and $g(x) = \frac{x^2 + 2x + 1}{x + 1}$.

a. Show that $\lim_{x \to +\infty} f(x) = +\infty$ and $\lim_{x \to +\infty} g(x) = +\infty$.

b. Evaluate $\lim_{x \to +\infty} [f(x) - g(x)]$, and show that the limit is not zero.

17. Use the algorithm for curve sketching to sketch the graph of the function $f(x) = \frac{2x^2 - 2x}{x^2 - 9}$.

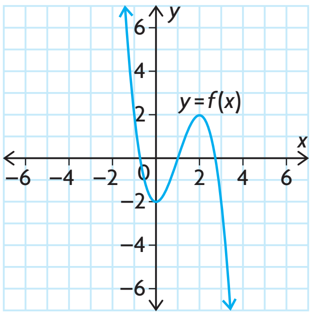
Mid-Chapter Review

1. Use a graphing calculator or graphing software to graph each of the following functions. Inspect the graph to determine where the function is increasing and where it is decreasing.
   a. $y = 3x^2 - 12x + 7$
   b. $y = 4x^3 - 12x^2 + 8$
   c. $f(x) = \dfrac{x + 2}{x + 3}$
   d. $f(x) = \dfrac{x^2 - 1}{x^2 + 3}$
2. Determine where $g(x) = 2x^3 - 3x^2 - 12x + 15$ is increasing and where it is decreasing.
3. Graph $f(x)$ if $f'(x) < 0$ when $x < -2$ and $x > 3$, $f'(x) > 0$ when $-2 < x < 3$, $f(-2) = 0$, and $f(3) = 5$.
4. Find all the critical numbers of each function.
   a. $y = -2x^2 + 16x - 31$
   b. $y = x^3 - 27x$
   c. $y = x^4 - 4x^2$
   d. $y = 3x^5 - 25x^3 + 60x$
   e. $y = \dfrac{x^2 - 1}{x^2 + 1}$
   f. $y = \dfrac{x}{x^2 + 2}$
5. For each function, find the critical numbers. Use the first derivative test to identify the local maximum and minimum values.
   a. $g(x) = 2x^3 - 9x^2 + 12x$
   b. $g(x) = x^3 - 2x^2 - 4x$
6. Find a value of k that gives $f(x) = x^2 + kx + 2$ a local minimum value of 1.
7. For $f(x) = x^4 - 32x + 4$, find the critical numbers, the intervals on which the function increases and decreases, and all the local extrema. Use graphing technology to verify your results.
8. Find the vertical asymptote(s) of the graph of each function. Describe the behaviour of $f(x)$ to the left and right of each asymptote.
   a. $f(x) = \dfrac{x - 1}{x + 2}$
   b. $f(x) = \dfrac{1}{9 - x^2}$
   c. $f(x) = \dfrac{x^2 - 4}{3x + 9}$
   d. $f(x) = \dfrac{2 - x}{3x^2 - 13x - 10}$
9. For each of the following, determine the equations of any horizontal asymptotes. Then state whether the curve approaches the asymptote from above or below.
   a. $y = \dfrac{3x - 1}{x + 5}$
   b. $f(x) = \dfrac{x^2 + 3x - 2}{(x - 1)^2}$
10. For each of the following, check for discontinuities and state the equation of any vertical asymptotes. Conduct a limit test to determine the behaviour of the curve on either side of the asymptote.
    a. $f(x) = \dfrac{x}{(x - 5)^2}$
    b. $f(x) = \dfrac{5}{x^2 + 9}$
    c. $f(x) = \dfrac{x - 2}{x^2 - 12x + 12}$

11. a. What does $f'(x) > 0$ imply about $f(x)$?
    b. What does $f'(x) < 0$ imply about $f(x)$?

12. A diver dives from the 3 m springboard. The diver's height above the water, in metres, at t seconds is $h(t) = -4.9t^2 + 9.5t + 2.2$.
    a. When is the height of the diver increasing? When is it decreasing?
    b. When is the velocity of the diver increasing? When is it decreasing?

13. The concentration, C, of a drug injected into the bloodstream t hours after injection can be modelled by $C(t) = \frac{t}{4} + 2t^{-2}$. Determine when the concentration of the drug is increasing and when it is decreasing.



14. Graph $y = f'(x)$ for the function shown at the left.

15. For each function $f(x)$,
    i. find the critical numbers
    ii. determine where the function increases and decreases
    iii. determine whether each critical number is at a local maximum, a local minimum, or neither
    iv. use all the information to sketch the graph

    a. $f(x) = x^2 - 7x - 18$
    b. $f(x) = -2x^3 + 9x^2 + 3$
    c. $f(x) = 2x^4 - 4x^2 + 2$
    d. $f(x) = x^5 - 5x$

16. Determine the equations of any vertical or horizontal asymptotes for each function. Describe the behaviour of the function on each side of any vertical or horizontal asymptote.

    a. $f(x) = \dfrac{x - 5}{2x + 1}$
    b. $g(x) = \dfrac{x^2 - 4x - 5}{(x + 2)^2}$
    c. $h(x) = \dfrac{x^2 + 2x - 15}{9 - x^2}$
    d. $m(x) = \dfrac{2x^2 + x + 1}{x + 4}$

17. Find each limit.

    a. $\lim\limits_{x \to \infty} \dfrac{3 - 2x}{3x}$
    b. $\lim\limits_{x \to \infty} \dfrac{x^2 - 2x + 5}{6x^2 + 2x - 1}$
    c. $\lim\limits_{x \to \infty} \dfrac{7 + 2x^2 - 3x^3}{x^3 - 4x^2 + 3x}$
    d. $\lim\limits_{x \to \infty} \dfrac{5 - 2x^3}{x^4 - 4x}$
    e. $\lim\limits_{x \to \infty} \dfrac{2x^5 - 1}{3x^4 - x^2 - 2}$
    f. $\lim\limits_{x \to \infty} \dfrac{x^2 + 3x - 18}{(x - 3)^2}$
    g. $\lim\limits_{x \to \infty} \dfrac{x^2 - 4x - 5}{x^2 - 1}$
    h. $\lim\limits_{x \to \infty} \left(5x + 4 - \dfrac{7}{x + 3}\right)$

Section 4.4—Concavity and Points of Inflection

In Chapter 3, you saw that the second derivative of a function has applications in problems involving velocity and acceleration or in general rates-of-change problems. Here we examine the use of the second derivative of a function in curve sketching.

INVESTIGATION 1 The purpose of this investigation is to examine the relationship between slopes of tangents and the second derivative of a function.

A. Sketch the graph of $f(x) = x^2$.

B. Determine $f'(x)$. Use $f'(x)$ to calculate the slope of the tangent to the curve at the points with the following x-coordinates: $x = -4, -3, -2, -1, 0, 1, 2, 3,$ and 4. Sketch each of these tangents.

C. Are these tangents above or below the graph of $y = f(x)$?

D. Describe the change in the slopes as x increases.

E. Determine $f''(x)$. How does the value of $f''(x)$ relate to the way in which the curve opens? How does the value of $f''(x)$ relate to the way $f'(x)$ changes as x increases?

F. Repeat parts B, C, and D for the graph of $f(x) = -x^2$.

G. How does the value of $f''(x)$ relate to the way in which the curve opens?

INVESTIGATION 2 The purpose of this investigation is to extend the results of Investigation 1 to other functions.

A. Sketch the graph of $f(x) = x^3$.

B. Determine all the values of x for which $f'(x) = 0$.

C. Determine intervals on the domain of the function such that $f''(x) < 0$, $f''(x) = 0$, and $f''(x) > 0$.

D. For values of x such that $f''(x) < 0$, how does the shape of the curve compare with your conclusions in Investigation 1?

E. Repeat part D for values of x such that $f''(x) > 0$.

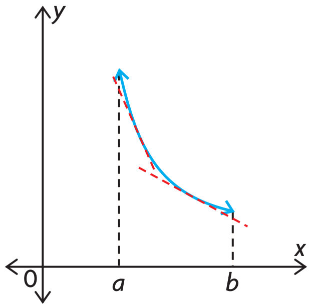
F. What happens when $f''(x) = 0$?

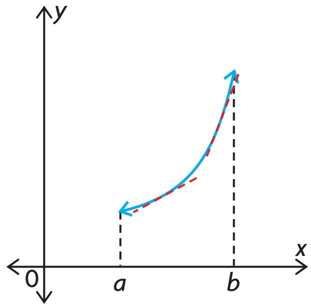
G. Using your observations from this investigation, sketch the graph of $y = x^3 - 12x$.

From these investigations, we can make a summary of the behaviour of the graphs.

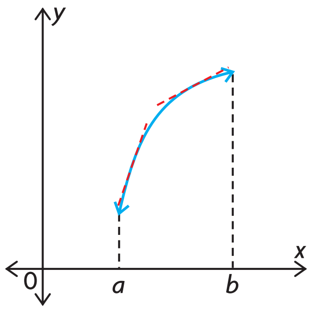
Concavity and the Second Derivative

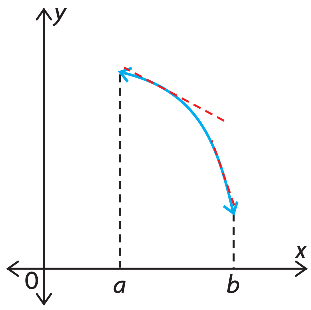
1. The graph of $y = f(x)$ is **concave up** on an interval $a \leq x \leq b$ in which the slopes of $f(x)$ are increasing. On this interval, $f''(x)$ exists and $f''(x) > 0$. The graph of the function is above the tangent at every point on the interval.



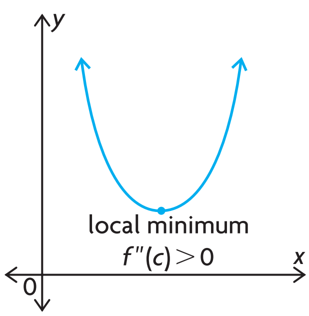


2. The graph of $y = f(x)$ is **concave down** on an interval $a \leq x \leq b$ in which the slopes of $f(x)$ are decreasing. On this interval, $f''(x)$ exists and $f''(x) < 0$. The graph of the function is below the tangent at every point on the interval.

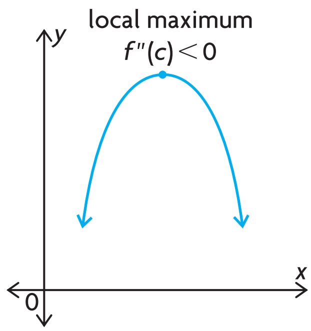




3. If $y = f(x)$ has a critical point at $x = c$, with $f'(c) = 0$, then the behaviour of $f(x)$ at $x = c$ can be analyzed through the use of the **second derivative test** by analyzing $f''(c)$, as follows:
   a. The graph is concave up, and $x = c$ is the location of a local minimum value of the function, if $f''(c) > 0$.

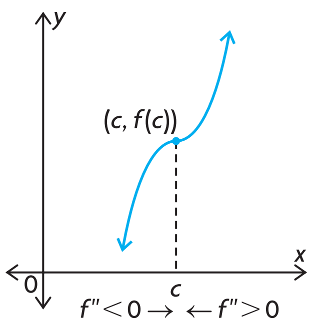


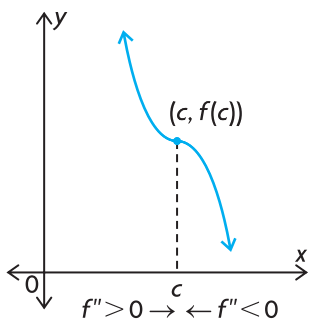
b. The graph is concave down, and $x = c$ is the location of a local maximum value of the function, if $f''(c) < 0$.



c. If $f''(c) = 0$, the nature of the critical point cannot be determined without further work.

4. A **point of inflection** occurs at $(c, f(c))$ on the graph of $y = f(x)$ if $f''(x)$ changes sign at $x = c$. That is, the curve changes from concave down to concave up, or vice versa.





5. All points of inflection on the graph of $y = f(x)$ must occur either where $\frac{d^2y}{dx^2}$ equals zero or where $\frac{d^2y}{dx^2}$ is undefined.

In the following examples, we will use these properties to sketch graphs of other functions.

EXAMPLE 1 Using the first and second derivatives to analyze a cubic function

Sketch the graph of $y = x^3 - 3x^2 - 9x + 10$.

Solution

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

Setting $\frac{dy}{dx} = 0$, we obtain

$$3(x^2 - 2x - 3) = 0$$
$$3(x - 3)(x + 1) = 0$$
$$x = 3 \text{ or } x = -1$$

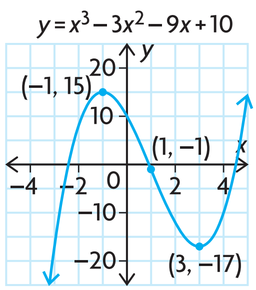
$$\frac{d^2y}{dx^2} = 6x - 6$$

Setting $\frac{d^2y}{dx^2} = 0$, we obtain $6x - 6 = 0$ or $x = 1$.

Now determine the sign of $f''(x)$ in the intervals determined by $x = 1$.

Interval	$x < 1$	$x = 1$	$x > 1$
$f''(x)$	< 0	0	> 0
Graph of $f(x)$	concave down	point of inflection	concave up
Sketch of $f(x)$			

Applying the second derivative test, at $x = 3$, we obtain the local minimum point, $(3, -17)$ and at $x = -1$, we obtain the local maximum point, $(-1, 15)$. The point of inflection occurs at $x = 1$ where $f(1) = -1$.
The graph can now be sketched.



EXAMPLE 2 Using the first and second derivatives to analyze a quartic function

Sketch the graph of $f(x) = x^4$.

Solution

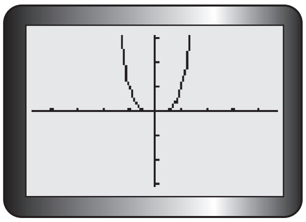
The first and second derivatives of $f(x)$ are $f'(x) = 4x^3$ and $f''(x) = 12x^2$.
Setting $f''(x) = 0$, we obtain
$12x^2 = 0$ or $x = 0$

But $x = 0$ is also obtained from $f'(x) = 0$.

Now determine the sign of $f''(x)$ on the intervals determined by $x = 0$.

Interval	$x < 0$	$x = 0$	$x > 0$
$f''(x)$	> 0	$= 0$	> 0
Graph of $f(x)$	concave up	?	concave up
Sketch of $f(x)$			

We conclude that the point $(0, 0)$ is *not* an inflection point because $f''(x)$ does not change sign at $x = 0$. However, since $x = 0$ is a critical number and $f'(x) < 0$ when $x < 0$ and $f'(x) > 0$ when $x > 0$, $(0, 0)$ is an absolute minimum.



EXAMPLE 3 Using the first and second derivatives to analyze a root function

Sketch the graph of the function $f(x) = x^{\frac{1}{3}}$.

Solution

The derivative of $f(x)$ is

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$$
$$= \frac{1}{3x^{\frac{2}{3}}}$$

Note that $f'(0)$ does not exist, so $x = 0$ is a critical number of $f(x)$. It is important to determine the behaviour of $f'(x)$ as $x \to 0$. Since $f'(x) > 0$ for all values of $x \neq 0$, and the denominator of $f'(x)$ is zero when $x = 0$, we have $\lim_{x \to 0} f'(x) = +\infty$. This means that there is a vertical tangent at $x = 0$. In addition, $f(x)$ is increasing for $x < 0$ and $x > 0$. As a result this graph has no local extrema.

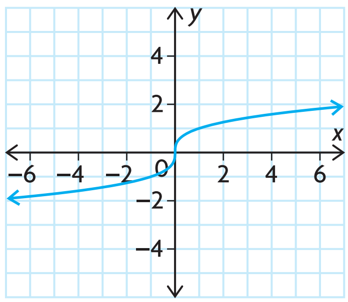
The second derivative of $f(x)$ is

$$f''(x) = -\frac{2}{9}x^{-\frac{5}{3}}$$
$$= -\frac{2}{9x^{\frac{5}{3}}}$$

Since $x^{\frac{5}{3}} > 0$ if $x > 0$, and $x^{\frac{5}{3}} < 0$ if $x < 0$, we obtain the following table:

Interval	$x < 0$	$x = 0$	$x > 0$
$f''(x)$	$\frac{-}{-} = +$	does not exist	$\frac{-}{+} = -$
$f(x)$			

The graph has a point of inflection when $x = 0$, even though $f'(0)$ and $f''(0)$ do not exist. Note that the curve crosses its tangent at $x = 0$.



EXAMPLE 4 Reasoning about points of inflection

Determine any points of inflection on the graph of $f(x) = \frac{1}{x^2 + 3}$.

Solution

The derivative of $f(x) = \frac{1}{x^2 + 3} = (x^2 + 3)^{-1}$ is $f'(x) = -2x(x^2 + 3)^{-2}$.
The second derivative is

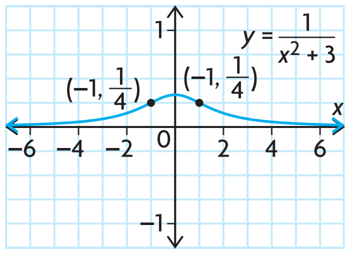
$$f''(x) = -2(x^2 + 3)^{-2} + 4x(x^2 + 3)^{-3}(2x)$$
$$= \frac{-2}{(x^2 + 3)^2} + \frac{8x^2}{(x^2 + 3)^3}$$
$$= \frac{-2(x^2 + 3) + 8x^2}{(x^2 + 3)^3}$$
$$= \frac{6x^2 - 6}{(x^2 + 3)^3}$$

Setting $f''(x) = 0$ gives $6x^2 - 6 = 0$ or $x = \pm 1$.

Determine the sign of $f''(x)$ on the intervals determined by $x = -1$ and $x = 1$.

Interval	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$f''(x)$	> 0	$= 0$	< 0	$= 0$	> 0
Graph of $f(x)$	concave up	point of inflection	concave down	point of inflection	concave up

Therefore, $\left(-1, \frac{1}{4}\right)$ and $\left(1, \frac{1}{4}\right)$ are points of inflection on the graph of $f(x)$.



IN SUMMARY

Key Ideas

- The graph of a function $f(x)$ is **concave up** on an interval if $f'(x)$ is increasing on the interval. The graph of a function $f(x)$ is **concave down** on an interval if $f'(x)$ is decreasing on the interval.
- A point of inflection is a point on the graph of $f(x)$ where the function changes from concave up to concave down, or vice versa. $f''(c) = 0$ or is undefined if $(c, f(c))$ is a point of inflection on the graph of $f(x)$.

Need to Know

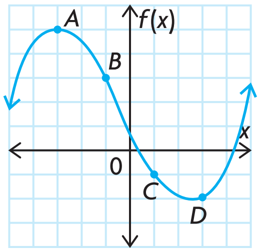
- **Test for concavity:** If $f(x)$ is a differentiable function whose second derivative exists on an open interval I, then
  - the graph of $f(x)$ is concave up on I if $f''(x) > 0$ for all values of x in I
  - the graph of $f(x)$ is concave down on I if $f''(x) < 0$ for all values of x in I
- **The second derivative test:** Suppose that $f(x)$ is a function for which $f''(c) = 0$, and the second derivative of $f(x)$ exists on an interval containing c.
  - If $f''(c) > 0$, then $f(c)$ is a local minimum value.
  - If $f''(c) < 0$, then $f(c)$ is a local maximum value.
  - If $f''(c) = 0$, then the test fails. Use the first derivative test.

Exercise 4.4

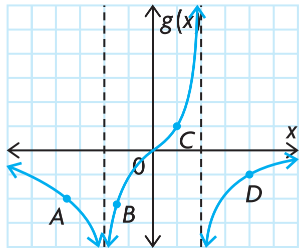
PART A

K 1. For each function, state whether the value of the second derivative is positive or negative at each of points A, B, C, and D.

a.



b.



2. Determine the critical points for each function, and use the second derivative test to decide if the point is a local maximum, a local minimum, or neither.

a. $y = x^3 - 6x^2 - 15x + 10$

b. $y = \dfrac{25}{x^2 + 48}$

c. $s = t + t^{-1}$

d. $y = (x - 3)^3 + 8$

3. Determine the points of inflection for each function in question 2. Then conduct a test to determine the change of sign in the second derivative.

4. Determine the value of the second derivative at the value indicated. State whether the curve lies above or below the tangent at this point.

a. $f(x) = 2x^3 - 10x + 3$ at $x = 2$

b. $g(x) = x^2 - \dfrac{1}{x}$ at $x = -1$

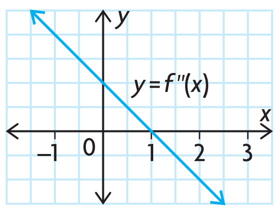
c. $p(w) = \dfrac{w}{\sqrt{w^2 + 1}}$ at $w = 3$

d. $s(t) = \dfrac{2t}{t - 4}$ at $t = -2$

PART B

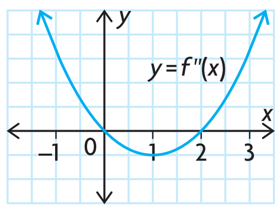
5. Each of the following graphs represents the second derivative, $f''(x)$, of a function $f(x)$:

a.



$f''(x)$ is a linear function.

b.



$f''(x)$ is a quadratic function.

For each of the graphs above, answer the following questions:

i. On which intervals is the graph of $f(x)$ concave up? On which intervals is the graph concave down?

ii. List the x-coordinates of all the points of inflection.

iii. Make a rough sketch of a possible graph of $f(x)$, assuming that $f(0) = 2$.

C 6. Describe how you would use the second derivative to determine a local minimum or maximum.

7. In the algorithm for curve sketching in Section 4.3, reword step 4 to include the use of the second derivative to test for local minimum or maximum values.

8. For each of the following functions,

   i. determine any points of inflection

   ii. use the results of part i, along with the revised algorithm, to sketch each function.

   a. $f(x) = x^4 + 4x$

   b. $g(w) = \dfrac{4w^2 - 3}{w^3}$

A 9. Sketch the graph of a function with the following properties:

- $f'(x) > 0$ when $x < 2$ and when $2 < x < 5$
- $f'(x) < 0$ when $x > 5$
- $f'(2) = 0$ and $f'(5) = 0$
- $f''(x) < 0$ when $x < 2$ and when $4 < x < 7$
- $f''(x) > 0$ when $2 < x < 4$ and when $x > 7$
- $f(0) = -4$

10. Find constants a, b, and c such that the function $f(x) = ax^3 + bx^2 + c$ will have a local extremum at $(2, 11)$ and a point of inflection at $(1, 5)$. Sketch the graph of $y = f(x)$.

PART C

11. Find the value of the constant b such that the function $f(x) = \sqrt{x + 1} + \frac{b}{x}$ has a point of inflection at $x = 3$.

T 12. Show that the graph of $f(x) = ax^4 + bx^3$ has two points of inflection. Show that the x-coordinate of one of these points lies midway between the x-intercepts.

13. a. Use the algorithm for curve sketching to sketch the function $y = \frac{x^3 - 2x^2 + 4x}{x^2 - 4}$.

    b. Explain why it is difficult to determine the oblique asymptote using a graphing calculator.

14. Find the inflection points, if any exist, for the graph of $f(x) = (x - c)^n$, for $n = 1, 2, 3$, and 4. What conclusion can you draw about the value of n and the existence of inflection points on the graph of f?

Section 4.5—An Algorithm for Curve Sketching

You now have the necessary skills to sketch the graphs of most elementary functions. However, you might be wondering why you should spend time developing techniques for sketching graphs when you have a graphing calculator. The answer is that, in doing so, you develop an understanding of the qualitative features of the functions you are analyzing. Also, for certain functions, maximum/minimum/inflection points are not obvious if the window setting is not optimal. In this section, you will combine the skills you have developed. Some of them use the calculus properties. Others were learned earlier. Putting all the skills together will allow you to develop an approach that leads to simple, yet accurate, sketches of the graphs of functions.

An Algorithm for Sketching the Graph of $y = f(x)$

Note: As each piece of information is obtained, use it to build the sketch.

1: Determine any discontinuities or limitations in the domain. For discontinuities, investigate the function's values on either side of the discontinuity.
2: Determine any vertical asymptotes.
3: Determine any intercepts.
4: Determine any critical numbers by finding where $\frac{dy}{dx} = 0$ or where $\frac{dy}{dx}$ is undefined.
5: Determine the intervals of increase/decrease, and then test critical points to see whether they are local maxima, local minima, or neither.
6: Determine the behaviour of the function for large positive and large negative values of x. This will identify horizontal asymptotes, if they exist. Identify if the functions values approach the horizontal asymptote from above or below.
7: Determine $\frac{d^2y}{dx^2}$ and test for points of inflection using the intervals of concavity.
8: Determine any oblique asymptotes. Identify if the functions values approach the obliques asymptote from above or below.
9: Complete the sketch using the above information.

When using this algorithm, keep two things in mind:

1. You will not use all the steps in every situation. Use only the steps that are essential.

2. You are familiar with the basic shapes of many functions. Use this knowledge when possible.

INVESTIGATION Use the algorithm for curve sketching to sketch the graph of each of the following functions. After completing your sketch, use graphing technology to verify your results.

a. $y = x^4 - 3x^2 + 2x$

b. $y = \dfrac{x}{x^2 - 1}$

EXAMPLE 1 Sketching an accurate graph of a polynomial function

Use the algorithm for curve sketching to sketch the graph of $f(x) = -3x^3 - 2x^2 + 5x$.

Solution

This is a polynomial function, so there are no discontinuities and no asymptotes. The domain is $\{x \in \mathbf{R}\}$. **Analyze $f(x)$.** Determine any intercepts.

x-intercept, $y = 0$

$$\begin{aligned} -3x^3 - 2x^2 + 5x &= 0 \\ -x(3x^2 + 2x - 5) &= 0 \\ -x(3x + 5)(x - 1) &= 0 \end{aligned}$$

$x = 0, x = -\frac{5}{3}, x = 1$

$(0, 0), \left(-\frac{5}{3}, 0\right), (1, 0)$

y-intercept, $x = 0$

$y = 0$

$(0, 0)$

Now determine the critical points.

Analyze $f'(x)$.

$f'(x) = -9x^2 - 4x + 5$

Setting $f'(x) = 0$, we obtain

$$\begin{aligned} -9x^2 - 4x + 5 &= 0 \\ -(9x^2 + 4x - 5) &= 0 \\ -(9x - 5)(x + 1) &= 0 \end{aligned}$$

$x = \frac{5}{9}$ or $x = -1$

When we sketch the function, we can use approximate values $x = 0.6$ and $y = 1.6$ for $x = \frac{5}{9}$ and $f\left(\frac{5}{9}\right)$.

Analyze $f''(x)$.

$f''(x) = -18x - 4$

At $x = \frac{5}{9}$,

$$\begin{aligned} f''\left(\frac{5}{9}\right) &= -18\left(\frac{5}{9}\right) - 4 \\ &= -10 - 4 \\ &= -14 \\ &< 0 \end{aligned}$$

At $x = -1$,

$$\begin{aligned} f''(-1) &= -18(-1) - 4 \\ &= 18 - 4 \\ &= 14 \\ &> 0 \end{aligned}$$

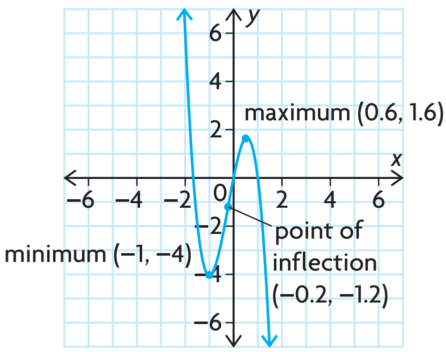
Therefore, by the second derivative test $x = \frac{5}{9}$ gives a local maximum and $x = -1$ gives a local minimum. Since this is a polynomial function, $f(x)$ must be decreasing when $x < -1$, increasing when $-1 < x < \frac{5}{9}$ and decreasing when $x > \frac{5}{9}$. For a point of inflection, $f''(x) = 0$ and changes sign.

$-18x - 4 = 0$ or $x = -\dfrac{2}{9}$

Now we determine the sign of $f''(x)$ in, the intervals determined by $x = -\frac{2}{9}$.
A point of inflection occurs at about $(-0.2, -1.2)$.

We can now draw our sketch.

Interval	$x < -\frac{2}{9}$	$x = -\frac{2}{9}$	$x > -\frac{2}{9}$
$f''(x)$	> 0	0	< 0
Graph of $f(x)$	concave up	point of inflection	concave down



EXAMPLE 2 Sketching an accurate graph of a rational function

Sketch the graph of $f(x) = \frac{x - 4}{x^2 - x - 2}$.

Solution

Analyze $f(x)$.

$f(x)$ is a rational function.

Determine any intercepts.

x-intercept, $y = 0$

$$\frac{x - 4}{x^2 - x - 2} = 0$$

$$x - 4 = 0$$

$$x = 4$$

$(4, 0)$

y-intercept, $x = 0$

$$y = \frac{0 - 4}{0 - 0 - 2}$$

$$y = \frac{-4}{-2}$$

$$y = 2$$

$(0, 2)$

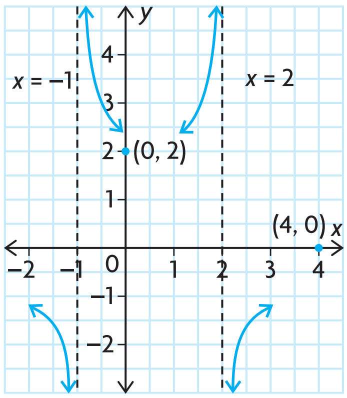
Determine any asymptotes.
The function is not defined if

$$x^2 - x - 2 = 0$$
$$(x - 2)(x + 1) = 0$$
$$x = 2 \text{ and } x = -1$$

The domain is $\{x \in \mathbf{R} \mid x \neq 2 \text{ and } x \neq -1\}$.
There are vertical asymptotes at $x = 2$ and $x = -1$.
Using $f(x) = \frac{x - 4}{x^2 - x - 2}$, we examine function values near the asymptotes.

$$\lim_{x \to -1^-} f(x) = -\infty \qquad \lim_{x \to -1^+} f(x) = +\infty$$
$$\lim_{x \to 2^-} f(x) = +\infty \qquad \lim_{x \to 2^+} f(x) = -\infty$$

Sketch the information you have so far, as shown.



Analyze $f'(x)$. Now determine the critical points.

$$f(x) = (x - 4)(x^2 - x - 2)^{-1}$$
$$f'(x) = (1)(x^2 - x - 2)^{-1} + (x - 4)(-1)(x^2 - x - 2)^{-2}(2x - 1)$$
$$= \frac{1}{x^2 - x - 2} - \frac{(x - 4)(2x - 1)}{(x^2 - x - 2)^2}$$
$$= \frac{(x^2 - x - 2)}{(x^2 - x - 2)^2} - \frac{(2x^2 - 9x + 4)}{(x^2 - x - 2)^2}$$
$$= \frac{-x^2 + 8x - 6}{(x^2 - x - 2)^2}$$
$$f'(x) = 0 \text{ if } -x^2 + 8x - 6 = 0$$
$$x = \frac{8 \pm 2\sqrt{10}}{2}$$
$$x = 4 \pm \sqrt{10}$$

Since we are sketching, approximate values 7.2 and 0.8 are acceptable. These values give the approximate points $(7.2, 0.1)$ and $(0.8, 1.5)$.

Interval	$(-\infty, -1)$	$(-1, 0.8)$	$(0.8, 2)$	$(2, 7.2)$	$(7.2, \infty)$
$-x^2 + 8x - 6$	$-$	$-$	$+$	$+$	$-$
$(x^2 - x - 2)^2$	$+$	$+$	$+$	$+$	$+$
$f'(x)$	< 0	< 0	> 0	> 0	< 0
$f(x)$	decreasing	decreasing	increasing	increasing	decreasing

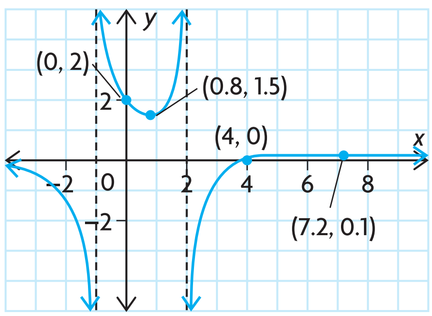
From the information obtained, we can see that $(7.2, 0.1)$ is likely a local maximum and $(0.8, 1.5)$ is likely a local minimum. To verify this using the second derivative test is a difficult computational task. Instead, verify using the first derivative test, as follows.
$x \doteq 0.8$ gives the local minimum. $x \doteq 7.2$ gives the local maximum.
Now check the end behaviour of the function.

$\lim_{x \to +\infty} f(x) = 0$ but $y > 0$ always.
$\lim_{x \to -\infty} f(x) = 0$ but $y < 0$ always.

Therefore, $y = 0$ is a horizontal asymptote. The curve approaches from above on the right and below on the left.

There is a point of inflection beyond $x = 7.2$, since the curve opens down at that point but changes as x becomes larger. The amount of work necessary to

determine the point is greater than the information we gain, so we leave it undone. (If you wish to check it, it occurs for $x \doteq 10.4$.) The finished sketch is given below and, because it is a sketch, it is not to scale.



IN SUMMARY

Key Idea

- The first and second derivatives of a function give information about the shape of the graph of the function.

Need to Know

Sketching the Graph of a Polynomial or Rational Function

1. Use the function to
   - determine the domain and any discontinuities
   - determine the intercepts
   - find any asymptotes, and determine function behaviour relative to these asymptotes
2. Use the first derivative to
   - find the critical numbers
   - determine where the function is increasing and where it is decreasing
   - identify any local maxima or minima
3. Use the second derivative to
   - determine where the graph is concave up and where it is concave down
   - find any points of inflection

The second derivative can also be used to identify local maxima and minima.

4. Calculate the values of y that correspond to critical points and points of inflection. Use the information above to sketch the graph.

Remember that you will not use all the steps in every situation! Use only the steps that are necessary to give you a good idea of what the graph will look like.

Exercise 4.5

PART A

1. If a polynomial function of degree three has a local minimum, explain how the function's values behave as $x \to +\infty$ and as $x \to -\infty$. Consider all cases.

C 2. How many local maximum and local minimum values are possible for a polynomial function of degree three, four, or n? Explain.

3. Determine whether each function has vertical asymptotes. If it does, state the equations of the asymptotes.

   a. $y = \dfrac{x}{x^2 + 4x + 3}$ b. $y = \dfrac{5x - 4}{x^2 - 6x + 12}$ c. $y = \dfrac{3x + 2}{x^2 - 6x + 9}$

PART B

K 4. Use the algorithm for curve sketching to sketch the following:

a. $y = x^3 - 9x^2 + 15x + 30$

b. $f(x) = -4x^3 + 18x^2 + 3$

c. $y = 3 + \frac{1}{(x + 2)^2}$

d. $f(x) = x^4 - 4x^3 - 8x^2 + 48x$

e. $y = \frac{2x}{x^2 - 25}$

f. $f(x) = \frac{1}{x^2 - 4x}$

g. $y = \frac{6x^2 - 2}{x^3}$

h. $f(x) = \frac{x + 3}{x^2 - 4}$

i. $y = \frac{x^2 - 3x + 6}{x - 1}$

j. $f(x) = (x - 4)^{\frac{2}{3}}$

5. Verify your results for question 4 using graphing technology.

A 6. Determine the constants a, b, c, and d so that the curve defined by $y = ax^3 + bx^2 + cx + d$ has a local maximum at the point $(2, 4)$ and a point of inflection at the origin. Sketch the curve.

7. Given the following results of the analysis of a function, sketch a possible graph for the function:

a. $f(0) = 0$, the horizontal asymptote is $y = 2$, the vertical asymptote is $x = 3$, and $f'(x) < 0$ and $f''(x) < 0$ for $x < 3$; $f'(x) < 0$ and $f''(x) > 0$ for $x > 3$.

b. $f(0) = 6, f(-2) = 0$ the horizontal asymptote is $y = 7$, the vertical asymptote is $x = -4$, and $f'(x) > 0$ and $f''(x) > 0$ for $x < -4$; $f'(x) > 0$ and $f''(x) < 0$ for $x > -4$.

PART C

8. Sketch the graph of $f(x) = \frac{k - x}{k^2 + x^2}$, where k is any positive constant.

9. Sketch the curve defined by $g(x) = x^{\frac{1}{3}}(x + 3)^{\frac{2}{3}}$.

10. Find the horizontal asymptotes for each of the following:

a. $f(x) = \frac{x}{\sqrt{x^2 + 1}}$

b. $g(t) = \sqrt{t^2 + 4t} - \sqrt{t^2 + t}$

T 11. Show that, for any cubic function of the form $y = ax^3 + bx^2 + cx + d$, there is a single point of inflection, and the slope of the curve at that point is $c - \frac{b^2}{3a}$.

CAREER LINK WRAP-UP Investigate and Apply

CHAPTER 4: PREDICTING STOCK VALUES

In the Career Link earlier in the chapter, you investigated a graphical model used to predict stock values for a new stock. A brand new stock is also called an initial public offering, or IPO. Remember that, in this model, the period immediately after the stock is issued offers excess returns on the stock—that is, the stock is selling for more than it is really worth.

One such model for a class of Internet IPOs predicts the percent overvaluation of a stock as a function of time as $R(t) = 250\left(\frac{t^2}{(2.718)^{3t}}\right)$, where $R(t)$ is the overvaluation in percent and t is the time in months after the initial issue.

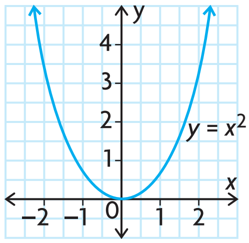
a. Use the information provided by the first derivative, second derivative, and asymptotes to prepare advice for clients as to when they should expect a signal to prepare to buy or sell (inflection point), the exact time when they should buy or sell (local maximum/minimum), and any false signals prior to a horizontal asymptote. Explain your reasoning.

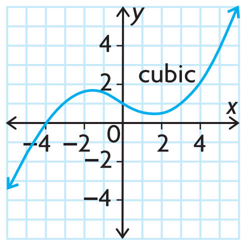
b. Make a sketch of the function without using a graphing calculator.

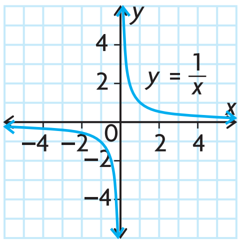
Key Concepts Review

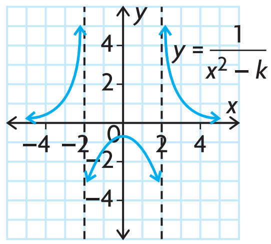
In this chapter, you saw that calculus can help you sketch graphs of polynomial and rational functions. Remember that concepts you learned in earlier studies are useful, and that calculus techniques help with sketching. Basic shapes should always be kept in mind. Use these, together with the algorithm for curve sketching, and always use your accumulated knowledge.

Basic Shapes to Remember









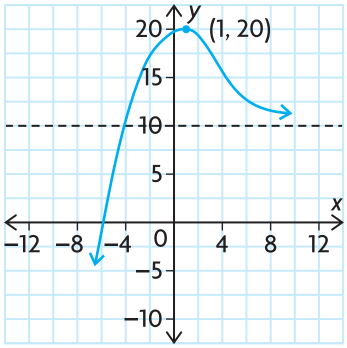
Sketching the Graph of a Polynomial or Rational Function

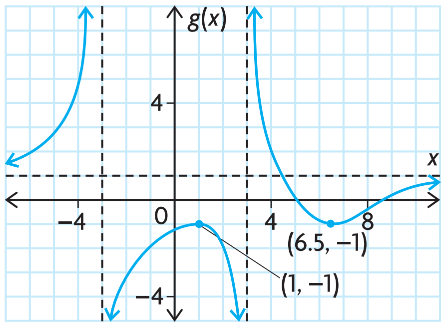
1. Use the function to
   - determine the domain and any discontinuities
   - determine the intercepts
   - find any asymptotes, and determine function behaviour relative to these asymptotes
2. Use the first derivative to
   - find the critical numbers
   - determine where the function is increasing and where it is decreasing
   - identify any local maxima or minima
3. Use the second derivative to
   - determine where the graph is concave up and where it is concave down
   - find any points of inflection

   The second derivative can also be used to identify local maxima and minima.
4. Calculate the values of y that correspond to critical points and points of inflection. Use the information above to sketch the graph.

Review Exercise

1. For each of the following graphs, state
   i. the intervals where the function is increasing
   ii. the intervals where the function is decreasing
   iii. the points where the tangent to the function is horizontal

a. 

b. 

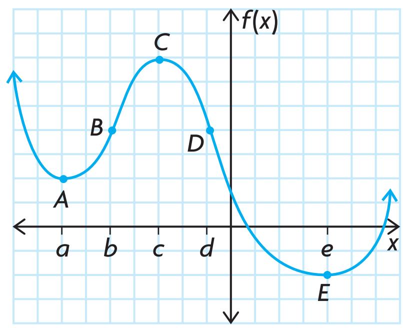
2. Is it always true that an increasing function is concave up in shape? Explain.

3. Determine the critical points for each function. Determine whether the critical point is a local maximum or local minimum and whether or not the tangent is parallel to the x-axis.

   a. $f(x) = -2x^3 + 9x^2 + 20$
   b. $f(x) = x^4 - 8x^3 + 18x^2 + 6$
   c. $h(x) = \dfrac{x - 3}{x^2 + 7}$
   d. $g(x) = (x - 1)^{\frac{1}{3}}$

4. The graph of the function $y = f(x)$ has local extrema at points A, C, and E and points of inflection at B and D. If a, b, c, d, and e are the x-coordinates of the points, state the intervals on which the following conditions are true:

   a. $f'(x) > 0$ and $f''(x) > 0$
   b. $f'(x) > 0$ and $f''(x) < 0$
   c. $f'(x) < 0$ and $f''(x) > 0$
   d. $f'(x) < 0$ and $f''(x) < 0$



5. For each of the following, check for discontinuities and state the equation of any vertical asymptotes. Conduct a limit test to determine the behaviour of the curve on either side of the asymptote.

a. $y = \frac{2x}{x - 3}$

b. $g(x) = \frac{x - 5}{x + 5}$

c. $f(x) = \frac{x^2 - 2x - 15}{x + 3}$

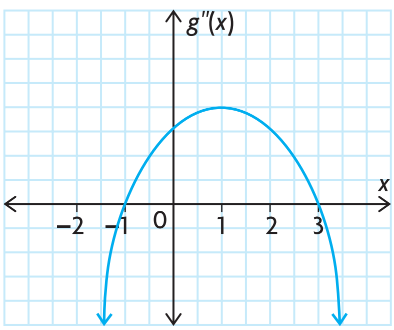
d. $g(x) = \frac{5}{x^2 - x - 20}$

6. Determine the point of inflection on the curve defined by $y = x^3 + 5$. Show that the tangent line at this point crosses the curve.

7. Sketch a graph of a function that is differentiable on the interval $-3 \le x \le 5$ and satisfies the following conditions:
   - There are local maxima at $(-2, 10)$ and $(3, 4)$.
   - The function f is decreasing on the intervals $-2 < x < 1$ and $3 \le x \le 5$.
   - The derivative $f'(x)$ is positive for $-3 \le x < -2$ and for $1 < x < 3$.
   - $f(1) = -6$

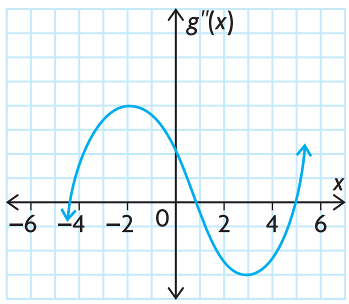
8. Each of the following graphs represents the second derivative, $g''(x)$, of a function $g(x)$:

a.



$g''(x)$ is a quadratic function.

b.



$g''(x)$ is a cubic function.

i. On what intervals is the graph of $g(x)$ concave up? On what intervals is the graph concave down?

ii. List the x-coordinates of the points of inflection.

iii. Make a rough sketch of a possible graph for $g(x)$, assuming that $g(0) = -3$.

9. a. If the graph of the function $g(x) = \frac{ax + b}{(x - 1)(x - 4)}$ has a horizontal tangent at point $(2, -1)$, determine the values of a and b.
   b. Sketch the function g.

10. Sketch each function using suitable techniques.
   a. $y = x^4 - 8x^2 + 7$
   b. $f(x) = \dfrac{3x - 1}{x + 1}$
   c. $g(x) = \dfrac{x^2 + 1}{4x^2 - 9}$
   d. $y = x(x - 4)^3$
   e. $h(x) = \dfrac{x}{x^2 - 4x + 4}$
   f. $f(t) = \dfrac{t^2 - 3t + 2}{t - 3}$

11. a. Determine the conditions on parameter k such that the function $f(x) = \frac{2x + 4}{x^2 - k^2}$ will have critical points.
   b. Select a value for k that satisfies the constraint established in part a, and sketch the section of the curve that lies in the domain $|x| \leq k$.

12. Determine the equation of the oblique asymptote in the form $y = mx + b$ for each function, and then show that $\lim_{x \to +\infty} [y - f(x)] = 0$.
   a. $f(x) = \dfrac{2x^2 - 7x + 5}{2x - 1}$
   b. $f(x) = \dfrac{4x^3 - x^2 - 15x - 50}{x^2 - 3x}$

13. Determine the critical numbers and the intervals on which $g(x) = (x^2 - 4)^2$ is increasing or decreasing.

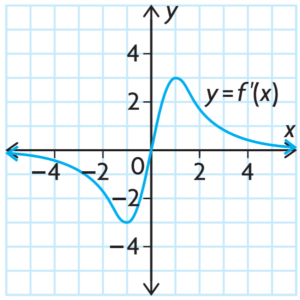
14. Use the second derivative test to identify all maximum and minimum values of $f(x) = x^3 + \frac{3}{2}x^2 - 7x + 5$ on the interval $-4 \leq x \leq 3$.

15. Use the y-intercept, local extrema, intervals of concavity, and points of inflection to graph $f(x) = 4x^3 + 6x^2 - 24x - 2$.

16. Let $p(x) = \frac{3x^3 - 5}{4x^2 + 1}$, $q(x) = \frac{3x - 1}{x^2 - 2x - 3}$, $r(x) = \frac{x^2 - 2x - 8}{x^2 - 1}$, and $s(x) = \frac{x^3 + 2x}{x - 2}$.
   a. Determine the asymptotes for each function, and identify their type (vertical, horizontal, or oblique).
   b. Graph $y = r(x)$, showing clearly the asymptotes and the intercepts.

17. If $f(x) = \frac{x^3 + 8}{x}$, determine the domain, intercepts, asymptotes, intervals of increase and decrease, and concavity. Locate any critical points and points of inflection. Use this information to sketch the graph of $f(x)$.

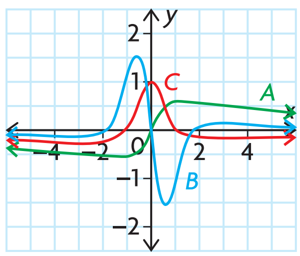
18. Explain how you can use this graph of $y = f'(x)$ to sketch a possible graph of the original function, $y = f(x)$.



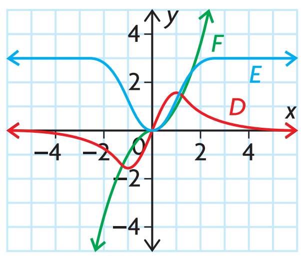
19. For $f(x) = \frac{5x}{(x - 1)^2}$, show that $f'(x) = \frac{-5(x + 1)}{(x - 1)^3}$ and $f''(x) = \frac{100(x + 2)}{(x - 1)^4}$. Use the function and its derivatives to determine the domain, intercepts, asymptotes, intervals of increase and decrease, and concavity, and to locate any local extrema and points of inflection. Use this information to sketch the graph of f.

20. The graphs of a function and its derivatives, $y = f(x)$, $y = f'(x)$, and $y = f''(x)$, are shown on each pair of axes. Which is which? Explain how you can tell.

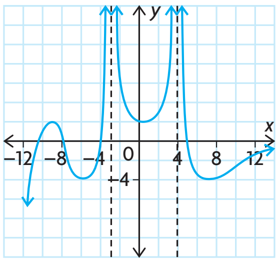
a.



b.



Chapter 4 Test



1. The graph of function $y = f(x)$ is shown at the left.
   a. Estimate the intervals where the function is increasing.
   b. Estimate the intervals where $f'(x) < 0$.
   c. Estimate the coordinates of the critical points.
   d. Estimate the equations of any vertical asymptotes.
   e. What is the value of $f''(x)$ on the interval $-4 < x < 4$?
   f. If $x \geq -6$, estimate the intervals where $f'(x) < 0$ and $f''(x) > 0$.
   g. Identify a point of inflection, and state the approximate ordered pair for the point.
2. a. Determine the critical points of the function $g(x) = 2x^4 - 8x^3 - x^2 + 6x$.
   b. Classify each critical point in part a.
3. Sketch the graph of a function with the following properties:
   - There are local extrema at $(-1, 7)$ and $(3, 2)$.
   - There is a point of inflection at $(1, 4)$.
   - The graph is concave down only when $x < 1$.
   - The x-intercept is -4, and the y-intercept is 6.
4. Check the function $g(x) = \frac{x^2 + 7x + 10}{(x - 3)(x + 2)}$ for discontinuities. Conduct appropriate tests to determine if asymptotes exist at the discontinuity values. State the equations of any asymptotes and the domain of $g(x)$.
5. Sketch a graph of a function f with all of the following properties:
   - The graph is increasing when $x < -2$ and when $-2 < x < 4$.
   - The graph is decreasing when $x > 4$.
   - $f'(-2) = 0, f'(4) = 0$
   - The graph is concave down when $x < -2$ and when $3 < x < 9$.
   - The graph is concave up when $-2 < x < 3$ and when $x > 9$.
6. Use at least five curve-sketching techniques to explain how to sketch the graph of the function $f(x) = \frac{2x + 10}{x^2 - 9}$. Sketch the graph on graph paper.
7. The function $f(x) = x^3 + bx^2 + c$ has a critical point at $(-2, 6)$.
   a. Find the constants b and c.
   b. Sketch the graph of $f(x)$ using only the critical points and the second derivative test.

Chapter 5

DERIVATIVES OF EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS

The world's population experiences exponential growth—the rate of growth becomes more rapid as the size of the population increases. Can this be explained in the language of calculus? Well, the rate of growth of the population is described by an exponential function, and the derivative of the population with respect to time is a constant multiple of the population at any time *t*. There are also many situations that can be modelled by trigonometric functions, whose derivative also provides a model for instantaneous rate of change at any time *t*. By combining the techniques in this chapter with the derivative rules seen earlier, we can find the derivative of an exponential or trigonometric function that is combined with other functions. Logarithmic functions and exponential functions are inverses of each other, and, in this chapter, you will also see how their graphs and properties are related to each other.

CHAPTER EXPECTATIONS

In this chapter, you will

- define *e* and the derivative of $y = e^x$, **Section 5.1**
- determine the derivative of the general exponential function $y = b^x$, **Section 5.2**
- compare the graph of an exponential function with the graph of its derivative, **Sections 5.1, 5.2**
- solve optimization problems using exponential functions, **Section 5.3**
- investigate and determine the derivatives of sinusoidal functions, **Section 5.4**
- determine the derivative of the tangent function, **Section 5.5**
- solve rate of change problems involving exponential and trigonometric function models using their derivatives, **Sections 5.1 to 5.5**

Review of Prerequisite Skills

In Chapter 5, you will be studying the derivatives of two classes of functions that occur frequently in calculus problems: exponential functions and trigonometric functions. To begin, we will review some of the properties of exponential and trigonometric functions.

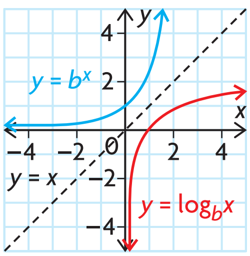
Properties of Exponents

- $b^m b^n = b^{m+n}$
- $\dfrac{b^m}{b^n} = b^{m-n}, b^n \neq 0$
- $(b^m)^n = b^{mn}$
- $b^{\log_b m} = m$
- $\log_b b^m = m$

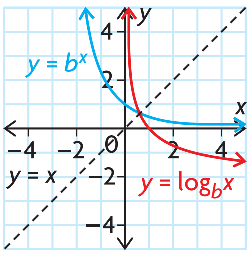
Properties of the Exponential Function, $y = b^x$

- The base b is positive and $b \neq 1$.
- The y-intercept is 1.
- The x-axis is a horizontal asymptote.
- The domain is the set of real numbers, **R**.
- The range is the set of positive real numbers.
- The exponential function is always increasing if $b > 1$.
- The exponential function is always decreasing if $0 < b < 1$.
- The inverse of $y = b^x$ is $x = b^y$.
- The inverse is called the logarithmic function and is written as $\log_b x = y$.

Graphs of $y = \log_b x$ and $y = b^x$



for $b > 1$



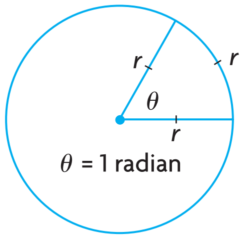
for $0 < b < 1$

- If $b^m = n$ for $b > 0$, then $\log_b n = m$.

Radian Measure

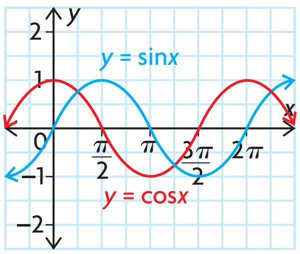
A radian is the measure of an angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.

π radians $= 180°$



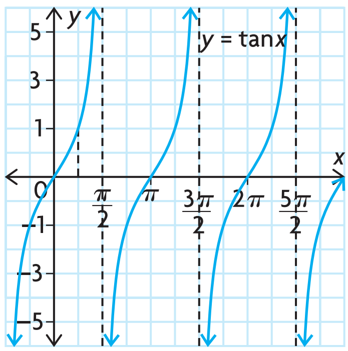
Sine and Cosine Functions
$f(x) = \sin x$ and $f(x) = \cos x$

Domain	$x \in \mathbf{R}$	
Range	$-1 \le \sin x \le 1$	
	$-1 \le \cos x \le 1$	
Periodicity	$\sin(x + 2\pi) = \sin x$	
	$\cos(x + 2\pi) = \cos x$	



Tangent Function
$f(x) = \tan x$

Domain	$\left\{x \in \mathbf{R} \mid x \ne \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots\right\}$
Range	$\{y \in \mathbf{R}\}$
Periodicity	$\tan(x + \pi) = \tan x$



Transformations of Sinusoidal Functions

For $y = a \sin k(x - d) + c$ and $y = a \cos k(x - d) + c$,

- the amplitude is $|a|$
- the period is $\frac{2\pi}{|k|}$
- the horizontal shift is d, and
- the vertical translation is c

Trigonometric Identities

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Reflection Identities

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

Cofunction Identities

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

Exercise

1. Evaluate each of the following:

a. 3^{-2} b. $32^{\frac{2}{5}}$ c. $27^{-\frac{2}{3}}$ d. $\left(\frac{2}{3}\right)^{-2}$

2. Express each of the following in the equivalent logarithmic form:

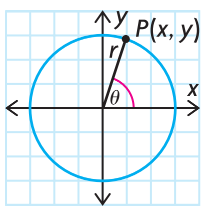
a. $5^4 = 625$ c. $x^3 = 3$ e. $3^8 = z$

b. $4^{-2} = \frac{1}{16}$ d. $10^w = 450$ f. $a^b = T$

3. Sketch the graph of each function, and state its x-intercept.

a. $y = \log_{10}(x + 2)$ b. $y = 5^{x+3}$

4. Refer to the following figure. State the value of each trigonometric ratio below.



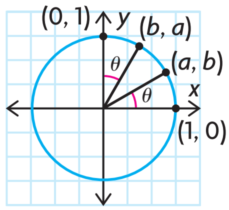
a. $\sin\theta$ b. $\cos\theta$ c. $\tan\theta$

5. Convert the following angles to radian measure:

a. 360°   c. −90°   e. 270°   g. 225°

b. 45°   d. 30°   f. −120°   h. 330°

6. Refer to the following figure. State the value of each trigonometric ratio below.



a. $\sin\theta$ c. $\cos\theta$ e. $\cos\left(\frac{\pi}{2}-\theta\right)$

b. $\tan\theta$ d. $\sin\left(\frac{\pi}{2}-\theta\right)$ f. $\sin(-\theta)$

7. The value of $\sin\theta$, $\cos\theta$, or $\tan\theta$ is given. Determine the values of the other two functions if θ lies in the given interval.

a. $\sin\theta = \frac{5}{13}, \frac{\pi}{2} \le \theta \le \pi$ c. $\tan\theta = -2, \frac{3\pi}{2} \le \theta \le 2\pi$

b. $\cos\theta = -\frac{2}{3}, \pi \le \theta \le \frac{3\pi}{2}$ d. $\sin\theta = 1, 0 \le \theta \le \pi$

8. State the period and amplitude of each of the following:

a. $y = \cos 2x$ d. $y = \frac{2}{7}\cos(12x)$

b. $y = 2\sin\frac{x}{2}$ e. $y = 5\sin\left(\theta - \frac{\pi}{6}\right)$

c. $y = -3\sin(\pi x) + 1$ f. $y = |3\sin x|$

9. Sketch the graph of each function over two complete periods.

a. $y = \sin 2x + 1$ b. $y = 3\cos\left(x + \frac{\pi}{2}\right)$

10. Prove the following identities:

a. $\tan x + \cot x = \sec x \csc x$ b. $\frac{\sin x}{1-\sin^2 x} = \tan x \sec x$

11. Solve the following equations, where $x \in [0, 2\pi]$.

a. $3\sin x = \sin x + 1$ b. $\cos x - 1 = -\cos x$

CAREER LINK Investigate

CHAPTER 5: RATE-OF-CHANGE MODELS IN MICROBIOLOGY

While many real-life situations can be modelled fairly well by polynomial functions, there are some situations that are best modelled by other types of functions, including exponential, logarithmic, and trigonometric functions. Because determining the derivative of a polynomial function is simple, finding the rate of change for models described by polynomial functions is also simple. Often the rate of change at various times is more important to the person studying the scenario than the value of the function is. In this chapter, you will learn how to differentiate exponential and trigonometric functions, increasing the number of function types you can use to model real-life situations and, in turn, analyze using rates of change.

Case Study—Microbiologist

Microbiologists contribute their expertise to many fields, including medicine, environmental science, and biotechnology. Enumerating, the process of counting bacteria, allows microbiologists to build mathematical models that predict populations after a given amount of time has elapsed. Once they can predict a population accurately, the model can be used in medicine, for example, to predict the dose of medication required to kill a certain bacterial infection. The data in the table shown was used by a microbiologist to produce a polynomial-based mathematical model to predict population $p(t)$ as a function of time t, in hours, for the growth of a certain strain of bacteria:

$$p(t) = 1000\left(1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5\right)$$

Time (h)	Population
0	1000
0.5	1649
1.0	2718
1.5	4482
2.0	7389

DISCUSSION QUESTIONS

1. How well does the function fit the data? Use the data, the equation, a graph, and/or a graphing calculator to comment on the "goodness of fit."

2. Use $p(t)$ and $p'(t)$ to determine the following:

a) the population after 0.5 h and the rate at which the population is growing at this time.

b) the population after 1.0 h and the rate at which the population is growing at this time.

3. What pattern did you notice in your calculations? Explain this pattern by examining the terms of the equation to find the reason why.

The polynomial function in this case study is an approximation of a special function in mathematics, natural science, and economics, $f(x) = e^x$, where e has a value of 2.718 28…. At the end of this chapter, you will complete a task on rates of change of exponential growth in a biotechnology case study.

Section 5.1—Derivatives of Exponential Functions, $y = e^x$

Many mathematical relations in the world are nonlinear. We have already studied various polynomial and rational functions and their rates of change. Another type of nonlinear model is the exponential function. Exponential functions are often used to model rapid change. Compound interest, population growth, the intensity of an earthquake, and radioactive decay are just a few examples of exponential change.

In this section, we will study the exponential function $y = e^x$ and its derivative. The number e is a special irrational number, like the number π. It is called the natural number, or Euler's number in honour of the Swiss mathematician Leonhard Euler (pronounced "oiler"), who lived from 1707 to 1783. We use a rational approximation for e of about 2.718. The rules developed thus far have been applied to polynomial functions and rational functions. We are now going to show how the derivative of an exponential function can be found.

INVESTIGATION

Tech *Support*

To evaluate powers of e, such as e^{-2}, press 2ND LN − 2 ENTER.

In this investigation, you will

- graph the exponential function $f(x) = e^x$ and its derivative
- determine the relationship between the exponential function and its derivative

A. Consider the function $f(x) = e^x$. Create a table similar to the one shown below. Complete the $f(x)$ column by using a graphing calculator to calculate the values of e^x for the values of x provided. Round all values to three decimal places.

x	$f(x)$	$f'(x)$
−2	0.135	
−1		
0		
1		
2		
3		

B. Graph the function $f(x) = e^x$.

C. Use a graphing calculator to calculate the value of the derivative $f'(x)$ at each of the given points.

To calculate $f'(x)$, press MATH and scroll down to **8:nDeriv(** under the MATH menu. Press ENTER and the display on the screen will be **nDeriv(**. To find the derivative, key in the expression e^x, the variable x, and the x-value at which you want the derivative, for example, to determine $\frac{d}{dx}(e^x)$ at $x = -2$, the display will be **nDeriv(** e^x, X, -2). Press ENTER, and the approximate value of $f'(-2)$ will be returned.

D. What do you notice about the values of $f(x)$ and $f'(x)$?

E. Draw the graph of the derivative function $f'(x)$ on the same set of axes as $f(x)$. How do the two graphs compare?

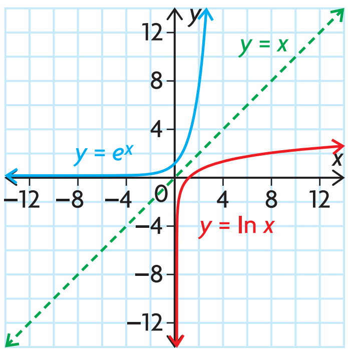
F. Try a few other values of x to see if the pattern continues.

G. What conclusion can you make about the function $f(x) = e^x$ and its derivative?

Properties of $y = e^x$

Since $y = e^x$ is an exponential function, it has the same properties as other exponential functions you have studied.

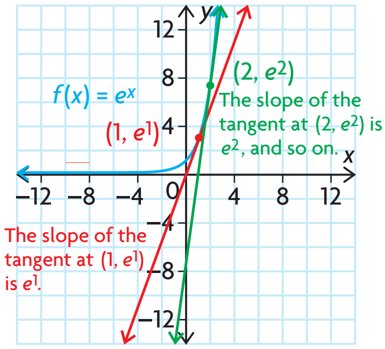
Recall that the logarithm function is the inverse of the exponential function. For example, $y = \log_2 x$ is the inverse of $y = 2^x$. The function $y = e^x$ also has an inverse, $y = \log_e x$. Their graphs are reflections in the line $y = x$. The function $y = \log_e x$ can be written as $y = \ln x$ and is called the **natural logarithm function.**



All the properties of exponential functions and logarithmic functions you are familiar with also apply to $y = e^x$ and $y = \ln x$.

$y = e^x$	$y = \ln x$
• The domain is $\{x \in \mathbf{R}\}$.	• The domain is $\{x \in \mathbf{R} \mid x > 0\}$.
• The range is $\{y \in \mathbf{R} \mid y > 0\}$.	• The range is $\{y \in \mathbf{R}\}$.
• The function passes through (0, 1).	• The function passes through (1, 0).
• $e^{\ln x} = x, x > 0$.	• $\ln e^x = x, x \in \mathbf{R}$.
• The line $y = 0$ is the horizontal asymptote.	• The line $x = 0$ is the vertical asymptote.

From the investigation, you should have noticed that all the values of the derivative $f'(x)$ were exactly the same as those of the original function $f(x) = e^x$. This is a very significant result, since this function is its own derivative—that is, $f(x) = f'(x)$. Since the derivative also represents the slope of the tangent at any given point, the function $f(x) = e^x$ has the special property that the slope of the tangent at a point is the value of the function at this point.



Derivative of $f(x) = e^x$
For the function $f(x) = e^x$, $f'(x) = e^x$.

EXAMPLE 1 **Selecting a strategy to differentiate a composite function involving e^x**

Determine the derivative of $f(x) = e^{3x}$.

Solution

To find the derivative, use the chain rule.

$$\frac{df(x)}{dx} = \frac{d(e^{3x})}{d(3x)}\frac{d(3x)}{dx}$$
$$= e^{3x} \times 3$$
$$= 3e^{3x}$$

Derivative of a Composite Function Involving e^x
In general, if $f(x) = e^{g(x)}$, then $f'(x) = e^{g(x)}\,g'(x)$ by the chain rule.

EXAMPLE 2 Derivatives of exponential functions involving e^x

Determine the derivative of each function.

a. $g(x) = e^{x^2 - x}$

b. $f(x) = x^2 e^x$

Solution

a. To find the derivative of $g(x) = e^{x^2 - x}$, we use the chain rule.

$$\begin{aligned} \frac{dg(x)}{dx} &= \frac{d(e^{x^2 - x})}{dx} \\ &= \frac{d(e^{x^2 - x})}{d(x^2 - x)} \times \frac{d(x^2 - x)}{dx} && \text{(Chain rule)} \\ &= e^{x^2 - x}(2x - 1) \end{aligned}$$

b. Using the product rule,

$$\begin{aligned} f'(x) &= \frac{d(x^2)}{dx} \times e^x + x^2 \times \frac{de^x}{dx} && \text{(Product rule)} \\ &= 2xe^x + x^2e^x && \text{(Factor)} \\ &= e^x(2x + x^2) \end{aligned}$$

EXAMPLE 3 Selecting a strategy to determine the value of the derivative

Given $f(x) = 3e^{x^2}$, determine $f'(-1)$.

Solution

First, find an expression for the derivative of $f'(x)$.

$$\begin{aligned} f'(x) &= \frac{d(3e^{x^2})}{d(x^2)} \frac{dx^2}{dx} && \text{(Chain rule)} \\ &= 3e^{x^2}(2x) \\ &= 6xe^{x^2} \end{aligned}$$

Then $f'(-1) = -6e$.

Answers are usually left as exact values in this form. If desired, numeric approximations can be obtained from a calculator. Here, using the value of e provided by the calculator, we obtain the answer -16.3097, rounded to four decimal places.

EXAMPLE 4 Connecting the derivative of an exponential function to the slope of a tangent

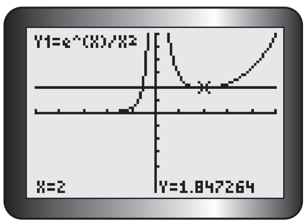
Determine the equation of the line tangent to $y = \frac{e^x}{x^2}$, where $x = 2$.

Solution

Use the derivative to determine the slope of the required tangent.

$$\begin{aligned}
y &= \frac{e^x}{x^2} && \text{(Rewrite as a product)} \\
&= x^{-2}e^x \\
\frac{dy}{dx} &= (-2x^{-3})e^x + x^{-2}e^x && \text{(Product rule)} \\
&= \frac{-2e^x}{x^3} + \frac{e^x}{x^2} && \text{(Determine a common denominator)} \\
&= \frac{-2e^x}{x^3} + \frac{xe^x}{x^3} && \text{(Simplify)} \\
&= \frac{-2e^x + xe^x}{x^3} && \text{(Factor)} \\
&= \frac{(-2 + x)e^x}{x^3}
\end{aligned}$$

When $x = 2$, $y = \frac{e^2}{4}$. When $x = 2$, $\frac{dy}{dx} = 0$ and the tangent is horizontal. Therefore, the equation of the required tangent is $y = \frac{e^2}{4}$. A calculator yields the following graph for $y = \frac{e^x}{x^2}$, and we see the horizontal tangent at $x = 2$. The number Y = 1.847264 in the display is an approximation to the exact number $\frac{e^2}{4}$.



How does the derivative of the general exponential function $g(x) = b^x$ compare with the derivative of $f(x) = e^x$? We will answer this question in Section 5.2.

IN SUMMARY

Key Ideas

- For $f(x) = e^x$, $f'(x) = e^x$.

  In Leibniz notation, $\frac{d}{dx}(e^x) = e^x$.
- For $f(x) = e^{g(x)}$, $f'(x) = e^{g(x)} \times g'(x)$.

  In Leibniz notation, $\frac{d(e^{g(x)})}{dx} = \frac{d(e^{g(x)})}{d(g(x))}\frac{d(g(x))}{dx}$.
- The slope of the tangent at a point on the graph of $y = e^x$ equals the value of the function at this point.

Need to Know

- The rules for differentiating functions, such as the product, quotient, and chain rules, also apply to combinations involving exponential functions of the form $f(x) = e^{g(x)}$.
- e is called Euler's number or the natural number, where $e \doteq 2.718$.

Exercise 5.1

PART A

1. Why can you not use the power rule for derivatives to differentiate $y = e^x$?
2. Differentiate each of the following:

   a. $y = e^{3x}$
   b. $s = e^{3t-5}$
   c. $y = 2e^{10t}$
   d. $y = e^{-3x}$
   e. $y = e^{5-6x+x^2}$
   f. $y = e^{\sqrt{x}}$

K 3. Determine the derivative of each of the following:

   a. $y = 2e^{x^3}$
   b. $y = xe^{3x}$
   c. $f(x) = \frac{e^{-x^3}}{x}$
   d. $f(x) = \sqrt{x}e^x$
   e. $h(t) = et^2 + 3e^{-t}$
   f. $g(t) = \frac{e^{2t}}{1 + e^{2t}}$

4. a. If $f(x) = \frac{1}{3}(e^{3x} + e^{-3x})$, calculate $f'(1)$.
   b. If $f(x) = e^{-\left(\frac{1}{x+1}\right)}$, calculate $f'(0)$.
   c. If $h(z) = z^2(1 + e^{-z})$, calculate $h'(-1)$.

5. a. Determine the equation of the tangent to the curve defined by $y = \frac{2e^x}{1 + e^x}$ at the point $(0, 1)$.
   b. Use graphing technology to graph the function in part a., and draw the tangent at $(0, 1)$.
   c. Compare the equation in part a. with the equation generated by graphing technology. Do they agree?

PART B

6. Determine the equation of the tangent to the curve $y = e^{-x}$ at the point where $x = -1$. Graph the original curve and the tangent.

7. Determine the equation of the tangent to the curve defined by $y = xe^{-x}$ at the point $A(1, e^{-1})$.

8. Determine the coordinates of all points at which the tangent to the curve defined by $y = x^2e^{-x}$ is horizontal.

9. If $y = \frac{5}{2}(e^{\frac{x}{5}} + e^{-\frac{x}{5}})$, prove that $y'' = \frac{y}{25}$.

10. a. For the function $y = e^{-3x}$, determine $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, and $\frac{d^3y}{dx^3}$.

    b. From the pattern in part a., state the value of $\frac{d^ny}{dx^n}$.

11. Determine the first and second derivatives of each function.

    a. $y = -3e^x$
    b. $y = xe^{2x}$
    c. $y = e^x(4 - x)$

A 12. The number, N, of bacteria in a culture at time t, in hours, is $N(t) = 1000\left[30 + e^{-\frac{t}{30}}\right]$

    a. What is the initial number of bacteria in the culture?
    b. Determine the rate of change in the number of bacteria at time t.
    c. How fast is the number of bacteria changing when $t = 20$?
    d. Determine the largest number of bacteria in the culture during the interval $0 \leq t \leq 50$.
    e. What is happening to the number of bacteria in the culture as time passes?

13. The distance s, in metres, fallen by a skydiver t seconds after jumping (and before the parachute opens) is $s = 160\left(\frac{1}{4}t - 1 + e^{-\frac{t}{4}}\right)$.

    a. Determine the velocity, v, at time t.
    b. Show that acceleration is given by $a = 10 - \frac{1}{4}v$.
    c. Determine $v_T = \lim_{t\to\infty} v$. This is the “terminal” velocity, the constant velocity attained when the air resistance balances the force of gravity.
    d. At what time is the velocity 95% of the terminal velocity? How far has the skydiver fallen at that time?

C 14. a. Use a table of values and successive approximation to evaluate each of the following:

    i. $\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)$
    ii. $\lim_{x\to 0}(1 + x)^{\frac{1}{x}}$

    b. Discuss your results.

PART C

T 15. Use the definition of the derivative to evaluate each limit.

a. $\lim_{h \to 0} \frac{e^h - 1}{h}$

b. $\lim_{h \to 0} \frac{e^{2+h} - e^2}{h}$

16. For what values of m does the function $y = Ae^{mt}$ satisfy the following equation?

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

17. The hyperbolic functions are defined as $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

a. Prove $\frac{d(\sinh x)}{dx} = \cosh x$.

b. Prove $\frac{d(\cosh x)}{dx} = \sinh x$.

c. Prove $\frac{d(\tanh x)}{dx} = \frac{1}{(\cosh x)^2}$ if $\tanh x = \frac{\sinh x}{\cosh x}$.

Extension: Graphing the Hyperbolic Function

1. Use graphing technology to graph $y = \cosh x$ by using the definition $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

CATALOG

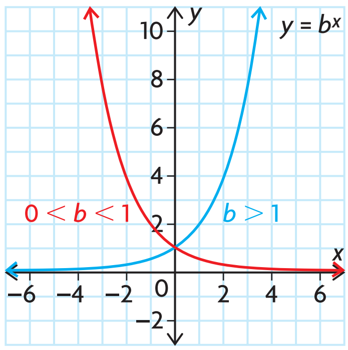
2. Press 2ND 0 for the list of CATALOG items, and select **cosh(** to investigate if cosh is a built-in function.
3. In the same window as problem 1, graph $y = 1.25x^2 + 1$ and $y = 1.05x^2 + 1$. Investigate changes in the coefficient a in the equation $y = ax^2 + 1$ to see if you can create a parabola that will approximate the hyperbolic cosine function.

18. a. Another expression for e is $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots$. Evaluate this expression using four, five, six, and seven consecutive terms of this expression. (*Note:* 2! is read "two factorial"; $2! = 2 \times 1$ and $5! = 5 \times 4 \times 3 \times 2 \times 1$.)

b. Explain why the expression for e in part a. is a special case of $e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$. What is the value of x?

Section 5.2—The Derivative of the General Exponential Function, $y = b^x$

In the previous section, we investigated the exponential function $y = e^x$ and its derivative. The exponential function has a special property—it is its own derivative. The graph of the derivative function is the same as the graph of $y = e^x$. In this section, we will look at the general exponential function $y = b^x$ and its derivative.



INVESTIGATION

In this investigation, you will

- graph and compare the general exponential function and its derivative using the slopes of the tangents at various points and with different bases
- determine the relationship between the general exponential function and its derivative by means of a special ratio

A. Consider the function $f(x) = 2^x$. Create a table with the headings shown below. Use the equation of the function to complete the $f(x)$ column.

x	$f(x)$	$f'(x)$	$\frac{f'(x)}{f(x)}$
−2			
−1			
0			
1			
2			
3			

B. Graph the function $f(x) = 2^x$.

C. Calculate the value of the derivative $f'(x)$ at each of the given points to three decimal places. To calculate $f'(x)$, use the **nDeriv(** function. (See the investigation in Section 5.1 for detailed instructions.)

D. Draw the graph of the derivative function on the same set of axes as $f(x)$ using the given x values and the corresponding values of $f'(x)$.

E. Compare the graph of the derivative with the graph of $f(x)$.

F. i. Calculate the ratio $\frac{f'(x)}{f(x)}$, and record these values in the last column of your table.
   ii. What do you notice about this ratio for the different values of x?
   iii. Is the ratio greater or less than 1?

G. Repeat parts A to F for the function $f(x) = 3^x$.

H. Compare the ratio $\frac{f'(x)}{f(x)}$ for the functions $f(x) = 2^x$ and $f(x) = 3^x$.

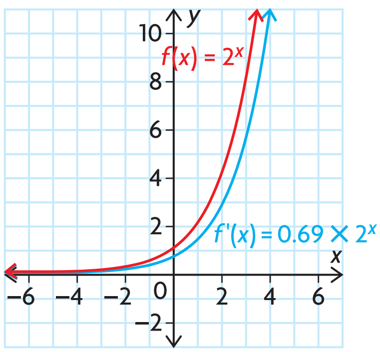
I. Repeat parts A to F for the function $f(x) = b^x$ using different values of b. Does the pattern you found for $f(x) = 2^x$ and $f(x) = 3^x$ continue?

J. What conclusions can you make about the general exponential function and its derivative?

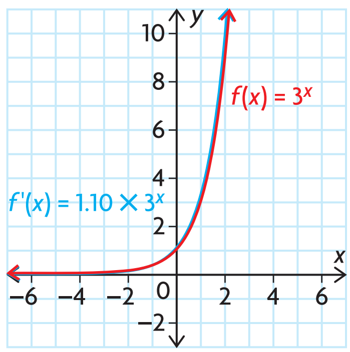
Properties of $y = b^x$

In this investigation, you worked with the functions $f(x) = 2^x$ and $f(x) = 3^x$, and their derivatives. You should have made the following observations:

- For the function $f(x) = 2^x$, the ratio $\frac{f'(x)}{f(x)}$ is approximately equal to 0.69.
- The derivative of $f(x) = 2^x$ is approximately equal to 0.69×2^x.
- For the function $f(x) = 3^x$, the ratio $\frac{f'(x)}{f(x)}$ is approximately equal to 1.10.
- The derivative of $f(x) = 3^x$ is approximately equal to 1.10×3^x.



The derivative of $f(x) = 2^x$ is an exponential function.
The graph of $f'(x)$ is a vertical compression of the graph of $f(x)$.



The derivative of $f(x) = 3^x$ is an exponential function.
The graph of $f'(x)$ is a vertical stretch of the graph of $f(x)$.

In general, for the exponential function $f(x) = b^x$, we can conclude that

- $f(x)$ and $f'(x)$ are both exponential functions
- the slope of the tangent at a point on the curve is proportional to the value of the function at this point
- $f'(x)$ is a vertical stretch or compression of $f(x)$, dependent on the value of b
- the ratio $\frac{f'(x)}{f(x)}$ is a constant and is equivalent to the stretch/compression factor

We can use the definition of the derivative to determine the derivative of the exponential function $f(x) = b^x$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0}\frac{b^{x+h} - b^x}{h} && \text{(Substitution)} \\
&= \lim_{h\to 0}\frac{b^x \times b^h - b^x}{h} && \text{(Properties of the exponential function)} \\
&= \lim_{h\to 0}\frac{b^x(b^h - 1)}{h} && \text{(Common factor)}
\end{aligned}$$

The factor b^x is constant as $h \to 0$ and does not depend on h. Therefore, $f'(x) = b^x \lim\limits_{h\to 0}\frac{b^h - 1}{h}$.

Consider the functions from our investigation:

- For $f(x) = 2^x$, we determined that $f'(x) \doteq 0.69 \times 2^x$ and so $\lim\limits_{h\to 0}\frac{2^h - 1}{h} \doteq 0.69$.
- For $f(x) = 3^x$, we determined that $f'(x) \doteq 1.10 \times 3^x$ and so $\lim\limits_{h\to 0}\frac{3^h - 1}{h} \doteq 1.10$.

In the previous section, for $f(x) = e^x$, we determined that $f'(x) = e^x$ and so $\lim\limits_{h\to 0}\frac{e^h - 1}{h} = 1$.

Can we find a way to determine this constant of proportionality without using a table of values?

The derivative of $f(x) = e^x$ might give us a hint at the answer to this question. From the previous section, we know that $f'(x) = 1 \times e^x$.

We also know that $\log_e e = 1$, or $\ln e = 1$.
Now consider ln 2 and ln 3.

$\ln 2 \doteq 0.693\ 147$ and $\ln 3 \doteq 1.098\ 612$

These match the constants $\frac{f'(x)}{f(x)}$ that we determined in our investigation.

This leads to the following conclusion:

Derivative of $f(x) = b^x$

$$\lim_{h \to 0} \frac{b^h - 1}{h} = \ln b \text{ and if } f(x) = b^x \text{, then } f'(x) = (\ln b) \times b^x$$

EXAMPLE 1 **Selecting a strategy to determine derivatives involving b^x**

Determine the derivative of

a. $f(x) = 5^x$

b. $f(x) = 5^{3x-2}$

Solution

a. $f(x) = 5^x$

Use the derivative of $f(x) = b^x$.

$f'(x) = (\ln 5) \times 5^x$

b. To differentiate $f(x) = 5^{3x-2}$, use the chain rule and the derivative of $f(x) = b^x$.

$f(x) = 5^{3x-2}$

We have $f(x) = 5^{g(x)}$ with $g(x) = 3x - 2$.

Then $g'(x) = 3$

Now, $f'(x) = 5^{3x-2} \times (\ln 5) \times 3$

$ = 3(5^{3x-2})\ln 5$

Derivative of $f(x) = b^{g(x)}$

For $f(x) = b^{g(x)}$, $f'(x) = b^{g(x)}(\ln b)(g'(x))$

EXAMPLE 2 **Solving a problem involving an exponential model**

On January 1, 1850, the population of Goldrushtown was 50 000. The size of the population since then can be modelled by the function $P(t) = 50\ 000(0.98)^t$, where t is the number of years since January 1, 1850.

a. What was the population of Goldrushtown on January 1, 1900?

b. At what rate was the population of Goldrushtown changing on January 1, 1900? Was it increasing or decreasing at that time?

Solution

a. January 1, 1900, is exactly 50 years after January 1, 1850, so we let $t = 50$.

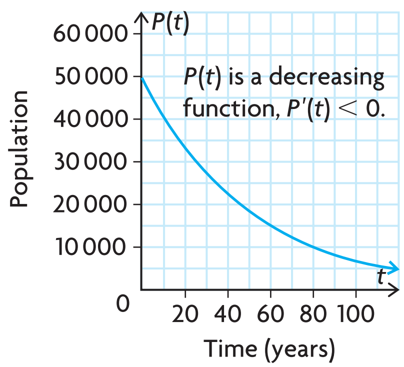
$$P(50) = 50\,000(0.98)^{50}$$
$$= 18\,208.484$$

The population on January 1, 1900, was approximately 18 208.

b. To determine the rate of change in the population, we require the derivative of P.

$$P'(t) = 50\,000(0.98)^t \ln(0.98)$$
$$P'(50) = 50\,000(0.98)^{50} \ln(0.98)$$
$$\doteq -367.861$$

Hence, after 50 years, the population was decreasing at a rate of approximately 368 people per year. (We expected the rate of change to be negative, because the original population function was a decaying exponential function since the base was less than 1.)



IN SUMMARY

Key Ideas

- If $f(x) = b^x$, then $f'(x) = b^x \times \ln b$.

  In Leibniz notation, $\frac{d}{dx}(b^x) = b^x \times \ln b$.
- If $f(x) = b^{g(x)}$, then $f'(x) = b^{g(x)} \times \ln b \times g'(x)$.

  In Leibniz notation, $\frac{d}{dx}(b^{g(x)}) = \frac{d(b^{g(x)})}{d(g(x))}\frac{d(g(x))}{dx}$.

Need to Know

- $\lim_{h \to 0} \frac{b^h - 1}{h} = \ln b$
- When you are differentiating a function that involves an exponential function, use the rules given above, along with the sum, difference, product, quotient, and chain rules as required.

Exercise 5.2

PART A

K

1. Differentiate each of the following functions:
   a. $y = 2^{3x}$
   b. $y = 3.1^x + x^3$
   c. $s = 10^{3t-5}$
   d. $w = 10^{(5-6n+n^2)}$
   e. $y = 3^{x^2+2}$
   f. $y = 400(2)^{x+3}$

2. Determine the derivative of each function.
   a. $y = x^5 \times (5)^x$
   b. $y = x(3)^{x^2}$
   c. $v = \dfrac{2^t}{t}$
   d. $f(x) = \dfrac{\sqrt{3^x}}{x^2}$

3. If $f(t) = 10^{3t-5} \times e^{2t^2}$, determine the values of t so that $f'(t) = 0$.

PART B

4. Determine the equation of the tangent to $y = 3(2^x)$ at $x = 3$.

5. Determine the equation of the tangent to $y = 10^x$ at $(1, 10)$.

A

6. A certain radioactive material decays exponentially. The percent, P, of the material left after t years is given by $P(t) = 100(1.2)^{-t}$.
   a. Determine the half-life of the substance.
   b. How fast is the substance decaying at the point where the half-life is reached?

T

7. Historical data show that the amount of money sent out of Canada for interest and dividend payments during the period from 1967 to 1979 can be approximated by the model $P = (5 \times 10^8)e^{0.200\,15t}$, where t is measured in years ($t = 0$ in 1967) and P is the total payment in Canadian dollars.
   a. Determine and compare the rates of increase for the years 1968 and 1978.
   b. Assuming this trend continues, compare the rate of increase for 1988 with the rate of increase for 1998.
   c. Check the Statistics Canada website to see if the rates of increase predicted by this model were accurate for 1988 and 1998.

8. Determine the equation of the tangent to the curve $y = 2^{-x^2}$ at the point on the curve where $x = 0$. Graph the curve and the tangent at this point.

PART C

C

9. The velocity of a car is given by $v(t) = 120(1 - 0.85^t)$. Graph the function. Describe the acceleration of the car.

Section 5.3—Optimization Problems Involving Exponential Functions

In earlier chapters, you considered numerous situations in which you were asked to optimize a given situation. As you learned, to optimize means to determine values of variables so that a function representing quantities such as cost, area, number of objects, or distance can be minimized or maximized.

Here we will consider further optimization problems, using exponential function models.

EXAMPLE 1 **Solving an optimization problem involving an exponential model**

The effectiveness of studying for an exam depends on how many hours a student studies. Some experiments show that if the effectiveness, E, is put on a scale of 0 to 10, then $E(t) = 0.5[10 + te^{-\frac{t}{20}}]$, where t is the number of hours spent studying for an examination. If a student has up to 30 h for studying, how many hours are needed for maximum effectiveness?

Solution

We wish to find the maximum value of the function $E(t) = 0.5[10 + te^{-\frac{t}{20}}]$, on the interval $0 \leq t \leq 30$.

First find critical numbers by determining $E'(t)$.

$$\begin{aligned} E'(t) &= 0.5\left(e^{-\frac{t}{20}} + t\left(-\frac{1}{20}e^{-\frac{t}{20}}\right)\right) \qquad \text{(Product and chain rules)} \\ &= 0.5e^{-\frac{t}{20}}\left(1 - \frac{t}{20}\right) \end{aligned}$$

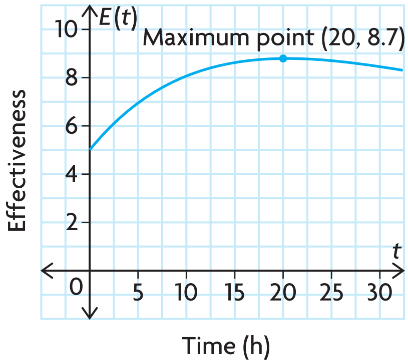
E' is defined for $t \in \mathbf{R}$, and $e^{-\frac{t}{20}} > 0$ for all values of t. So, $E'(t) = 0$ when $1 - \frac{t}{20} = 0$.
Therefore, $t = 20$ is the only critical number.

To determine the maximum effectiveness, we use the algorithm for finding extreme values.

$$\begin{aligned} E(0) &= 0.5(10 + 0e^{0}) = 5 \\ E(20) &= 0.5(10 + 20e^{-1}) \doteq 8.7 \\ E(30) &= 0.5(10 + 30e^{-1.5}) \doteq 8.3 \end{aligned}$$

Therefore, the maximum effectiveness measure of 8.7 is achieved when a student studies 20 h for the exam.

Examining the graph of the function $E(t) = 0.5[10 + te^{-\frac{t}{20}}]$ confirms our result.



EXAMPLE 2 Using calculus techniques to analyze an exponential business model

A mathematical consultant determines that the proportion of people who will have responded to the advertisement of a new product after it has been marketed for t days is given by $f(t) = 0.7(1 - e^{-0.2t})$. The area covered by the advertisement contains 10 million potential customers, and each response to the advertisement results in revenue to the company of \$0.70 (on average), excluding the cost of advertising. The advertising costs \$30 000 to produce and a further \$5000 per day to run.

a. Determine $\lim_{t\to\infty} f(t)$, and interpret the result.

b. What percent of potential customers have responded after seven days of advertising?

c. Write the function $P(t)$ that represents the average profit after t days of advertising. What is the average profit after seven days?

d. For how many full days should the advertising campaign be run in order to maximize the average profit? Assume an advertising budget of \$200 000.

Solution

a. As $t \to \infty$, $e^{-0.2t} \to 0$, so $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} 0.7(1 - e^{-0.2t}) = 0.7$. This result means that if the advertising is left in place indefinitely (forever), 70% of the population will respond.

b. $f(7) = 0.7(1 - e^{-0.2(7)}) \doteq 0.53$

After seven days of advertising, about 53% of the population has responded.

c. The average profit is the difference between the average revenue received from all customers responding to the ad and the advertising costs. Since the area covered by the ad contains 10 million potential customers, the number of customers responding to the ad after t days is $10^7[0.7(1 - e^{-0.2t})] = 7 \times 10^6(1 - e^{-0.2t})$.

The average revenue to the company from these respondents is
$R(t) = 0.7[7 \times 10^6(1 - e^{-0.2t})] = 4.9 \times 10^6(1 - e^{-0.2t})$.
The advertising costs for t days are $C(t) = 30\,000 + 5000t$.
Therefore, the average profit earned after t days of advertising is given by

$$\begin{aligned} P(t) &= R(t) - C(t) \\ &= 4.9 \times 10^6(1 - e^{-0.2t}) - 30\,000 - 5000t \end{aligned}$$

After seven days of advertising, the average profit is

$$\begin{aligned} P(7) &= 4.9 \times 10^6(1 - e^{-0.2(7)}) - 30\,000 - 5000(7) \\ &\doteq 3\,627\,000 \end{aligned}$$

d. If the total advertising budget is \$200 000, then we require that

$$\begin{aligned} 30\,000 + 5000t &\le 200\,000 \\ 5000t &\le 170\,000 \\ t &\le 34 \end{aligned}$$

We wish to maximize the average profit function $P(t)$ on the interval $0 \le t \le 34$.
For critical numbers, determine $P'(t)$.

$$\begin{aligned} P'(t) &= 4.9 \times 10^6(0.2e^{-0.2t}) - 5000 \\ &= 9.8 \times 10^5 e^{-0.2t} - 5000 \end{aligned}$$

$P'(t)$ is defined for $t \in \mathbf{R}$. Let $P'(t) = 0$.

$$9.8 \times 10^5 e^{-0.2t} - 5000 = 0$$

$$e^{-0.2t} = \frac{5000}{9.8 \times 10^5} \qquad \text{(Isolate } e^{-0.2t}\text{)}$$

$$e^{-0.2t} \doteq 0.005\,102\,04 \qquad \text{(Take the ln of both sides)}$$

$$-0.2t = \ln(0.005\,102\,04) \qquad \text{(Solve)}$$

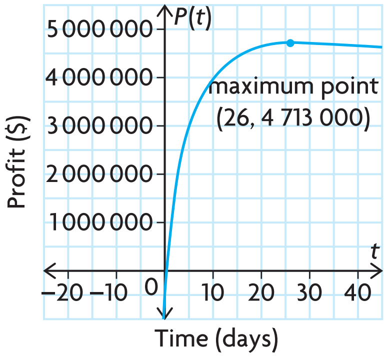
$$t \doteq 26$$

To determine the maximum average profit, we evaluate.

$$\begin{aligned} P(26) &= 4.9 \times 10^6(1 - e^{-0.2(26)}) - 30\,000 - 5000(26) \\ &\doteq 4\,713\,000 \\ P(0) &= 4.9 \times 10^6(1 - e^0) - 30\,000 - 0 \\ &= -30\,000 \text{ (They're losing money!)} \\ P(34) &= 4.9 \times 10^6(1 - e^{-0.2(34)}) - 30\,000 - 5000(34) \\ &\doteq 4\,695\,000 \end{aligned}$$

The maximum average profit of \$4 713 000 occurs when the ad campaign runs for 26 days.

Examining the graph of the function $P(t)$ confirms our result.



IN SUMMARY

Key Ideas

- Optimizing means determining the values of the independent variable so that the values of a function that models a situation can be minimized or maximized.
- The techniques used to optimize an exponential function model are the same as those used to optimize polynomial and rational functions.

Need to Know

- Apply the algorithm introduced in Chapter 3 to solve an optimization problem:

1. Understand the problem, and identify quantities that can vary. Determine a function in one variable that represents the quantity to be optimized.
2. Determine the domain of the function to be optimized, using the information given in the problem.
3. Use the algorithm for finding extreme values (from Chapter 3) to find the absolute maximum or minimum value of the function on the domain.
4. Use your result from step 3 to answer the original problem.
5. Graph the original function using technology to confirm your results.

Exercise 5.3

PART A

1. Use graphing technology to graph each of the following functions. From the graph, find the absolute maximum and absolute minimum values of the given functions on the indicated intervals.
   a. $f(x) = e^{-x} - e^{-3x}$ on $0 \leq x \leq 10$
   b. $m(x) = (x + 2)e^{-2x}$ on $x \in [-4, 4]$

2. a. Use the algorithm for finding extreme values to determine the absolute maximum and minimum values of the functions in question 1.
   b. Explain which approach is easier to use for the functions in question 1.

3. The squirrel population in a small self-contained forest was studied by a biologist. The biologist found that the squirrel population, P, measured in hundreds, is a function of time, t, where t is measured in weeks. The function is $P(t) = \frac{20}{1 + 3e^{-0.02t}}$.
   a. Determine the population at the start of the study, when $t = 0$.
   b. The largest population the forest can sustain is represented mathematically by the limit as $t \to \infty$. Determine this limit.
   c. Determine the point of inflection.
   d. Graph the function.
   e. Explain the meaning of the point of inflection in terms of squirrel population growth.

PART B

4. The net monthly profit, in dollars, from the sale of a certain item is given by the formula $P(x) = 10^6[1 + (x - 1)e^{-0.001x}]$, where x is the number of items sold.
   a. Determine the number of items that yield the maximum profit. At full capacity, the factory can produce 2000 items per month.
   b. Repeat part a., assuming that, at most, 500 items can be produced per month.

K 5. Suppose that the monthly revenue in thousands of dollars, for the sale of x hundred units of an electronic item is given by the function $R(x) = 40x^2e^{-0.4x} + 30$, where the maximum capacity of the plant is 800 units. Determine the number of units to produce in order to maximize revenue.

6. A rumour spreads through a population in such a way that t hours after the rumour starts, the percent of people involved in passing it on is given by $P(t) = 100\left(e^{-t} - e^{-4t}\right)$. What is the highest percent of people involved in spreading the rumour within the first 3 h? When does this occur?

7. Small countries trying to develop an industrial economy rapidly often try to achieve their objectives by importing foreign capital and technology. Statistics Canada data show that when Canada attempted this strategy from 1867 to 1967, the amount of U.S. investment in Canada increased from about $\$15 \times 10^6$ to $\$280\,305 \times 10^6$. This increase in foreign investment can be represented by the simple mathematical model $C(t) = 0.015 \times 10^9 e^{0.075\,33t}$, where t represents the number of years (starting with 1867 as zero) and C represents the total capital investment from U.S. sources in dollars.
   a. Graph the curve for the 100-year period.
   b. Compare the growth rate of U.S. investment in 1947 with the rate in 1967.
   c. Determine the growth rate of investment in 1967 as a percent of the amount invested.
   d. If this model is used up to 1977, calculate the total U.S. investment and the growth rate in this year.
   e. Use the Internet to determine the actual total U.S. investment in 1977, and calculate the error in the model.
   f. If the model is used up to 2007, calculate the expected U.S. investment and the expected growth rate.

A 8. A colony of bacteria in a culture grows at a rate given by $N(t) = 2^{\frac{t}{5}}$, where N is the number of bacteria t minutes from the beginning. The colony is allowed to grow for 60 min, at which time a drug is introduced to kill the bacteria. The number of bacteria killed is given by $K(t) = e^{\frac{t}{3}}$, where K bacteria are killed at time t minutes.
   a. Determine the maximum number of bacteria present and the time at which this occurs.
   b. Determine the time at which the bacteria colony is obliterated.

9. Lorianne is studying for two different exams. Because of the nature of the courses, the measure of study effectiveness on a scale from 0 to 10 for the first course is $E_1 = 0.6(9 + te^{-\frac{t}{20}})$, while the measure for the second course is $E_2 = 0.5(10 + te^{-\frac{t}{10}})$. Lorianne is prepared to spend up to 30 h, in total, studying for the exams. The total effectiveness is given by $f(t) = E_1 + E_2$. How should this time be allocated to maximize total effectiveness?

C 10. Explain the steps you would use to determine the absolute extrema of $f(x) = x - e^{2x}$ on the interval $x \in [-2, 2]$.

T 11. a. For $f(x) = x^2e^x$, determine the intervals of increase and decrease.
   b. Determine the absolute minimum value of $f(x)$.

12. Find the maximum and minimum values of each function. Graph each function.

a. $y = e^x + 2$

b. $y = xe^x + 3$

c. $y = 2xe^{2x}$

d. $y = 3xe^{-x} + x$

13. The profit function of a commodity is $P(x) = xe^{-0.5x^2}$, where $x > 0$. Find the maximum value of the function if x is measured in hundreds of units and P is measured in thousands of dollars.

14. You have just walked out the front door of your home. You notice that it closes quickly at first and then closes more slowly. In fact, a model of the movement of the door is given by $d(t) = 200\,t(2)^{-t}$, where d is the number of degrees between the door frame and the door at t seconds.

a. Graph this relation.

b. Determine when the speed of the moving door is increasing and decreasing.

c. Determine the maximum speed of the moving door.

d. At what point would you consider the door closed?

PART C

15. Suppose that, in question 9, Lorianne has only 25 h to study for the two exams. Is it possible to determine the time to be allocated to each exam? If so, how?

16. Although it is true that many animal populations grow exponentially for a period of time, it must be remembered that the food available to sustain the population is limited and the population will level off because of this. Over a period of time, the population will level out to the maximum attainable value, L. One mathematical model to describe a population that grows exponentially at the beginning and then levels off to a limiting value, L, is the **logistic model**. The equation for this model is $P = \frac{aL}{a + (L - a)e^{-kLt}}$, where the independent variable t represents the time and P represents the size of the population. The constant a is the size of the population at $t = 0$, L is the limiting value of the population, and k is a mathematical constant.

a. Suppose that a biologist starts a cell colony with 100 cells and finds that the limiting size of the colony is 10 000 cells. If the constant $k = 0.0001$, draw a graph to illustrate this population, where t is in days.

b. At what point in time does the cell colony stop growing exponentially? How large is the colony at this point?

c. Compare the growth rate of the colony at the end of day 3 with the growth rate at the end of day 8. Explain what is happening.

Mid-Chapter Review

1. Determine the derivative of each function.
   a. $y = 5e^{-3x}$
   b. $y = 7e^{\frac{1}{7}x}$
   c. $y = x^3e^{-2x}$
   d. $y = (x - 1)^2e^x$
   e. $y = (x - e^{-x})^2$
   f. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

2. A certain radioactive substance decays exponentially over time. The amount of a sample of the substance that remains, P, after t years is given by $P(t) = 100e^{-5t}$, where P is expressed as a percent.
   a. Determine the rate of change of the function, $\frac{dP}{dt}$.
   b. What is the rate of decay when 50% of the original sample has decayed?

3. Determine the equation of the tangent to the curve $y = 2 - xe^x$ at the point where $x = 0$.

4. Determine the first and second derivatives of each function.
   a. $y = -3e^x$
   b. $y = xe^{2x}$
   c. $y = e^x(4 - x)$

5. Determine the derivative of each function.
   a. $y = 8^{2x+5}$
   b. $y = 3.2(10)^{0.2x}$
   c. $f(x) = x^2 2^x$
   d. $H(x) = 300(5)^{3x-1}$
   e. $q(x) = 1.9^x + x^{1.9}$
   f. $f(x) = (x - 2)^2 \times 4^x$

6. The number of rabbits in a forest at time t, in months, is $R(t) = 500\left[10 + e^{-\frac{t}{10}}\right]$.
   a. What is the initial number of rabbits in the forest?
   b. Determine the rate of change of the number of rabbits at time t.
   c. How fast is the number of rabbits changing after one year?
   d. Determine the largest number of rabbits in the forest during the first three years.
   e. Use graphing technology to graph R versus t. Give physical reasons why the population of rabbits might behave this way.

7. A drug is injected into the body in such a way that the concentration, C, in the blood at time t hours is given by the function $C(t) = 10(e^{-2t} - e^{-3t})$. At what time does the highest concentration occur within the first 5 h?

8. Given $y = c(e^{kx})$, for what values of k does the function represent growth? For what values of k does the function represent decay?

9. The rapid growth in the number of a species of insect is given by $P(t) = 5000e^{0.02t}$, where t is the number of days.
   a. What is the initial population $(t = 0)$?
   b. How many insects will there be after a week?
   c. How many insects will there be after a month (30 days)?
10. If you have ever travelled in an airplane, you probably noticed that the air pressure in the airplane varied. The atmospheric pressure, y, varies with the altitude, x kilometres, above Earth. For altitudes up to 10 km, the pressure in millimetres of mercury (mm Hg) is given by $y = 760e^{-0.125x}$. What is the atmospheric pressure at each distance above Earth?
    a. 5 km b. 7 km c. 9 km
11. A radioactive substance decays in such a way that the amount left after t years is given by $A = 100e^{-0.3t}$. The amount, A, is expressed as a percent. Find the function, A', that describes the rate of decay. What is the rate of decay when 50% of the substance is gone?
12. Given $f(x) = xe^x$, find all the x values for which $f'(x) > 0$. What is the significance of this?
13. Find the equation of the tangent to the curve $y = 5^{-x^2}$ at the point on the curve where $x = 1$. Graph the curve and the tangent at this point.
14. a. Determine an equation for $A(t)$, the amount of money in the account at any time t.
    b. Find the derivative $A'(t)$ of the function.
    c. At what rate is the amount growing at the end of two years? At what rate is it growing at the end of five years and at the end of 10 years?
    d. Is the rate constant?
    e. Determine the ratio of $\frac{A'(t)}{A(t)}$ for each value that you determined for $A'(t)$.
    f. What do you notice?
15. The function $y = e^x$ is its own derivative. It is not the only function, however, that has this property. Show that for every value of c, $y = c\,(e^x)$ has the same property.

Section 5.4—The Derivatives of $y = \sin x$ and $y = \cos x$

In this section, we will investigate to determine the derivatives of $y = \sin x$ and $y = \cos x$.

INVESTIGATION 1

A. Using a graphing calculator, graph $y = \sin x$, where x is measured in radians. Use the following WINDOW settings:
- Xmin = 0, Xmax = 9.4, Xscl = $\pi \div 2$
- Ymin = -3.1, Ymax = 3.1, Yscl = 1

Enter $y = \sin x$ into Y1, and graph the function.

Tech Support

To calculate $\frac{dy}{dx}$ at a point, press 2ND TRACE 6 and enter the desired x-coordinate of your point. Then press ENTER.

B. Use the CALC function (with value or $\frac{dy}{dx}$ selected) to compute y and $\frac{dy}{dx}$, respectively, for $y = \sin x$. Record these values in a table like the following (correct to four decimal places):

x	$\sin x$	$\frac{d}{dx}(\sin x)$
0		
0.5		
1.0		
:		
:		
:		
6.5		

C. Create another column, to the right of the $\frac{d}{dx}(\sin x)$ column, with $\cos x$ as the heading. Using your graphing calculator, graph $y = \cos x$ with the same window settings as above.

Tech Support

For help calculating a value of a function using a graphing calculator, see Technical Appendix p. 598.

D. Compute the values of $\cos x$ for $x = 0, 0.5, 1.0, \ldots, 6.5$, correct to four decimal places. Record the values in the $\cos x$ column.

E. Compare the values in the $\frac{d}{dx}(\sin x)$ column with those in the $\cos x$ column, and write a concluding equation.

INVESTIGATION 2 A. Using your graphing calculator, graph $y = \cos x$, where x is measured in radians. Use the following WINDOW settings:

- Xmin = 0, Xmax = 9.4, Xscl = $\pi \div 2$
- Ymin = -3.1, Ymax = 3.1, Yscl = 1

Enter $y = \cos x$ into Y1, and graph the function.

B. Use the CALC function (with value or $\frac{dy}{dx}$ selected) to compute y and $\frac{dy}{dx}$, respectively, for $y = \cos x$. Record these values, correct to four decimal places, in a table like the following:

x	$\cos x$	$\frac{d}{dx}(\cos x)$
0		
0.5		
1.0		
:		
:		
:		
6.5		

C. Create another column to the right of the $\frac{d}{dx}(\cos x)$ column with $-\sin x$ as the heading. Using your graphing calculator, graph $y = -\sin x$ with the same window settings as above.

D. Compute the values of $-\sin x$ for $x = 0, 0.5, 1.0, \ldots, 6.5$, correct to four decimal places. Record the values in the $-\sin x$ column.

E. Compare the values in the $\frac{d}{dx}(\cos x)$ column with those in the $-\sin x$ column, and write a concluding equation.

Investigations 1 and 2 lead to the following conclusions:

Derivatives of Sinusoidal Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

EXAMPLE 1 Selecting a strategy to determine the derivative of a sinusoidal function

Determine $\frac{dy}{dx}$ for each function.

a. $y = \cos 3x$

b. $y = x \sin x$

Solution

a. To differentiate this function, use the chain rule.

$$y = \cos 3x$$

$$\frac{dy}{dx} = \frac{d(\cos 3x)}{d(3x)} \times \frac{d(3x)}{dx} \qquad \text{(Chain rule)}$$

$$= -\sin 3x \times (3)$$

$$= -3 \sin 3x$$

b. To find the derivative, use the product rule.

$$y = x \sin x$$

$$\frac{dy}{dx} = \frac{dx}{dx} \times \sin x + x\frac{d(\sin x)}{dx} \qquad \text{(Product rule)}$$

$$= (1) \times \sin x + x \cos x$$

$$= \sin x + x \cos x$$

EXAMPLE 2 Reasoning about the derivatives of sinusoidal functions

Determine $\frac{dy}{dx}$ for each function.

a. $y = \sin x^2$

b. $y = \sin^2 x$

Solution

a. To differentiate this composite function, use the chain rule and change of variable.

Here, the inner function is $u = x^2$, and the outer function is $y = \sin u$.

Then, $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ (Chain rule)

$$= (\cos u)(2x) \qquad \text{(Substitute)}$$

$$= 2x \cos x^2$$

b. Since $y = \sin^2 x = (\sin x)^2$, we use the chain rule with $y = u^2$, where $u = \sin x$.

Then, $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ (Chain rule)

$$= (2u)(\cos x) \qquad \text{(Substitute)}$$

$$= 2 \sin x \cos x$$

With practice, you will learn how to apply the chain rule without the intermediate step of introducing the variable u. For $y = \sin x^2$, for example, you can skip this step and immediately write $\frac{dy}{dx} = (\cos x^2)(2x)$.

Derivatives of Composite Sinusoidal Functions

If $y = \sin f(x)$, then $\frac{dy}{dx} = \cos f(x) \times f'(x)$.

In Leibniz notation, $\frac{d}{dx}(\sin f(x)) = \frac{d(\sin f(x))}{d(f(x))} \times \frac{d(f(x))}{dx} = \cos f(x) \times \frac{d(f(x))}{dx}$.

If $y = \cos f(x)$, then $\frac{dy}{dx} = -\sin f(x) \times f'(x)$.

In Leibniz notation, $\frac{d}{dx}(\cos f(x)) = \frac{d(\cos f(x))}{d(f(x))} \times \frac{d(f(x))}{dx} = -\sin f(x) \times \frac{d(f(x))}{dx}$.

EXAMPLE 3 Differentiating a composite cosine function

Determine $\frac{dy}{dx}$ for $y = \cos(1 + x^3)$.

Solution

$$y = \cos(1 + x^3)$$

$$\frac{dy}{dx} = \frac{d[\cos(1 + x^3)]}{d(1 + x^3)} \times \frac{d(1 + x^3)}{dx} \qquad \text{(Chain rule)}$$

$$= -\sin(1 + x^3)(3x^2)$$

$$= -3x^2 \sin(1 + x^3)$$

EXAMPLE 4 Differentiating a combination of functions

Determine y' for $y = e^{\sin x + \cos x}$.

Solution

$$y = e^{\sin x + \cos x}$$

$$y' = \frac{d(e^{\sin x + \cos x})}{d(\sin x + \cos x)} \times \frac{d(\sin x + \cos x)}{dx} \qquad \text{(Chain rule)}$$

$$= e^{\sin x + \cos x}(\cos x - \sin x)$$

EXAMPLE 5 **Connecting the derivative of a sinusoidal function to the slope of a tangent**

Determine the equation of the tangent to the graph of $y = x \cos 2x$ at $x = \frac{\pi}{2}$.

Solution

When $x = \frac{\pi}{2}$, $y = \frac{\pi}{2} \cos \pi = -\frac{\pi}{2}$.

The point of tangency is $\left(\frac{\pi}{2}, -\frac{\pi}{2}\right)$.

The slope of the tangent at any point on the graph is given by

$$\frac{dy}{dx} = \frac{dx}{dx} \times \cos 2x + x \times \frac{d(\cos 2x)}{dx} \qquad \text{(Product and chain rules)}$$

$$= (1)(\cos 2x) + x(-\sin 2x)(2) \qquad \text{(Simplify)}$$

$$= \cos 2x - 2x \sin 2x$$

At $x = \frac{\pi}{2}$, $\frac{dy}{dx} = \cos \pi - \pi(\sin \pi)$ (Evaluate)

$$= -1$$

The equation of the tangent is

$$y + \frac{\pi}{2} = -\left(x - \frac{\pi}{2}\right) \text{ or } y = -x.$$

EXAMPLE 6 **Connecting the derivative of a sinusoidal function to its extreme values**

Determine the maximum and minimum values of the function $f(x) = \cos^2 x$ on the interval $x \in [0, 2\pi]$.

Solution

By the algorithm for finding extreme values, the maximum and minimum values occur at points on the graph where $f'(x) = 0$ or at endpoints of the interval.
The derivative of $f(x)$ is

$$f'(x) = 2(\cos x)(-\sin x) \qquad \text{(Chain rule)}$$

$$= -2 \sin x \cos x$$

$$= -\sin 2x \qquad \text{(Using the double angle identity)}$$

Solving $f'(x) = 0$,

$$-\sin 2x = 0$$

$$\sin 2x = 0$$

$$2x = 0, \pi, 2\pi, 3\pi, \text{ or } 4\pi$$

$$\text{so } x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \text{ or } 2\pi$$

We evaluate $f(x)$ at the critical numbers. (In this example, the endpoints of the interval are included.)

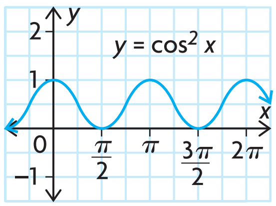
x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$f(x) = \cos^2 x$	1	0	1	0	1

The maximum value is 1 when $x = 0, \pi$, or 2π. The minimum value is 0 when $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

The above solution is verified by our knowledge of the cosine function. For the function $y = \cos x$,
- the domain is $x \in \mathbf{R}$
- the range is $-1 \leq \cos x \leq 1$

For the given function $y = \cos^2 x$,
- the domain is $x \in \mathbf{R}$
- the range is $0 \leq \cos^2 x \leq 1$

Therefore, the maximum value is 1 and the minimum value is 0.



IN SUMMARY

Key Idea

- The derivatives of sinusoidal functions are found as follows:
  - $\frac{d(\sin x)}{dx} = \cos x$ and $\frac{d(\cos x)}{dx} = -\sin x$
  - If $y = \sin f(x)$, then $\frac{dy}{dx} = \cos f(x) \times f'(x)$.
  - If $y = \cos f(x)$, then $\frac{dy}{dx} = -\sin f(x) \times f'(x)$.

Need to Know

- When you are differentiating a function that involves sinusoidal functions, use the rules given above, along with the sum, difference, product, quotient, and chain rules as required.

Exercise 5.4

PART A

K 1. Determine $\frac{dy}{dx}$ for each of the following:

a. $y = \sin 2x$
b. $y = 2\cos 3x$
c. $y = \sin(x^3 - 2x + 4)$
d. $y = 2\cos(-4x)$
e. $y = \sin 3x - \cos 4x$
f. $y = 2^x + 2\sin x - 2\cos x$
g. $y = \sin(e^x)$
h. $y = 3\sin(3x + 2\pi)$
i. $y = x^2 + \cos x + \sin \frac{\pi}{4}$
j. $y = \sin \frac{1}{x}$

2. Differentiate the following functions:

a. $y = 2\sin x \cos x$
b. $y = \frac{\cos 2x}{x}$
c. $y = \cos(\sin 2x)$
d. $y = \frac{\sin x}{1 + \cos x}$
e. $y = e^x(\cos x + \sin x)$
f. $y = 2x^3 \sin x - 3x \cos x$

PART B

3. Determine an equation for the tangent at the point with the given x-coordinate for each of the following functions:

a. $f(x) = \sin x, x = \frac{\pi}{3}$
b. $f(x) = x + \sin x, x = 0$
c. $f(x) = \cos(4x), x = \frac{\pi}{4}$
d. $f(x) = \sin 2x + \cos x, x = \frac{\pi}{2}$
e. $f(x) = \cos\left(2x + \frac{\pi}{3}\right), x = \frac{\pi}{4}$
f. $f(x) = 2\sin x \cos x, x = \frac{\pi}{2}$

C 4. a. If $f(x) = \sin^2 x$ and $g(x) = 1 - \cos^2 x$, explain why $f'(x) = g'(x)$.
b. If $f(x) = \sin^2 x$ and $g(x) = 1 + \cos^2 x$, how are $f'(x)$ and $g'(x)$ related?

5. Differentiate each function.

a. $v(t) = \sin^2(\sqrt{t})$
b. $v(t) = \sqrt{1 + \cos t + \sin^2 t}$
c. $h(x) = \sin x \sin 2x \sin 3x$
d. $m(x) = (x^2 + \cos^2 x)^3$

6. Determine the absolute extreme values of each function on the given interval. (Verify your results with graphing technology.)
   a. $y = \cos x + \sin x, 0 \le x \le 2\pi$
   b. $y = x + 2\cos x, -\pi \le x \le \pi$
   c. $y = \sin x - \cos x, x \in [0, 2\pi]$
   d. $y = 3\sin x + 4\cos x, x \in [0, 2\pi]$

A 7. A particle moves along a line so that, at time t, its position is $s(t) = 8\sin 2t$.
   a. For what values of t does the particle change direction?
   b. What is the particle's maximum velocity?

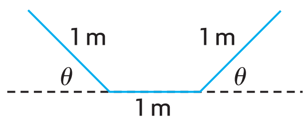
8. a. Graph the function $f(x) = \cos x + \sin x$.
   b. Determine the coordinates of the point where the tangent to the curve of $f(x)$ is horizontal, on the interval $0 \le x \le \pi$.

9. Determine expressions for the derivatives of $\csc x$ and $\sec x$.

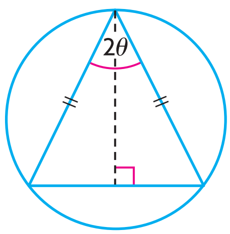
10. Determine the slope of the tangent to the curve $y = \cos 2x$ at point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$.

11. A particle moves along a line so that at time t, its position is $s = 4\sin 4t$.
   a. When does the particle change direction?
   b. What is the particle's maximum velocity?
   c. What is the particle's minimum distance from the origin? What is its maximum distance from the origin?

T 12. An irrigation channel is constructed by bending a sheet of metal that is 3 m wide, as shown in the diagram. What angle θ will maximize the cross-sectional area (and thus the capacity) of the channel?



13. An isosceles triangle is inscribed in a circle of radius **R**. Find the value of θ that maximizes the area of the triangle.



PART C

14. If $y = A\cos kt + B\sin kt$, where A, B, and k are constants, show that $y'' + k^2y = 0$.

Section 5.5—The Derivative of $y = \tan x$

In this section, we will study the derivative of the remaining primary trigonometric function—tangent.

Since this function can be expressed in terms of sine and cosine, we can find its derivative using the product rule.

EXAMPLE 1 **Reasoning about the derivative of the tangent function**

Determine $\frac{dy}{dx}$ for $y = \tan x$.

Solution

$$\begin{aligned} y &= \tan x \\ &= \frac{\sin x}{\cos x} \\ &= (\sin x)(\cos x)^{-1} \\ \frac{dy}{dx} &= \frac{d(\sin x)}{dx} \times (\cos x)^{-1} + \sin x \times \frac{d(\cos x)^{-1}}{dx} && \text{(Product rule)} \\ &= (\cos x)(\cos x)^{-1} + \sin x\,(-1)(\cos x)^{-2}(-\sin x) && \text{(Chain rule)} \\ &= 1 + \frac{\sin^2 x}{\cos^2 x} \\ &= 1 + \tan^2 x && \text{(Using the Pythagorean identity)} \\ &= \sec^2 x \end{aligned}$$

Therefore, $\frac{d(\tan x)}{dx} = \sec^2 x$

EXAMPLE 2 **Selecting a strategy to determine the derivative of a composite tangent function**

Determine $\frac{dy}{dx}$ for $y = \tan(x^2 + 3x)$.

Solution

$$\begin{aligned} y &= \tan(x^2 + 3x) \\ \frac{dy}{dx} &= \frac{d\tan(x^2 + 3x)}{d(x^2 + 3x)} \times \frac{d(x^2 + 3x)}{dx} && \text{(Chain rule)} \\ &= \sec^2(x^2 + 3x) \times (2x + 3) \\ &= (2x + 3)\sec^2(x^2 + 3x) \end{aligned}$$

Derivatives of Composite Functions Involving $y = \tan x$

If $y = \tan f(x)$, then $\frac{dy}{dx} = \sec^2 f(x) \times f'(x)$.

In Leibniz notation, $\frac{d}{dx}(\tan f(x)) = \frac{d(\tan f(x))}{d(f(x))} \times \frac{df(x))}{dx} = \sec^2(f(x)) \times \frac{d(f(x))}{dx}$.

EXAMPLE 3

Determining the derivative of a combination of functions

Determine $\frac{dy}{dx}$ for $y = (\sin x + \tan x)^4$.

Solution

$$y = (\sin x + \tan x)^4$$

$$\frac{dy}{dx} = 4(\sin x + \tan x)^3(\cos x + \sec^2 x)$$ (Chain rule)

EXAMPLE 4

Determining the derivative of a product involving the tangent function

Determine $\frac{dy}{dx}$ for $y = x \tan(2x - 1)$.

Solution

$$y = x \tan(2x - 1)$$

$$\frac{dy}{dx} = (1)\tan(2x - 1) + (x)\sec^2(2x - 1)\frac{d(2x - 1)}{dx}$$ (Product and chain rules)

$$= \tan(2x - 1) + 2x \sec^2(2x - 1)$$

IN SUMMARY

Key Idea

- The derivatives of functions involving the tangent function are found as follows:
  - $\frac{d(\tan x)}{dx} = \sec^2 x$
  - $\frac{d}{dx}(\tan f(x)) = \sec^2 f(x) \times f'(x)$

Need to Know

- Trigonometric identities can be used to write one expression as an equivalent expression and then differentiate. In some cases, the new function will be easier to work with.

Exercise 5.5

PART A

K 1. Determine $\frac{dy}{dx}$ for each of the following:

a. $y = \tan 3x$

b. $y = 2\tan x - \tan 2x$

c. $y = \tan^2(x^3)$

d. $y = \frac{x^2}{\tan \pi x}$

e. $y = \tan(x^2) - \tan^2 x$

f. $y = 3\sin 5x \tan 5x$

A 2. Determine an equation for the tangent to each function at the point with the given x-coordinate.

a. $f(x) = \tan x, x = \frac{\pi}{4}$

b. $f(x) = 6\tan x - \tan 2x, x = 0$

PART B

3. Determine y' for each of the following:

a. $y = \tan(\sin x)$

b. $y = [\tan(x^2 - 1)]^{-2}$

c. $y = \tan^2(\cos x)$

d. $y = (\tan x + \cos x)^2$

e. $y = \sin^3 x \tan x$

f. $y = e^{\tan\sqrt{x}}$

4. Determine $\frac{d^2y}{dx^2}$ for each of the following:

a. $y = \sin x \tan x$

b. $y = \tan^2 x$

5. Determine all the values of x, $0 \leq x \leq 2\pi$, for which the slope of the tangent to $f(x) = \sin x \tan x$ is zero.

6. Determine the local maximum point on the curve $y = 2x - \tan x$, where $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

T 7. Prove that $y = \sec x + \tan x$ is always increasing on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

8. Determine the equation of the line that is tangent to $y = 2\tan x$, where $x = \frac{\pi}{4}$.

C 9. If you forget the expression that results when differentiating the tangent function, explain how you can derive this derivative using an identity.

PART C

10. Determine the derivative of $\cot x$.

11. Determine $f''(x)$, where $f(x) = \cot 4x$.

CAREER LINK WRAP-UP Investigate and Apply

CHAPTER 5: RATE-OF-CHANGE MODELS IN MICROBIOLOGY

A simplified model for bacterial growth is $P(t) = P_0 e^{rt}$, where $P(t)$ is the population of the bacteria colony after t hours, P_0 is the initial population of the colony (the population at $t = 0$), and r determines the growth rate of the colony. The model is simple because it does not account for limited resources, such as space and nutrients. As time increases, so does the population, but there is no bound on the population. While a model like this can describe a population for a short period of time or can be made to describe a population for a longer period of time by adjusting conditions in a laboratory experiment, in general, populations are better described by more complex models.

To determine how the population of a particular type of bacteria will grow over time under controlled conditions, a microbiologist observes the initial population and the population every half hour for 8 h. (The microbiologist also controls the environment in which the colony is growing to make sure that temperature and light conditions remain constant and ensures that the amount of nutrients available to the colony as it grows is sufficient for the increasing population.)

After analyzing the population data, the microbiologist determines that the population of the bacteria colony can be modelled by the equation $P(t) = 500\,e^{0.1t}$.

a. What is the initial population of the bacteria colony?

b. What function describes the instantaneous rate of change in the bacteria population after t hours?

c. What is the instantaneous rate of change in the population after 1 h? What is the instantaneous rate of change after 8 h?

d. How do your answers for part c. help you make a prediction about how long the bacteria colony will take to double in size? Make a prediction for the number of hours the population will take to double, using your answers for part c. and/or other information.

e. Determine the actual doubling time—the time that the colony takes to grow to twice its initial population. (***Hint:*** Solve for t when $P(t) = 1000$.)

f. Compare your prediction for the doubling time with the calculated value. If your prediction was not close to the actual value, what factors do you think might account for the difference?

g. When is the instantaneous rate of change equal to 500 bacteria per hour?

Key Concepts Review

In this chapter, we introduced a new base for exponential functions, namely the number e, where $e \doteq 2.718\ 281$. We examined the derivatives of the exponential functions along with the primary trigonometric functions. You should now be able to apply all the rules of differentiation that you learned in Chapter 2 to expressions that involve the exponential, sine, cosine, and tangent functions combined with polynomial and rational functions.

We also examined some applications of exponential and trigonometric functions. The calculus techniques that are used to determine instantaneous rates of change, equations of tangent lines, and absolute extrema for polynomial and rational functions, can also be used for exponential and trigonometric functions.

Derivative Rules for Exponential Functions

- $\frac{d}{dx}(e^x) = e^x$ and $\frac{d}{dx}(e^{g(x)}) = e^{g(x)} \times g'(x)$
- $\frac{d}{dx}(b^x) = b^x \ln b$ and $\frac{d}{dx}(b^{g(x)}) = b^{g(x)}(\ln b)g'(x)$

Derivative Rules for Primary Trigonometric Functions

- $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\sin f(x)) = \cos f(x) \times f'(x)$
- $\frac{d}{dx}(\cos x) = -\sin x$ and $\frac{d}{dx}(\cos f(x)) = -\sin f(x) \times f'(x)$
- $\frac{d}{dx}(\tan x) = \sec^2 x$ and $\frac{d}{dx}(\tan f(x)) = \sec^2 f(x) \times f'(x)$

Review Exercise

1. Differentiate each of the following:
   a. $y = 6 - e^x$
   b. $y = 2x + 3e^x$
   c. $y = e^{2x+3}$
   d. $y = e^{-3x^2+5x}$
   e. $y = xe^x$
   f. $s = \dfrac{e^t - 1}{e^t + 1}$
2. Determine $\frac{dy}{dx}$ for each of the following:
   a. $y = 10^x$
   b. $y = 4^{3x^2}$
   c. $y = (5x)(5^x)$
   d. $y = (x^4)2^x$
   e. $y = \dfrac{4x}{4^x}$
   f. $y = \dfrac{5^{\sqrt{x}}}{x}$
3. Differentiate each of the following:
   a. $y = 3\sin 2x - 4\cos 2x$
   b. $y = \tan 3x$
   c. $y = \dfrac{1}{2 - \cos x}$
   d. $y = x\tan 2x$
   e. $y = (\sin 2x)e^{3x}$
   f. $y = \cos^2 2x$
4. a. Given the function $f(x) = \frac{e^x}{x}$, solve the equation $f'(x) = 0$.
   b. Discuss the significance of the solution you found in part a.
5. a. If $f(x) = xe^{-2x}$, find $f'\left(\frac{1}{2}\right)$.
   b. Explain what this number represents.
6. Determine the second derivative of each of the following:
   a. $y = xe^x - e^x$
   b. $y = xe^{10x}$
7. If $y = \frac{e^{2x} - 1}{e^{2x} + 1}$, prove that $\frac{dy}{dx} = 1 - y^2$.
8. Determine the equation of the tangent to the curve defined by $y = x - e^{-x}$ that is parallel to the line represented by $3x - y - 9 = 0$.
9. Determine the equation of the tangent to the curve $y = x\sin x$ at the point where $x = \frac{\pi}{2}$.
10. An object moves along a line so that, at time t, its position is $s = \frac{\sin t}{3 + \cos 2t}$, where s is the displacement in metres. Calculate the object's velocity at $t = \frac{\pi}{4}$.

11. The number of bacteria in a culture, N, at time t is given by $N(t) = 2000\left[30 + te^{-\frac{t}{20}}\right]$.

    a. When is the rate of change of the number of bacteria equal to zero?

    b. If the bacterial culture is placed in a colony of mice, the number of mice that become infected, M, is related to the number of bacteria present by the equation $M(t) = \sqrt[3]{N + 1000}$. After 10 days, how many mice are infected per day?

12. The concentrations of two medicines in the bloodstream t hours after injection are $c_1(t) = te^{-t}$ and $c_2(t) = t^2e^{-t}$.

    a. Which medicine has the larger maximum concentration?

    b. Within the first half hour, which medicine has the larger maximum concentration?

13. Differentiate.

    a. $y = (2 + 3e^{-x})^3$

    b. $y = x^e$

    c. $y = e^{e^x}$

    d. $y = (1 - e^{5x})^5$

14. Differentiate.

    a. $y = 5^x$

    b. $y = (0.47)^x$

    c. $y = (52)^{2x}$

    d. $y = 5(2)^x$

    e. $y = 4(e)^x$

    f. $y = -2(10)^{3x}$

15. Determine y'.

    a. $y = \sin 2^x$

    b. $y = x^2 \sin x$

    c. $y = \sin\left(\frac{\pi}{2} - x\right)$

    d. $y = \cos x \sin x$

    e. $y = \cos^2 x$

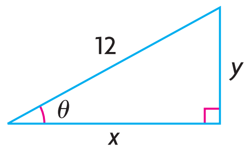
    f. $y = \cos x \sin^2 x$

16. Determine the equation of the tangent to the curve $y = \cos x$ at $\left(\frac{\pi}{2}, 0\right)$.

17. An object is suspended from the end of a spring. Its displacement from the equilibrium position is $s = 8 \sin(10\pi t)$ at time t. Calculate the velocity and acceleration of the object at any time t, and show that $\frac{d^2s}{dt^2} + 100\pi^2 s = 0$.

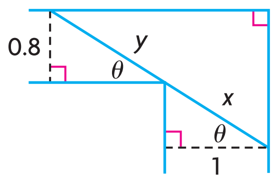
18. The position of a particle is given by $s = 5\cos\left(2t + \frac{\pi}{4}\right)$ at time t. What are the maximum values of the displacement, the velocity, and the acceleration?

19. The hypotenuse of a right triangle is 12 cm in length. Calculate the measures of the unknown angles in the triangle that will maximize its perimeter.

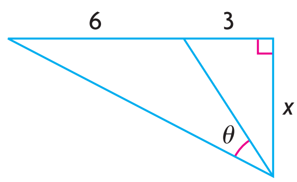


20. A fence is 1.5 m high and is 1 m from a wall. A ladder must start from the ground, touch the top of the fence, and rest somewhere on the wall. Calculate the minimum length of the ladder.

21. A thin rigid pole needs to be carried horizontally around a corner joining two corridors, which are 1 m and 0.8 m wide. Calculate the length of the longest pole that can be carried around this corner.



22. When the rules of hockey were developed, Canada did not use the metric system. Thus, the distance between the goal posts was designated to be six feet (slightly less than 2 m). If Sidney Crosby is on the goal line, three feet outside one of the goal posts, how far should he go out (perpendicular to the goal line) to maximize the angle in which he can shoot at the goal?
Hint: Determine the values of x that maximize θ in the following diagram.



23. Determine $f''(x)$
    a. $f(x) = 4\sin^2(x - 2)$
    b. $f(x) = 2(\cos x)(\sec^2 x)$

Chapter 5 Test

1. Determine the derivative $\frac{dy}{dx}$ for each of the following:
   a. $y = e^{-2x^2}$
   b. $y = 3^{x^2+3x}$
   c. $y = \dfrac{e^{3x} + e^{-3x}}{2}$
   d. $y = 2\sin x - 3\cos 5x$
   e. $y = \sin^3(x^2)$
   f. $y = \tan\sqrt{1 - x}$
2. Determine the equation of the tangent to the curve defined by $y = 2e^{3x}$ that is parallel to the line defined by $-6x + y = 2$.
3. Determine the equation of the tangent to $y = e^x + \sin x$ at $(0, 1)$.
4. The velocity of a certain particle that moves in a straight line under the influence of forces is given by $v(t) = 10e^{-kt}$, where k is a positive constant and $v(t)$ is in centimetres per second.
   a. Show that the acceleration of the particle is proportional to a constant multiple of its velocity. Explain what is happening to the particle.
   b. What is the initial velocity of the particle?
   c. At what time is the velocity equal to half the initial velocity? What is the acceleration at this time?
5. Determine $f''(x)$.
   a. $f(x) = \cos^2 x$
   b. $f(x) = \cos x \cot x$
6. Determine the absolute extreme values of $f(x) = \sin^2 x$, where $x \in [0, \pi]$.
7. Calculate the slope of the tangent line that passes through $y = 5^x$, where $x = 2$. Express your answer to two decimal places.
8. Determine all the maximum and minimum values of $y = xe^x + 3e^x$.
9. $f(x) = 2\cos x - \sin 2x$ where $x \in [-\pi, \pi]$
   a. Determine all critical number for $f(x)$ on the given interval.
   b. Determine the intervals where $f(x)$ is increasing and where it is decreasing.
   c. Determine all local maximum and minimum values of $f(x)$ on the given interval.
   d. Use the information you found above to sketch the curve.

Chapter 5 Test

1. Determine the derivative $\frac{dy}{dx}$ for each of the following:

   a. $y = e^{-2x^2}$

   b. $y = 3^{x^2+3x}$

   c. $y = \dfrac{e^{3x} + e^{-3x}}{2}$

   d. $y = 2 \sin x - 3 \cos 5x$

   e. $y = \sin^3 (x^2)$

   f. $y = \tan\sqrt{1 - x}$

2. Determine the equation of the tangent to the curve defined by $y = 2e^{3x}$ that is parallel to the line defined by $-6x + y = 2$.

3. Determine the equation of the tangent to $y = e^x + \sin x$ at $(0, 1)$.

4. The velocity of a certain particle that moves in a straight line under the influence of forces is given by $v(t) = 10e^{-kt}$, where k is a positive constant and $v(t)$ is in centimetres per second.

   a. Show that the acceleration of the particle is proportional to a constant multiple of its velocity. Explain what is happening to the particle.

   b. What is the initial velocity of the particle?

   c. At what time is the velocity equal to half the initial velocity? What is the acceleration at this time?

5. Determine $f''(x)$.

   a. $f(x) = \cos^2 x$

   b. $f(x) = \cos x \cot x$

6. Determine the absolute extreme values of $f(x) = \sin^2 x$, where $x \in [0, \pi]$.

7. Calculate the slope of the tangent line that passes through $y = 5^x$, where $x = 2$. Express your answer to two decimal places.

8. Determine all the maximum and minimum values of $y = xe^x + 3e^x$.

9. $f(x) = 2 \cos x - \sin 2x$ where $x \in [-\pi, \pi]$

   a. Determine all critical number for $f(x)$ on the given interval.

   b. Determine the intervals where $f(x)$ is increasing and where it is decreasing.

   c. Determine all local maximum and minimum values of $f(x)$ on the given interval.

   d. Use the information you found above to sketch the curve.

Cumulative Review of Calculus

1. Using the limit definition of the slope of a tangent, determine the slope of the tangent to each curve at the given point.

   a. $f(x) = 3x^2 + 4x - 5$, $(2, 15)$

   b. $f(x) = \dfrac{2}{x - 1}$, $(2, 2)$

   c. $f(x) = \sqrt{x + 3}$, $(6, 3)$

   d. $f(x) = 2^{5x}$, $(1, 32)$

2. The position, in metres, of an object is given by $s(t) = 2t^2 + 3t + 1$, where t is the time in seconds.

   a. Determine the average velocity from $t = 1$ to $t = 4$.

   b. Determine the instantaneous velocity at $t = 3$.

3. If $\lim\limits_{h \to 0} \dfrac{(4 + h)^3 - 64}{h}$ represents the slope of the tangent to $y = f(x)$ at $x = 4$, what is the equation of $f(x)$?

4. An object is dropped from the observation deck of the Skylon Tower in Niagara Falls, Ontario. The distance, in metres, from the deck at t seconds is given by $d(t) = 4.9t^2$.

   a. Determine the average rate of change in distance with respect to time from $t = 1$ to $t = 3$.

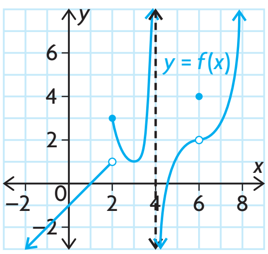
   b. Determine the instantaneous rate of change in distance with respect to time at 2 s.

   c. The height of the observation deck is 146.9 m. How fast is the object moving when it hits the ground?

5. The model $P(t) = 2t^2 + 3t + 1$ estimates the population of fish in a reservoir, where P represents the population, in thousands, and t is the number of years since 2000.

   a. Determine the average rate of population change between 2000 and 2008.

   b. Estimate the rate at which the population was changing at the start of 2005.



6. a. Given the graph of $f(x)$ at the left, determine the following:

   i. $f(2)$

   ii. $\lim\limits_{x \to 2^-} f(x)$

   iii. $\lim\limits_{x \to 2^+} f(x)$

   iv. $\lim\limits_{x \to 6} f(x)$

   b. Does $\lim\limits_{x \to 4} f(x)$ exist? Justify your answer.

7. Consider the following function:

$$f(x) = \begin{cases} x^2 + 1, & \text{if } x < 2 \\ 2x + 1, & \text{if } x = 2 \\ -x + 5, & \text{if } x > 2 \end{cases}$$

   Determine where $f(x)$ is discontinuous, and justify your answer.

8. Use algebraic methods to evaluate each limit (if it exists).

a. $\lim_{x\to 0} \frac{2x^2+1}{x-5}$

b. $\lim_{x\to 3} \frac{x-3}{\sqrt{x+6}-3}$

c. $\lim_{x\to -3} \frac{\frac{1}{x}+\frac{1}{3}}{x+3}$

d. $\lim_{x\to 2} \frac{x^2-4}{x^2-x-2}$

e. $\lim_{x\to 2} \frac{x-2}{x^3-8}$

f. $\lim_{x\to 0} \frac{\sqrt{x+4}-\sqrt{4-x}}{x}$

9. Determine the derivative of each function from first principles.

a. $f(x) = 3x^2 + x + 1$

b. $f(x) = \frac{1}{x}$

10. Determine the derivative of each function.

a. $y = x^3 - 4x^2 + 5x + 2$

b. $y = \sqrt{2x^3+1}$

c. $y = \frac{2x}{x+3}$

d. $y = (x^2+3)^2(4x^5+5x+1)$

e. $y = \frac{(4x^2+1)^5}{(3x-2)^3}$

f. $y = [x^2 + (2x+1)^3]^5$

11. Determine the equation of the tangent to $y = \frac{18}{(x+2)^2}$ at the point $(1, 2)$.

12. Determine the slope of the tangent to $y = x^2 + 9x + 9$ at the point where the curve intersects the line $y = 3x$.

13. In 1980, the population of Littletown, Ontario, was 1100. After a time t, in years, the population was given by $p(t) = 2t^2 + 6t + 1100$.

a. Determine $p'(t)$, the function that describes the rate of change of the population at time t.

b. Determine the rate of change of the population at the start of 1990.

c. At the beginning of what year was the rate of change of the population 110 people per year?

14. Determine f' and f'' for each function.

a. $f(x) = x^5 - 5x^3 + x + 12$

b. $f(x) = \frac{-2}{x^2}$

c. $f(x) = \frac{4}{\sqrt{x}}$

d. $f(x) = x^4 - \frac{1}{x^4}$

15. Determine the extreme values of each function on the given interval.

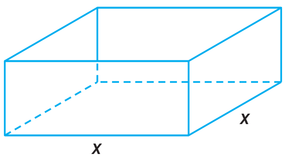
a. $f(x) = 1 + (x+3)^2, -2 \le x \le 6$

b. $f(x) = x + \frac{1}{\sqrt{x}}, 1 \le x \le 9$

c. $f(x) = \frac{e^x}{1+e^x}, x \in [0, 4]$

d. $f(x) = 2\sin 4x + 3, x \in [0, \pi]$

16. The position, at time t, in seconds, of an object moving along a line is given by $s(t) = 3t^3 - 40.5t^2 + 162t$ for $0 \leq t \leq 8$.
    a. Determine the velocity and the acceleration at any time t.
    b. When is the object stationary? When is it advancing? When is it retreating?
    c. At what time, t, is the velocity not changing?
    d. At what time, t, is the velocity decreasing?
    e. At what time, t, is the velocity increasing?
17. A farmer has 750 m of fencing. The farmer wants to enclose a rectangular area on all four sides, and then divide it into four pens of equal size with the fencing parallel to one side of the rectangle. What is the largest possible area of each of the four pens?
18. A cylindrical metal can is made to hold 500 mL of soup. Determine the dimensions of the can that will minimize the amount of metal required. (Assume that the top and sides of the can are made from metal of the same thickness.)
19. A cylindrical container, with a volume of 4000 cm^3, is being constructed to hold candies. The cost of the base and lid is $0.005/$cm^2$, and the cost of the side walls is $0.0025/$cm^2$. Determine the dimensions of the cheapest possible container.
20. An open rectangular box has a square base, with each side measuring x centimetres.



    a. If the length, width, and depth have a sum of 140 cm, find the depth in terms of x.
    b. Determine the maximum possible volume you could have when constructing a box with these specifications. Then determine the dimensions that produce this maximum volume.
21. The price of x MP3 players is $p(x) = 50 - x^2$, where $x \in \mathbf{N}$. If the total revenue, $R(x)$, is given by $R(x) = xp(x)$, determine the value of x that corresponds to the maximum possible total revenue.
22. An express railroad train between two cities carries 10 000 passengers per year for a one-way fare of $50. If the fare goes up, ridership will decrease because more people will drive. It is estimated that each $10 increase in the fare will result in 1000 fewer passengers per year. What fare will maximize revenue?
23. A travel agent currently has 80 people signed up for a tour. The price of a ticket is $5000 per person. The agency has chartered a plane seating 150 people at a cost of $250 000. Additional costs to the agency are incidental fees of $300 per person. For each $30 that the price is lowered, one new person will sign up. How much should the price per person be lowered to maximize the profit for the agency?

24. For each function, determine the derivative, all the critical numbers, and the intervals of increase and decrease.

a. $y = -5x^2 + 20x + 2$

b. $y = 6x^2 + 16x - 40$

c. $y = 2x^3 - 24x$

d. $y = \dfrac{x}{x - 2}$

25. For each of the following, determine the equations of any horizontal, vertical, or oblique asymptotes and all local extrema:

a. $y = \dfrac{8}{x^2 - 9}$

b. $y = \dfrac{4x^3}{x^2 - 1}$

26. Use the algorithm for curve sketching to sketch the graph of each function.

a. $f(x) = 4x^3 + 6x^2 - 24x - 2$

b. $y = \dfrac{3x}{x^2 - 4}$

27. Determine the derivative of each function.

a. $f(x) = (-4)e^{5x+1}$

b. $f(x) = xe^{3x}$

c. $y = 6^{3x-8}$

d. $y = e^{\sin x}$

28. Determine the equation of the tangent to the curve $y = e^{2x-1}$ at $x = 1$.

29. In a research laboratory, a dish of bacteria is infected with a particular disease. The equation $N(d) = (15d)e^{-\frac{d}{5}}$ models the number of bacteria, N, that will be infected after d days.

a. How many days will pass before the maximum number of bacteria will be infected?

b. Determine the maximum number of bacteria that will be infected.

30. Determine the derivative of each function.

a. $y = 2 \sin x - 3 \cos 5x$

b. $y = (\sin 2x + 1)^4$

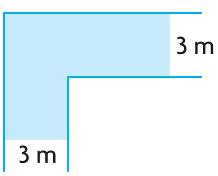
c. $y = \sqrt{x^2 + \sin 3x}$

d. $y = \dfrac{\sin x}{\cos x + 2}$

e. $y = \tan x^2 - \tan^2 x$

f. $y = \sin (\cos x^2)$

31. A tool shed, 250 cm high and 100 cm deep, is built against a wall. Calculate the shortest ladder that can reach from the ground, over the shed, to the wall behind.



32. A corridor that is 3 m wide makes a right-angle turn, as shown on the left. Find the longest rod that can be carried horizontally around this corner. Round your answer to the nearest tenth of a metre.

Chapter 6

INTRODUCTION TO VECTORS

Have you ever tried to swim across a river with a strong current or run into a head wind? Have you ever tried sailing across a windy lake? If your answer is yes, then you have experienced the effect of vector quantities. Vectors were developed in the late nineteenth century as mathematical tools for studying physics. In the following century, vectors became an essential tool for anyone using mathematics including social sciences. In order to navigate, pilots need to know what effect a crosswind will have on the direction in which they intend to fly. In order to build bridges, engineers need to know what load a particular design will support. In this chapter, you will learn more about vectors and how they represent quantities possessing both magnitude and direction.

CHAPTER EXPECTATIONS

In this chapter, you will

- represent vectors as directed line segments, **Section 6.1**
- recognize a vector as a quantity with both magnitude and direction, **Section 6.1**
- perform mathematical operations on geometric vectors, **Sections 6.2, 6.3**
- determine some properties of the operations performed on vectors, **Section 6.4**
- determine the Cartesian representation of a vector in two- and three-dimensional space, **Sections 6.5, 6.6, 6.7, 6.8**
- perform mathematical operations on algebraic vectors in two- and three-dimensional space, **Sections 6.6, 6.7, 6.8**

Review of Prerequisite Skills

In this chapter, you will be introduced to the concept of a vector, a mathematical entity having both magnitude and direction. You will examine geometric and algebraic representations of vectors in two- and three-dimensional space. Before beginning this introduction to vectors, you may wish to review some basic facts of trigonometry.

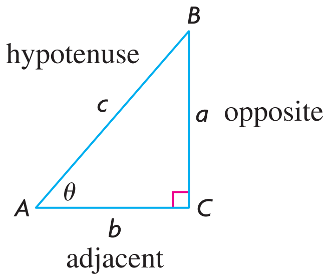
TRIGONOMETRIC RATIOS

In a right-angled triangle, as shown, $a^2 + b^2 = c^2$.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}$$

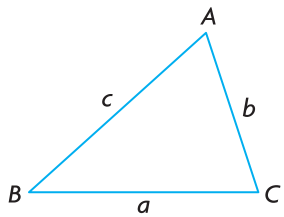
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b}$$



THE SINE LAW

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



THE COSINE LAW

$$a^2 = b^2 + c^2 - 2bc\cos A \quad \text{or} \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

SOLVING A TRIANGLE

- To solve a triangle means to find the measures of the sides and angles whose values are not given.
- Solving a triangle may require the use of trigonometric ratios, the Pythagorean theorem, the sine law, and/or the cosine law.

Exercise

1. State the exact value of each of the following:

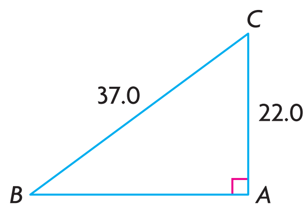
a. sin $60°$ c. cos $60°$ e. sin $135°$

b. tan $120°$ d. cos $30°$ f. tan $45°$

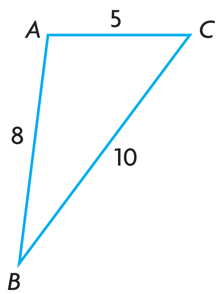
2. In $\triangle ABC$, $AB = 6$, $\angle B = 90°$, and $AC = 10$. State the exact value of tan A.

3. Solve $\triangle ABC$, to one decimal place.

a.



b.



4. In $\triangle XYZ$, $XY = 6$, $\angle X = 60°$, and $\angle Y = 70°$. Determine the values of XZ, YZ, and $\angle Z$, to two digit accuracy.

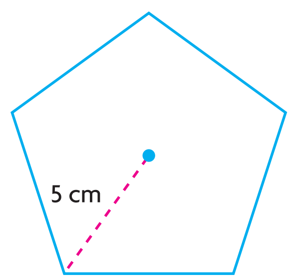
5. In $\triangle RST$, $RS = 4$, $RT = 7$, and $ST = 5$. Determine the measures of the angles to the nearest degree.

6. An aircraft control tower, T, is tracking two airplanes at points A, 3.5 km from T, and B, 6 km from T. If $\angle ATB = 70°$, determine the distance between the two airplanes to two decimal places.

7. Three ships are at points P, Q, and R such that $PQ = 2$ km, $PR = 7$ km, and $\angle QPR = 142°$. What is the distance between Q and R, to two decimal places.

8. Two roads intersect at an angle of 48º. A car and truck collide at the intersection, and then leave the scene of the accident. The car travels at 100 km/h down one road, while the truck goes 80 km/h down the other road. Fifteen minutes after the accident, a police helicopter locates the car and pulls it over. Twenty minutes after the accident, a police cruiser pulls over the truck. How far apart are the car and the truck at this time?

9. A regular pentagon has all sides equal and all central angles equal. Calculate, to the nearest tenth, the area of the pentagon shown.



CAREER LINK | Investigate

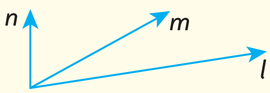
CHAPTER 6: FIGURE SKATING

Figure skaters are exceptional athletes and artists. Their motion while skating is also an illustration of the use of vectors. The ice they skate on is a nearly frictionless surface—so any force applied by the skater has a direct impact on speed, momentum, and direction. Vectors can be used to describe a figure skater's path on the ice. When the skater starts moving in a direction, she will continue moving in that direction and at that speed until she applies a force to change or stop her motion. This is more apparent with pairs figure skaters. To stay together, each skater must skate with close to the same speed as their partner in the same direction. If one skater uses less force or applies the force in a different direction, the skaters will either bump into each other or separate and fly away from each other. If they don't let go of each other, the opposing forces may cause them to spin.

Case Study—Throwing a Triple Salchow

The Triple Salchow throw is one of the more difficult moves in pairs figure skating. Both partners skate together in one direction with a lot of speed. Next, the male skater plants his feet to throw his partner and add his momentum to that of the female skater. She applies force with one skate to jump into the air. In order to make herself spin, she applies force at an angle to the initial direction and spins three times in the air before landing. There are three main vectors at work here. These vectors are the initial thrust of both skaters, the force the male skater applies to the female skater, and the vertical force of the jump.

Vector	Magnitude (size of the force)
Both skaters' initial thrust (l)	60
Female skater's change in direction to cause spin (m)	40
Female skater's vertical leap (n)	20



DISCUSSION QUESTIONS

1. What operation on vectors l and m do you think should be done to find the resulting thrust vector (a) for the female skater?
2. If we performed the same operation as in problem 1 to the vectors a and n, what would the resulting vector represent? You may assume that the angle between a and n is 90°.
3. Can you think of a three-dimensional figure that would represent all of the vectors l, m, and n at the same time, as well as the vectors found in the previous two problems? Give as complete a description as possible for this figure, including any properties you notice. For example, is it constructed from any familiar two-dimensional objects?

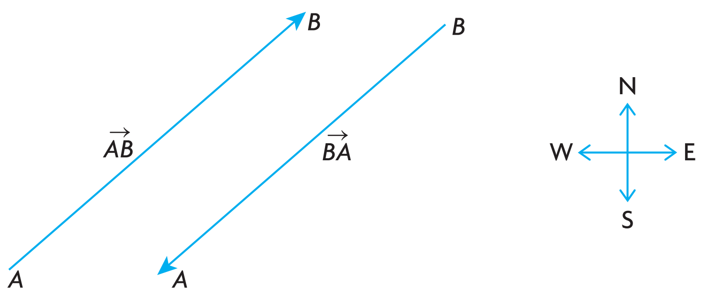
Section 6.1—An Introduction to Vectors

In mathematics and science, you often come in contact with different quantities. Some of these quantities, those whose **magnitude** (or size) can be completely specified by just one number, are called **scalars**. Some examples of scalars are age, volume, area, speed, mass, and temperature. On the other hand, some quantities (such as weight, velocity, or friction) require both a magnitude and a direction for a complete description and are called **vectors**.

Defining the Characteristics of Vectors

A vector can be represented by a directed line segment. A directed line segment has a length, called its magnitude, and a direction indicated by an arrowhead.

The diagram below can help to make the distinction between a vector and a scalar. If an airplane is travelling at a speed of 500 km/h, this description is useful, but for navigation and computational purposes, incomplete. If we add the fact that the airplane is travelling in a northeasterly direction, we now have a description of its velocity because we have specified both its speed and direction. This defines velocity as a vector quantity. If we refer to the speed of the airplane, we are describing it with just a single number, which defines speed as a scalar quantity.



Scale: 1 cm is equivalent to 100 km/h

In the diagram, $\overrightarrow{AB}$ is an example of a vector. In this case, it is a line segment running from A to B with its tail at A and head at B. Its actual size, or magnitude, is denoted by $|\overrightarrow{AB}|$. The magnitude of a vector is always non-negative. The vector $\overrightarrow{AB}$ could be used to represent the velocity of any airplane heading in a northeasterly direction at 500 km/h (using a scale of 1 cm to 100 km/h, i.e., $|\overrightarrow{AB}| = 5 \text{ cm} = 500 \text{ km/h}$). The direction of the "arrow" represents the direction of the airplane, and its length represents its speed.

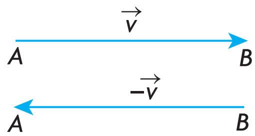
What is a Vector?

A vector is a mathematical quantity having both magnitude and direction.

In the diagram on the previous page, $\overrightarrow{BA}$ is a vector pointing from B to A. The vector $\overrightarrow{BA}$ represents an airplane travelling in a southwesterly direction at 500 km/h. Note that the magnitudes of the two vectors are equal, i.e., $|\overrightarrow{AB}| = |\overrightarrow{BA}|$, but that the vectors themselves are not equal because they point in opposite directions. For this reason, we describe these as **opposite vectors**.

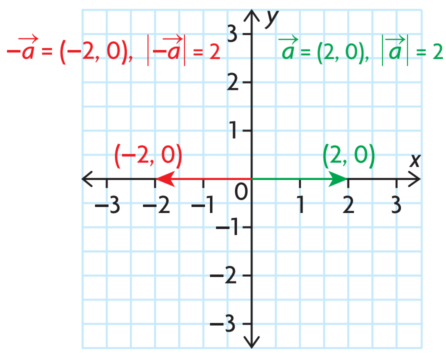
Opposite Vectors

Two vectors that are opposites have the same magnitude but point in opposite directions.



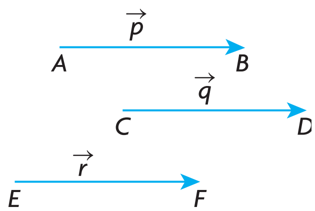
$\overrightarrow{AB}$ and $\overrightarrow{BA}$ are opposites, and $\overrightarrow{AB} = -\overrightarrow{BA}$. In this case, $|\overrightarrow{AB}| = |\overrightarrow{BA}|$ and the vectors are parallel but point in opposite directions. Vectors can also be represented with lower-case letters. In the diagram above, vectors $\vec{v}$ and $-\vec{v}$ have the same magnitude, i.e., $|\vec{v}| = |-\vec{v}|$, but point in opposite directions, so $\vec{v}$ and $-\vec{v}$ are also opposites.

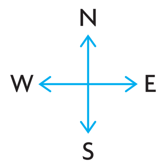
No mention has yet been made of using coordinate systems to represent vectors. In the diagram below, it is helpful to note that $\vec{a} = (2, 0)$ is a vector having its tail at the origin and head at $(2, 0)$; this vector has magnitude 2, i.e., $|\vec{a}| = 2$. Also, observe that $-\vec{a} = (-2, 0)$ and $|-\vec{a}| = 2$. The vectors $\vec{a}$ and $-\vec{a}$ are opposites.



It is not always appropriate or necessary to describe a quantity by both a magnitude and direction. For example, the description of the area of a square or rectangle does not require a direction. In referring to a person's age, it is clear what is meant by just the number. By their nature, quantities of this type do not have a direction associated with them and, thus, are not vectors.

Vectors are equal, or equivalent, if they have the same direction and the same magnitude. This means that the velocity vector for an airplane travelling in an easterly direction at 400 km/h could be represented by any of the three vectors in the following diagram.



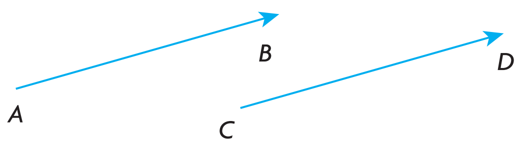


Scale: 1 cm is equivalent to 100 km/h

Notice that any one of these vectors could be translated to be **coincident** with either of the other two. (When vectors are translated, it means they are picked up and moved without changing either their direction or size.) This implies that the velocity vector of an airplane travelling at 400 km/h in an easterly direction from Calgary is identical to that of an airplane travelling at 400 km/h in an easterly direction from Toronto.

Note that, in the diagram above, we have also used lower-case letters to represent the three vectors. It is convenient to write the vector $\overrightarrow{AB}$ as $\vec{p}$, for example, and in this case $\vec{p} = \vec{q} = \vec{r}$.

Equal Vectors



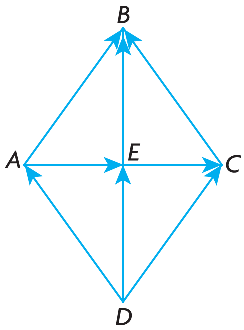
Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal (or equivalent) if and only if

1. $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel to each other, and the direction from A to B is the same as the direction from C to D
2. the magnitude of $\overrightarrow{AB}$ equals the magnitude of $\overrightarrow{CD}$, i.e., $|\overrightarrow{AB}| = |\overrightarrow{CD}|$

EXAMPLE 1 Connecting vectors to two-dimensional figures

Rhombus $ABCD$ is drawn and its two diagonals AC and BD are drawn as shown. Name vectors equal to each of the following.

a. $\overrightarrow{AB}$ b. $\overrightarrow{DA}$ c. $\overrightarrow{EB}$ d. $\overrightarrow{AE}$



Solution

A rhombus is a parallelogram with its opposite sides parallel and the four sides equal in length. Thus, $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{DA} = \overrightarrow{CB}$ and $|\overrightarrow{AB}| = |\overrightarrow{DC}| = |\overrightarrow{DA}| = |\overrightarrow{CB}|$. Note that $\overrightarrow{AB} \neq \overrightarrow{DA}$ because these vectors have different directions, even though they have equal magnitudes, i.e., $|\overrightarrow{AB}| = |\overrightarrow{DA}|$.

Since the diagonals in a rhombus bisect each other, $\overrightarrow{AE} = \overrightarrow{EC}$ and $\overrightarrow{EB} = \overrightarrow{DE}$. Note also that, if the arrow had been drawn from C to D instead of from D to C, the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ would be opposites and would not be equal, even though they are of the same length. If these vectors are opposites, then the relationship between them can be expressed as $\overrightarrow{AB} = -\overrightarrow{CD}$. This implies that these vectors have the same magnitude but opposite directions.

In summary: a. $\overrightarrow{AB} = \overrightarrow{DC}$ b. $\overrightarrow{DA} = \overrightarrow{CB}$ c. $\overrightarrow{EB} = \overrightarrow{DE}$ d. $\overrightarrow{AE} = \overrightarrow{EC}$

In our discussion of vectors thus far, we have illustrated our ideas with **geometric vectors**. Geometric vectors are those that are considered without reference to coordinate axes. The ability to use vectors in applications usually requires us to place them on a coordinate plane. These are referred to as **algebraic vectors**; they will be introduced in the exercises and examined in detail in Section 6.5. Algebraic vectors will become increasingly important in our work.

IN SUMMARY

Key Ideas

- A vector is a mathematical quantity having both magnitude and direction, for example velocity.
- A scalar is a mathematical quantity having only magnitude, for example, speed.

Need to Know

- $\overrightarrow{AB}$ represents a vector running from A to B, with its tail at A and head at B.
- $|\overrightarrow{AB}|$ represents the magnitude of a vector and is always non-negative.
- Two vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are opposite if they are parallel and have the same magnitude but opposite directions. It follows that $|\overrightarrow{AB}| = |\overrightarrow{BA}|$ and $\overrightarrow{AB} = -\overrightarrow{BA}$.
- Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal if they are parallel and have the same magnitude and the same direction. It follows that $|\overrightarrow{AB}| = |\overrightarrow{CD}|$ and $\overrightarrow{AB} = \overrightarrow{CD}$.

Exercise 6.1

PART A

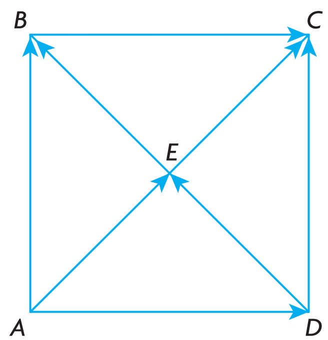
1. State whether each statement is true or false. Justify your decision.
   a. If two vectors have the same magnitude, then they are equal.
   b. If two vectors are equal, then they have the same magnitude.
   c. If two vectors are parallel, then they are either equal or opposite vectors.
   d. If two vectors have the same magnitude, then they are either equal or opposite vectors.

2. For each of the following, state whether the quantity is a scalar or a vector and give a brief explanation why: height, temperature, weight, mass, area, volume, distance, displacement, speed, force, and velocity.

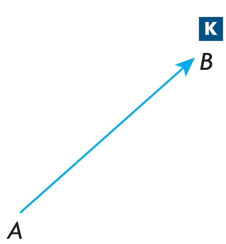
3. Friction is considered to be a vector because friction can be described as the force of resistance between two surfaces in contact. Give two examples of friction from everyday life, and explain why they can be described as vectors.

PART B

4. Square *ABCD* is drawn as shown below with the diagonals intersecting at *E*.



   a. State four pairs of equivalent vectors.
   b. State four pairs of opposite vectors.
   c. State two pairs of vectors whose magnitudes are equal but whose directions are perpendicular to each other.



K 5. Given the vector $\overrightarrow{AB}$ as shown, draw a vector

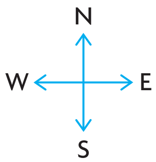
a. equal to $\overrightarrow{AB}$

b. opposite to $\overrightarrow{AB}$

c. whose magnitude equals $|\overrightarrow{AB}|$ but is not equal to $\overrightarrow{AB}$

d. whose magnitude is twice that of $\overrightarrow{AB}$ and in the same direction

e. whose magnitude is half that of $\overrightarrow{AB}$ and in the opposite direction



6. Using a scale of 1 cm to represent 10 km/h, draw a velocity vector to represent each of the following:

a. a bicyclist heading due north at 40 km/h

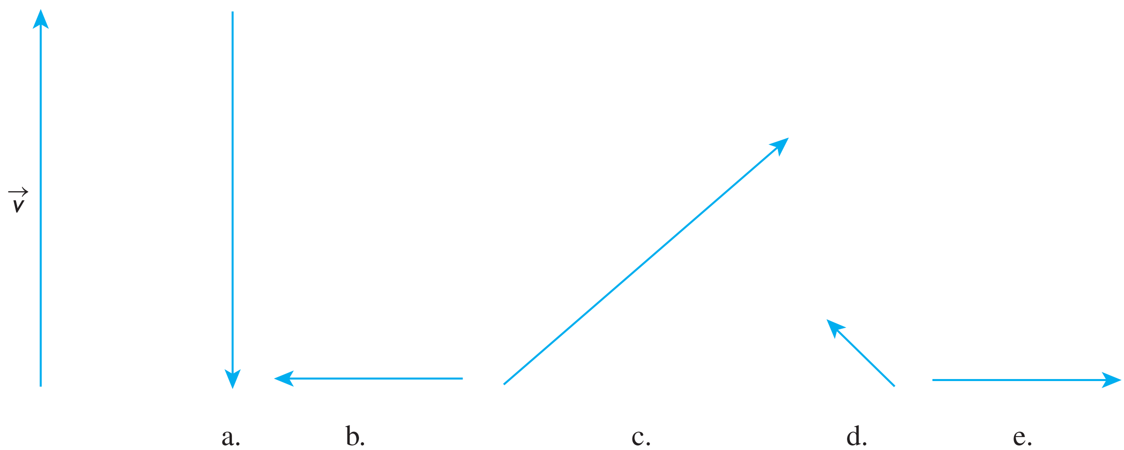
b. a car heading in a southwesterly direction at 60 km/h

c. a car travelling in a northeasterly direction at 100 km/h

d. a boy running in a northwesterly direction at 30 km/h

e. a girl running around a circular track travelling at 15 km/h heading due east

7. The vector shown, $\vec{v}$, represents the velocity of a car heading due north at 100 km/h. Give possible interpretations for each of the other vectors shown.



C 8. For each of the following vectors, describe the opposite vector.

a. an airplane flies due north at 400 km/h

b. a car travels in a northeasterly direction at 70 km/h

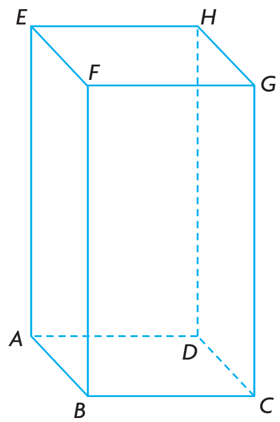
c. a bicyclist pedals in a northwesterly direction at 30 km/h

d. a boat travels due west at 25 km/h

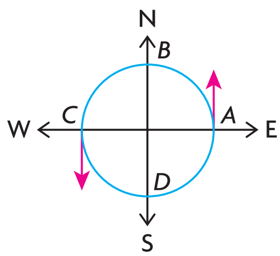
T 9. a. Given the square-based prism shown where $AB = 3$ cm and $AE = 8$ cm, state whether each statement is true or false. Explain.

i) $\overrightarrow{AB} = \overrightarrow{GH}$ ii) $|\overrightarrow{EA}| = |\overrightarrow{CG}|$ iii) $|\overrightarrow{AD}| = |\overrightarrow{DC}|$ iv) $\overrightarrow{AH} = \overrightarrow{BG}$

b. Calculate the magnitude of $\overrightarrow{BD}$, $\overrightarrow{BE}$, and $\overrightarrow{BH}$.



A 10. James is running around a circular track with a circumference of 1 km at a constant speed of 15 km/h. His velocity vector is represented by a vector tangent to the circle. Velocity vectors are drawn at points A and C as shown. As James changes his position on the track, his velocity vector changes.



a. Explain why James's velocity can be represented by a vector tangent to the circle.

b. What does the length of the vector represent?

c. As he completes a lap running at a constant speed, explain why James's velocity is different at every point on the circle.

d. Determine the point on the circle where James is heading due south.

e. In running his first lap, there is a point at which James is travelling in a northeasterly direction. If he starts at point A how long would it have taken him to get to this point?

f. At the point he has travelled $\frac{3}{8}$ of a lap, in what direction would James be heading? Assume he starts at point A.

PART C

11. $\overrightarrow{AB}$ is a vector whose tail is at $(-4, 2)$ and whose head is at $(-1, 3)$.

a. Calculate the magnitude of $\overrightarrow{AB}$.

b. Determine the coordinates of point D on vector $\overrightarrow{CD}$, if $C(-6, 0)$ and $\overrightarrow{CD} = \overrightarrow{AB}$.

c. Determine the coordinates of point E on vector $\overrightarrow{EF}$, if $F(3, -2)$ and $\overrightarrow{EF} = \overrightarrow{AB}$.

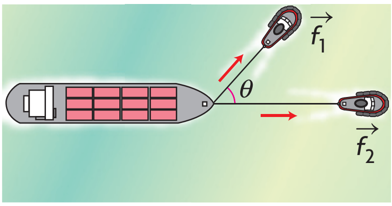
d. Determine the coordinates of point G on vector $\overrightarrow{GH}$, if $G(3, 1)$ and $\overrightarrow{GH} = -\overrightarrow{AB}$.

Section 6.2—Vector Addition

In this section, we will examine ways that vectors can be used in different physical situations. We will consider a variety of contexts and use them to help develop rules for the application of vectors.

Examining Vector Addition

Suppose that a cargo ship has a mechanical problem and must be towed into port by two tugboats. This situation is represented in the following diagram.



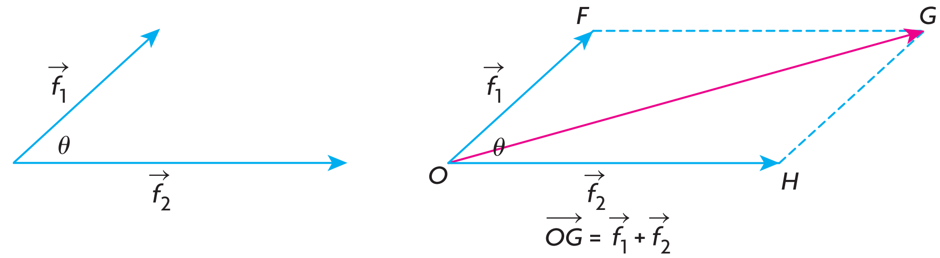
The force exerted by the first tugboat is denoted by $\vec{f_1}$ and that of the second tugboat as $\vec{f_2}$. They are denoted as vectors because these forces have both magnitude and direction. θ is the angle between the two forces shown in the diagram, where the vectors are placed tail to tail.

In considering this situation, a number of assumptions have been made:

1. The direction of the force exerted by each of the tugboats is indicated by the direction of the arrows.
2. The magnitude of the force exerted by each of the two tugboats is proportional to the length of the corresponding force vector. This means that the longer the force vector, the greater the exerted force.
3. The forces that have been exerted have been applied at a common point on the ship.

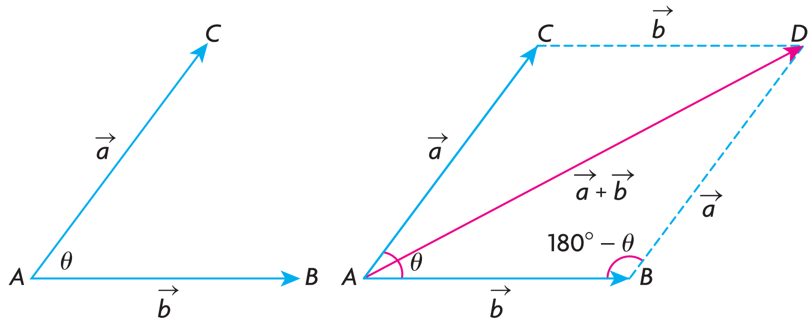
What we want to know is whether we can predict the direction the ship will move and with what force. Intuitively, we know that the ship will move in a direction somewhere between the direction of the forces, but because $|\vec{f_2}| > |\vec{f_1}|$ (the magnitude of the second force is greater than that of the first force), the boat should move closer to the direction of $\vec{f_2}$ rather than $\vec{f_1}$. The combined magnitude of the two forces should be greater than either of $|\vec{f_1}|$ or $|\vec{f_2}|$ but not equal to their sum, because they are pulling at an angle of θ to each other, i.e. they are not pulling in exactly the same direction.
There are several other observations to be made in this situation. The actions of the two tug boats are going to pull the ship in a way that combines the force vectors. The ship is going to be towed in a constant direction with a certain force, which, in effect, means the two smaller force vectors can be replaced with just

one vector. To find this single vector to replace $\vec{f_1}$ and $\vec{f_2}$, the parallelogram determined by these vectors is constructed. The main diagonal of the parallelogram is called the **resultant** or sum of these two vectors and represents the combined effect of the two vectors. The resultant of $\vec{f_1}$ and $\vec{f_2}$ has been shown in the following diagram as the diagonal, $\overrightarrow{OG}$, of the parallelogram.



The length (or magnitude) of each vector representing a force is proportional to the actual force exerted. After the tugboats exert their forces, the ship will head in the direction of $\overrightarrow{OG}$ with a force proportional to the length of $\overrightarrow{OG}$.

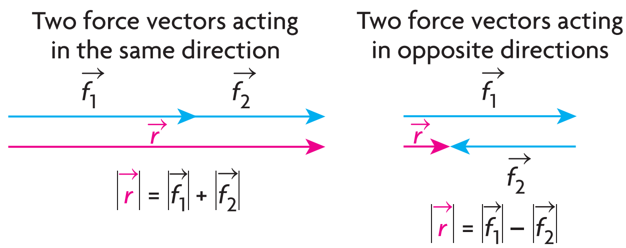
The Parallelogram Law for Adding Two Vectors



To determine the sum of the two vectors $\vec{a}$ and $\vec{b}$, complete the parallelogram formed by these two vectors when placed tail to tail. Their sum is the vector $\overrightarrow{AD}$, the diagonal of the constructed parallelogram,
$\vec{a} + \vec{b} = \overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$.

Consider the triangle formed by vectors $\vec{a}$, $\vec{b}$ and $\vec{a} + \vec{b}$. It is important to note that $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$. This means that the magnitude of the sum $\vec{a} + \vec{b}$ is less than or equal to the combined magnitudes of $\vec{a}$ and $\vec{b}$. The magnitude of $\vec{a} + \vec{b}$ is equal to the sum of the magnitudes of $\vec{a}$ and $\vec{b}$ only when these three vectors lie in the same direction.

In the tugboat example, this means the overall effect of the two tugboats is less than the sum of their individual efforts. If the tugs pulled in the same direction, the overall magnitude would be equal to the sum of their individual magnitudes. If they pulled in opposite directions, the overall magnitude would be their difference.

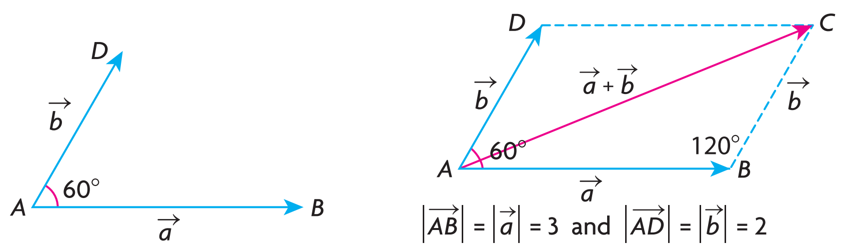


EXAMPLE 1 Selecting a strategy to determine the magnitude of a resultant vector

Given vectors $\vec{a}$ and $\vec{b}$ such that the angle between the two vectors is 60°, $|\vec{a}| = 3$, and $|\vec{b}| = 2$, determine $|\vec{a} + \vec{b}|$.

Solution

If it is stated that the angle between the vectors is θ, this means that the vectors are placed tail to tail and the angle between the vectors is θ. In this problem, the angle between the vectors is given to be 60°, so the vectors are placed tail to tail as shown.



To calculate the value of $|\vec{a} + \vec{b}|$, draw the diagonal of the related parallelogram. From the diagram, $\vec{AB} + \vec{BC} = \vec{a} + \vec{b} = \vec{AC}$. Note that the angle between $\vec{AB}$ and $\vec{BC}$ is 120°, the supplement of 60°.

Now, $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos(\angle ABC)$ (Cosine law)

$|\vec{a} + \vec{b}|^2 = 3^2 + 2^2 - 2(3)(2)\cos 120°$ (Substitution)

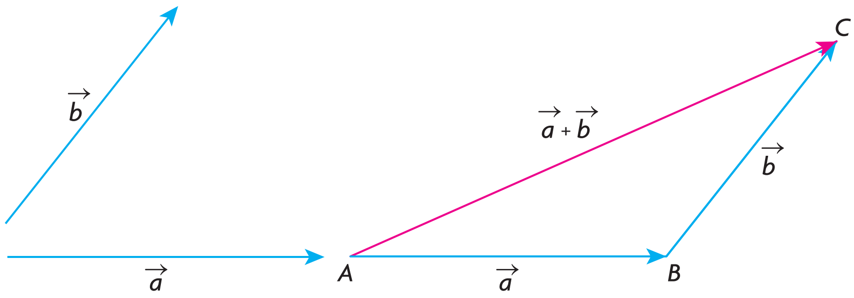
$$|\vec{a} + \vec{b}|^2 = 13 - 2(3)(2)\left(\frac{-1}{2}\right)$$

$$|\vec{a} + \vec{b}|^2 = 19$$

Therefore, $|\vec{a} + \vec{b}| = \sqrt{19} \doteq 4.36$.

When finding the sum of two or more vectors, it is not necessary to draw a parallelogram each time. In the following, we show how to add vectors using the triangle law of addition.

Triangle Law of Addition

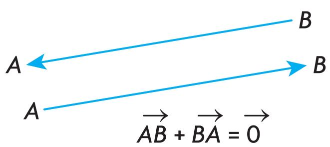


In the diagram, the sum of the vectors $\vec{a}$ and $\vec{b}$, $\vec{a} + \vec{b}$, is found by translating the tail of vector $\vec{b}$ to the head of vector $\vec{a}$. This could also have been done by translating $\vec{a}$ so that its tail was at the head of $\vec{b}$. In either case, the sum of the vectors $\vec{a}$ and $\vec{b}$ is $\overrightarrow{AC}$.

A way of thinking about the sum of two vectors is to imagine that, if you start at point A and walk to point B and then to C, you end up in the exact location as if you walked directly from point A to C. Thus, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

The Zero Vector

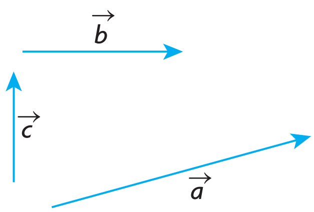
An observation that comes directly from the triangle law of addition is that when two opposite vectors are added, the resultant is the zero vector. This means that the combined effect of a vector and its opposite is the zero vector. In symbols, $\overrightarrow{AB} + \overrightarrow{BA} = \vec{0}$.



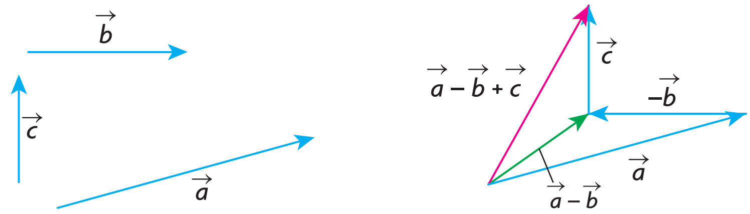
The zero vector has a magnitude of 0, i.e., $|\vec{0}| = 0$, and no defined direction.

EXAMPLE 2 Representing a combination of three vectors using the triangle law of addition

Suppose you are given the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ as shown below. Using these three vectors, sketch $\vec{a} - \vec{b} + \vec{c}$.

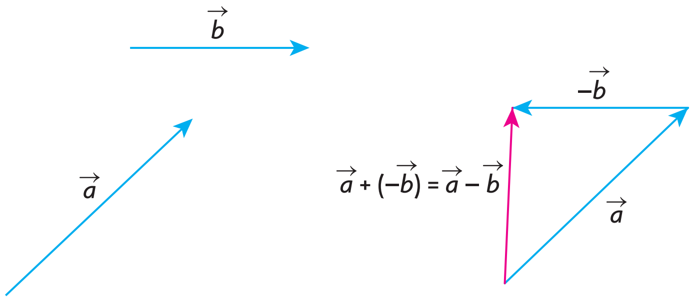


Solution



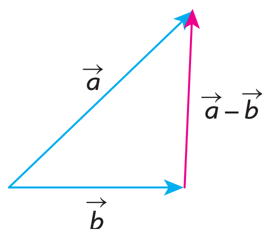
To draw the required vector, first draw $-\vec{b}$, the opposite of $\vec{b}$, and then place the vectors head to tail as shown. It should be emphasized that $\vec{a} - \vec{b}$ actually means $\vec{a} + (-\vec{b})$. Note that the required resultant vector $\vec{a} - \vec{b} + \vec{c}$ is also the resultant vector of $(\vec{a} - \vec{b}) + \vec{c}$ by the triangle law of addition.

The Difference of Two Vectors, $\vec{a} - \vec{b}$



In the diagram above, the difference between vectors $\vec{a}$ and $\vec{b}$ is found by adding the opposite vector $\vec{b}$ to $\vec{a}$ using the triangle law of addition.

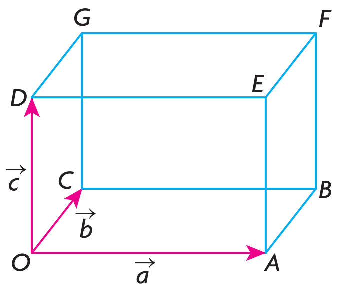
Another way to think about $\vec{a} - \vec{b}$ is to arrange the vectors tail to tail. In this case, $\vec{a} - \vec{b}$ is the vector that must be added to $\vec{b}$ to get $\vec{a}$. This is illustrated in the following diagram. Using the vectors above, the difference vector is the same as the one produced by adding the opposite.



The concept of addition and subtraction is applied in Example 3.

EXAMPLE 3 Representing a single vector as a combination of vectors

In the rectangular box shown below, $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OC} = \vec{b}$, and $\overrightarrow{OD} = \vec{c}$.



Express each of the following vectors in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$.

a. $\overrightarrow{BC}$ b. $\overrightarrow{GF}$ c. $\overrightarrow{OB}$ d. $\overrightarrow{AC}$ e. $\overrightarrow{BG}$ f. $\overrightarrow{OF}$

Solution

a. $\overrightarrow{BC}$ is the opposite of $\vec{a}$, so $\overrightarrow{BC} = -\vec{a}$.

b. $\overrightarrow{GF}$ is the same as $\vec{a}$, so $\overrightarrow{GF} = \vec{a}$.

c. In rectangle $OABC$, $\overrightarrow{OB}$ is the diagonal of the rectangle, so $\overrightarrow{OB} = \vec{a} + \vec{b}$.

d. Since $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$ and $\overrightarrow{AO} = -\vec{a}$, $\overrightarrow{AC} = -\vec{a} + \vec{b}$ or $\overrightarrow{AC} = \vec{b} - \vec{a}$.

e. Since $\overrightarrow{BG} = \overrightarrow{BC} + \overrightarrow{CG}$, $\overrightarrow{BC} = -\vec{a}$, and $\overrightarrow{CG} = \vec{c}$, $\overrightarrow{BG} = -\vec{a} + \vec{c}$ or $\overrightarrow{BG} = \vec{c} - \vec{a}$.

f. Since $\overrightarrow{OF} = \overrightarrow{OB} + \overrightarrow{BF}$, $\overrightarrow{OB} = \vec{a} + \vec{b}$, and $\overrightarrow{BF} = \vec{c}$, $\overrightarrow{OF} = (\vec{a} + \vec{b}) + \vec{c} = \vec{a} + \vec{b} + \vec{c}$.

In the next example, we demonstrate how vectors might be used in a situation involving velocity.

EXAMPLE 4 Solving a problem using vectors

An airplane heads due south at a speed of 300 km/h and meets a wind from the west at 100 km/h. What is the resultant velocity of the airplane (relative to the ground)?

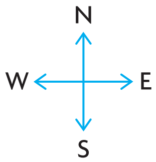
Solution

Let $\vec{v}$ represent the air speed of the airplane (velocity of the airplane without the wind).

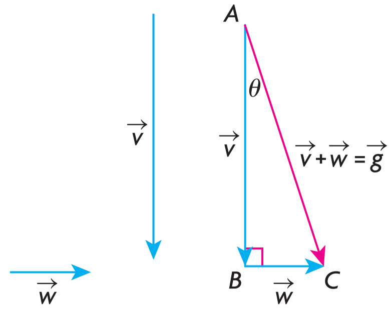
Let $\vec{w}$ represent the velocity of the wind.

Let $\vec{g}$ represent the ground speed of the airplane (the resultant velocity of the airplane with the wind taken into account relative to a fixed point on the ground).

The vectors are drawn so that their lengths are proportionate to their speed. That is to say, $|\vec{v}| = 3|\vec{w}|$.



In order to calculate the resultant ground velocity, the vectors are arranged head-to-tail as shown. The resultant ground velocity, $\vec{v} + \vec{w}$, is the hypotenuse of a right triangle.



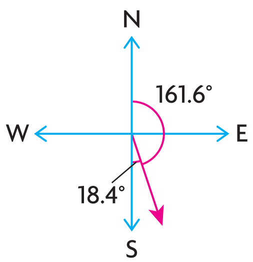
So, $|\vec{g}|^2 = |\vec{v} + \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2$

$$|\vec{v} + \vec{w}|^2 = 300^2 + 100^2$$
$$|\vec{v} + \vec{w}|^2 = 100\,000$$
$$|\vec{v} + \vec{w}| = \sqrt{100\,000}$$
$$\doteq 316.23$$

Since we are calculating the resultant ground velocity, we must also determine the new direction of the airplane. To do so, we must determine θ.

Thus, $\tan\theta = \dfrac{|\vec{w}|}{|\vec{v}|} = \dfrac{100}{300} = \dfrac{1}{3}$ and $\theta = \tan^{-1}\left(\dfrac{1}{3}\right) \doteq 18.4°$.

This means that the airplane is heading S18.4°E at a speed of 316.23 km/h. The wind has not only thrown the airplane off course, but it has also caused it to speed up. When we say the new direction of the airplane is S18.4°E, this means that the airplane is travelling in a south direction, 18.4° toward the east. This is illustrated in the following diagram.



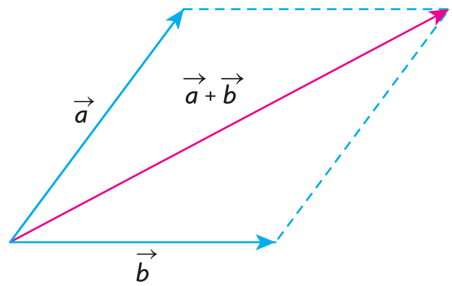
Other ways of stating this would be E71.6°S or a **bearing** of 161.6° (i.e., 161.6° rotated clockwise from due North).

In calculating the velocity of an object, such as an airplane, the velocity must always be calculated relative to some fixed object or some frame of reference. For example, if you are walking forward in an airplane at 5 km/h, your velocity relative to the airplane is 5 km/h in the same direction as the airplane, but relative to the ground, your velocity is 5 km/h in the same direction as the airplane plus the velocity of the airplane relative to the ground. If the airplane is travelling forward at 800 km/h, then your velocity relative to the ground is 805 km/h in the same direction as the airplane. In our example, the velocities given are measured relative to the ground, as is the final velocity. This is often referred to as the ground velocity.

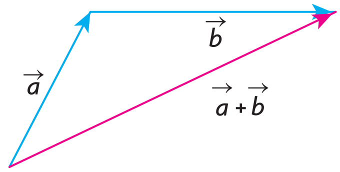
IN SUMMARY

Key Ideas

- To determine the sum of any two vectors $\vec{a}$ and $\vec{b}$, arranged tail-to-tail, complete the parallelogram formed by the two vectors. Their sum is the vector that is the diagonal of the constructed parallelogram.

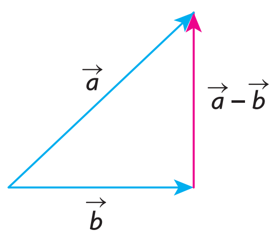


- The sum of the vectors $\vec{a}$ and $\vec{b}$ is also found by translating the tail of vector $\vec{b}$ to the head of vector $\vec{a}$. The resultant is the vector from the tail of $\vec{a}$ to the head of $\vec{b}$.



Need to Know

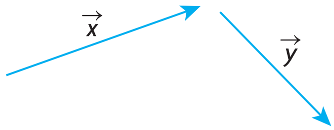
- When two opposite vectors are added, the resultant is the zero vector.
- The zero vector has a magnitude of 0 and no defined direction.
- To think about $\vec{a} - \vec{b}$, arrange the vectors tail to tail. $\vec{a} - \vec{b}$ is the vector that must be added to $\vec{b}$ to get $\vec{a}$. This is the vector from the head of $\vec{b}$ to the head of $\vec{a}$. This vector is also equivalent to $\vec{a} + \left(-\vec{b}\right)$.



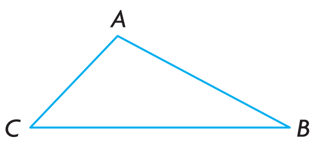
Exercise 6.2

PART A

1. The vectors $\vec{x}$ and $\vec{y}$ are drawn as shown below. Draw a vector equivalent to each of the following.



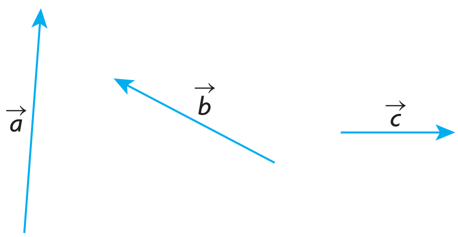
a. $\vec{x} + \vec{y}$ b. $\vec{x} - \vec{y}$ c. $\vec{y} - \vec{x}$ d. $(-\vec{y}) + (-\vec{x})$



2. Given $\triangle ABC$, draw appropriate arrows on the sides of the triangle and give a single vector equivalent for each of the following:

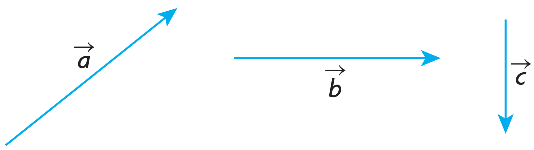
a. $\overrightarrow{BC} + \overrightarrow{CA}$ b. $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$ c. $\overrightarrow{AB} - \overrightarrow{AC}$ d. $-\overrightarrow{BC} + \overrightarrow{BA}$

3. Given the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, construct vectors equivalent to each of the following.



a. $\vec{a} + \vec{b} + \vec{c}$ b. $\vec{a} + \vec{b} - \vec{c}$ c. $\vec{a} - \vec{b} - \vec{c}$ d. $\vec{a} - \vec{b} + \vec{c}$

4. Vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are as shown.



a. Construct $\vec{a} + \left(\vec{b} + \vec{c}\right)$.
b. Construct $\left(\vec{a} + \vec{b}\right) + \vec{c}$.
c. Compare your results from parts a. and b.

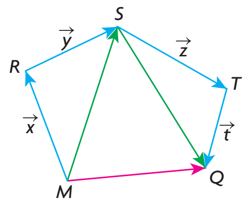
5. Each of the following vector expressions can be simplified and written as a single vector. Write the single vector corresponding to each expression and illustrate your answer with a sketch.

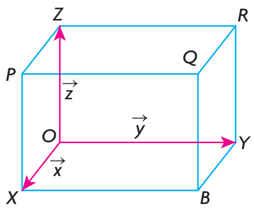
   a. $\overrightarrow{PQ} - \overrightarrow{RQ} + \overrightarrow{RS}$

   b. $\overrightarrow{PS} + \overrightarrow{RQ} - \overrightarrow{RS} - \overrightarrow{PQ}$

C

6. Explain why $(\vec{x} + \vec{y}) + (\vec{z} + \vec{t})$ equals $\overrightarrow{MQ}$ in the following diagram.



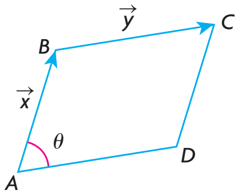


7. The rectangular box shown is labelled with $\overrightarrow{OX} = \vec{x}$, $\overrightarrow{OY} = \vec{y}$, and $\overrightarrow{OZ} = \vec{z}$.

   Express each of the following vectors in terms of $\vec{x}$, $\vec{y}$, and $\vec{z}$.

   a. $\overrightarrow{BY}$ b. $\overrightarrow{XB}$ c. $\overrightarrow{OB}$ d. $\overrightarrow{XY}$ e. $\overrightarrow{OQ}$ f. $\overrightarrow{QZ}$ g. $\overrightarrow{XR}$ h. $\overrightarrow{PO}$

8. In the diagram, $\vec{x}$ and $\vec{y}$ represent adjacent sides of a parallelogram.



   a. Draw vectors that are equivalent to $\vec{x} - \vec{y}$ and $\vec{y} - \vec{x}$.

   b. To calculate $|\vec{x} - \vec{y}|$, the formula $|\vec{x} - \vec{y}|^2 = |\vec{x}|^2 + |\vec{y}|^2 - 2|\vec{x}||\vec{y}|\cos\theta$ is used. Show, by drawing the vector $\vec{y} - \vec{x}$, that the formula for calculating $|\vec{y} - \vec{x}|$ is the same.

PART B

9. In still water, Maria can paddle at the rate of 7 km/h. The current in which she paddles has a speed of 4 km/h.

   a. At what velocity does she travel downstream?

   b. Using vectors, draw a diagram that illustrates her velocity going downstream.

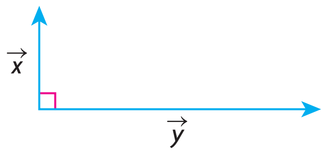
   c. If Maria changes her direction and heads upstream instead, what is her speed? Using vectors, draw a diagram that illustrates her velocity going upstream.

10. a. In the example involving a ship being towed by the two tugboats, draw $\vec{f_1}, \vec{f_2}, \theta$, and $\vec{f_1} + \vec{f_2}$.

b. Show that $|\vec{f_1} + \vec{f_2}| = \sqrt{|\vec{f_1}|^2 + |\vec{f_2}| + 2|\vec{f_1}||\vec{f_2}|\cos\theta}$.

A 11. A small airplane is flying due north at 150 km/h when it encounters a wind of 80 km/h from the east. What is the resultant ground velocity of the airplane?

K 12. $|\vec{x}| = 7$ and $|\vec{y}| = 24$. If the angle between these vectors is 90°, determine $|\vec{x} + \vec{y}|$ and calculate the angle between $\vec{x}$ and $\vec{x} + \vec{y}$.

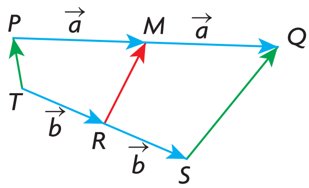


13. $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are two unit vectors (vectors with magnitude 1) with an angle of 150° between them. Calculate $|\overrightarrow{AB} + \overrightarrow{AC}|$.

T 14. $ABCD$ is a parallelogram whose diagonals BD and AC meet at the point E. Prove that $\overrightarrow{EA} + \overrightarrow{EB} + \overrightarrow{EC} + \overrightarrow{ED} = \vec{0}$.

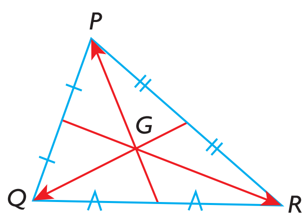
PART C

15. M is the midpoint of line segment PQ, and R is the midpoint of TS. If $\overrightarrow{PM} = \overrightarrow{MQ} = \vec{a}$ and $\overrightarrow{TR} = \overrightarrow{RS} = \vec{b}$, as shown, prove that $2\overrightarrow{RM} = \overrightarrow{TP} + \overrightarrow{SQ}$.



16. Two nonzero vectors, $\vec{a}$ and $\vec{b}$, are such that $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$. Show that $\vec{a}$ and $\vec{b}$ must represent the sides of a rectangle.

17. The three medians of $\triangle PQR$ meet at a common point G. The point G divides each median in a 2:1 ratio. Prove that $\overrightarrow{GP} + \overrightarrow{GQ} + \overrightarrow{GR} = \vec{0}$.



Section 6.3—Multiplication of a Vector by a Scalar

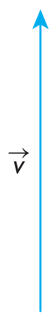
In this section, we will demonstrate the effect of multiplying a vector, $\vec{a}$, by a number k to produce a new vector, $k\vec{a}$. The number k used for multiplication is called a scalar and can be any real number. Previously, the distinction was made between scalars and vectors by saying that scalars have magnitude and not direction, whereas vectors have both. In this section, we are giving a more general meaning to the word *scalar* so that it means any real number. Since real numbers have magnitude (size) but not direction, this meaning is consistent with our earlier understanding.

Examining Scalar Multiplication

Multiplying $\vec{a}$ by different values of k can affect the direction and magnitude of a vector, depending on the values of k that are chosen. The following example demonstrates the effect on a velocity vector when it is multiplied by different scalars.

EXAMPLE 1 **Reasoning about the meaning of scalar multiplication**

An airplane is heading due north at 1000 km/h. The airplane's velocity is represented by $\vec{v}$. Draw the vectors $-\vec{v}$, $\frac{1}{2}\vec{v}$, and $-\frac{1}{2}\vec{v}$ and give an interpretation for each.



Scale: 1 cm is equivalent to 250 km/h

Solution

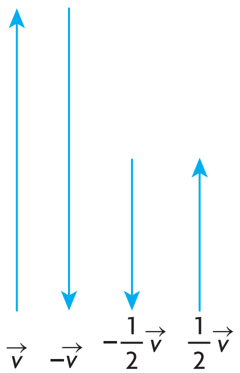
We interpret the vectors in the following way:

$\vec{v}$: the velocity vector for an airplane heading due north at 1000 km/h

$-\vec{v}$: the velocity vector for an airplane heading due south at 1000 km/h

$\frac{1}{2}\vec{v}$: the velocity vector for an airplane heading due north at 500 km/h

$-\frac{1}{2}\vec{v}$: the velocity vector for an airplane heading due south at 500 km/h



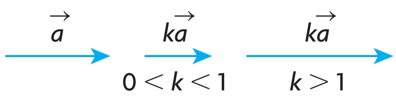
The previous example illustrates how multiplication of a vector by different values of a scalar k can change the magnitude and direction of a vector. The effect of multiplying a vector by a scalar is summarized as follows.

Multiplication of a Vector by a Scalar

For the vector $k\vec{a}$, where k is a scalar and $\vec{a}$ is a nonzero vector:

1. If $k > 0$, then $k\vec{a}$ is in the same direction as $\vec{a}$ with magnitude $k|\vec{a}|$.

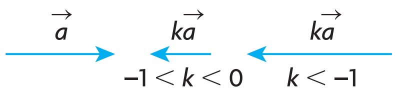
   For $k > 0$, two different possibilities will be considered and are illustrated in the following diagram:



   For $0 < k < 1$, the vector is shortened, and the direction stays the same. If $\vec{a}$ is as shown above, then $\frac{1}{2}\vec{a}$ is half the length of the original vector and in the same direction, i.e., $\left|\frac{1}{2}\vec{a}\right| = \frac{1}{2}|\vec{a}|$.

   For $k > 1$, the vector is lengthened, and the direction stays the same. If $\vec{a}$ is as shown above, then $\frac{3}{2}\vec{a}$ is one and a half times as long as $\vec{a}$ and in the same direction, i.e., $\left|\frac{3}{2}\vec{a}\right| = \frac{3}{2}|\vec{a}|$.

2. If $k < 0$, then $k\vec{a}$ is in the opposite direction as $\vec{a}$ with magnitude $|k||\vec{a}|$. Again, two situations will be considered for $k < 0$.



   For $-1 < k < 0$, the vector is shortened and changes to the opposite direction. If $\vec{a}$ is as shown above, then $-\frac{1}{2}\vec{a}$ is half the length of the original vector $\vec{a}$ but in the opposite direction, i.e., $\left|-\frac{1}{2}\vec{a}\right| = \frac{1}{2}|\vec{a}|$. In the situation where $k < -1$, the vector is lengthened and changes to the opposite direction. If $\vec{a}$ is as shown above, then $-\frac{3}{2}\vec{a}$ is one and a half times as long as $\vec{a}$ but in the opposite direction, i.e., $\left|-\frac{3}{2}\vec{a}\right| = \frac{3}{2}|\vec{a}|$.

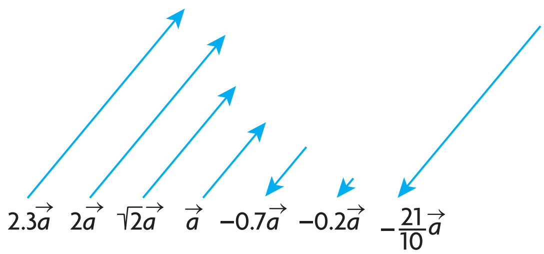
Collinear Vectors

A separate comment should be made about the cases $k = 0$ and $k = -1$.

If we multiply any vector $\vec{a}$ by the scalar 0, the result is always the zero vector, i.e., $0\vec{a} = \vec{0}$. Note that the right side of this equation is a vector, not a scalar.

When we multiply a vector by -1, i.e., $(-1)\vec{a}$, we normally write this as $-\vec{a}$. When any vector is multiplied by -1, its magnitude is unchanged but the direction changes to the opposite. For example, the vectors $-4\vec{a}$ and $4\vec{a}$ have the same magnitude (length) but are opposite.

The effect of multiplying a vector, $\vec{a}$, by different scalars is shown below.



When two vectors are parallel or lie on the same straight line, these vectors are described as being **collinear**. They are described as being collinear because they can be translated so that they lie in the same straight line. Vectors that are not collinear are not parallel. All of the vectors shown above are scalar multiples of $\vec{a}$ and are collinear. When discussing vectors, the terms *parallel* and *collinear* are used interchangeably.

> Two vectors u and v are collinear if and only if it is possible to find a nonzero scalar k such that $\vec{u} = k\vec{v}$.

In the following example, we combine concepts learned in the previous section with those introduced in this section.

EXAMPLE 2 Selecting a strategy to determine the magnitude and direction of a vector

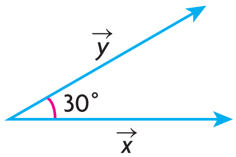
The vectors $\vec{x}$ and $\vec{y}$ are unit vectors (vectors with magnitude 1) that make an angle of 30° with each other.

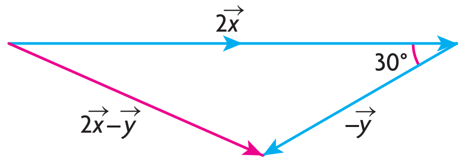
a. Calculate the value of $|2\vec{x} - \vec{y}|$.

b. Determine the direction of $2\vec{x} - \vec{y}$.

Solution

a.





To calculate the value of $|2\vec{x} - \vec{y}|$, construct $2\vec{x} - \vec{y}$ by drawing $2\vec{x}$ and $-\vec{y}$ head-to-tail and then adding them.

Using the cosine law, $|2\vec{x} - \vec{y}|^2 = |2\vec{x}|^2 + |-\vec{y}|^2 - 2|2\vec{x}||-\vec{y}|\cos 30°$

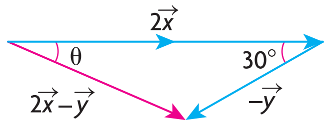
$$|2\vec{x} - \vec{y}|^2 = 2^2 + 1^2 - 2(2)(1)\frac{\sqrt{3}}{2}$$

$$|2\vec{x} - \vec{y}|^2 = 5 - 2\sqrt{3}$$

$$|2\vec{x} - \vec{y}| = \sqrt{5 - 2\sqrt{3}}$$

Therefore, $|2\vec{x} - \vec{y}| \doteq 1.24$.

b. To determine the direction of $2\vec{x} - \vec{y}$, we will calculate θ using the sine law and describe the direction relative to the
the
direction of $\vec{x}$.



$$\frac{\sin \theta}{|-\vec{y}|} = \frac{\sin 30°}{|2\vec{x} - \vec{y}|}$$

$$\frac{\sin \theta}{1} \doteq \frac{\sin 30°}{1.24}$$

$$\theta = \sin^{-1}\left(\frac{\sin 30°}{1.24}\right)$$

$$\theta \doteq 23.8°$$

Therefore, $2\vec{x} - \vec{y}$ has a direction of 23.8° rotated clockwise relative to $\vec{x}$.

In many practical situations that involve velocities, we use specialized notation to describe direction. In the following example, we use this notation along with scalar multiplication to help illustrate its meaning.

EXAMPLE 3 Representing velocity using vectors

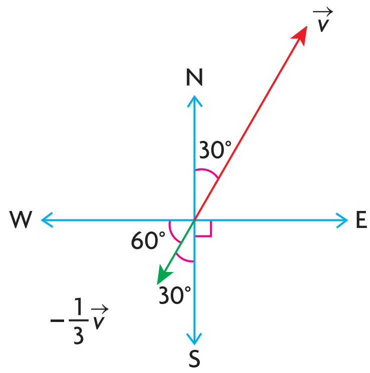
An airplane is flying in the direction $N30°E$ at an airspeed of 240 km/h. The velocity vector for this airplane is represented by $\vec{v}$.

a. Draw a sketch of $-\frac{1}{3}\vec{v}$ and state the direction of this vector.

b. For the vector $\frac{3}{2}\vec{v}$, state its direction and magnitude.

Solution

a.

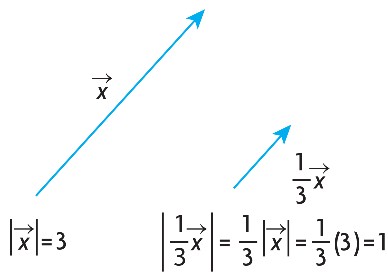


Scale: 1 cm is equivalent to 40 km/h

The vector $-\frac{1}{3}\vec{v}$ represents a speed of $\frac{1}{3}(240 \text{ km/h}) = 80$ km/h and points in the opposite direction as $\vec{v}$. The direction for this vector can be described as W60°S, S30°W, or a bearing of 210°.

b. The velocity vector $\frac{3}{2}\vec{v}$ represents a speed of $\frac{3}{2}(240 \text{ km/h}) = 360$ km/h in the same direction as $\vec{v}$.

It is sometimes useful to multiply the nonzero vector $\vec{x}$ by the scalar $\frac{1}{|\vec{x}|}$. When we multiply $\vec{x}$ by $\frac{1}{|\vec{x}|}$, we get the vector $\frac{1}{|\vec{x}|}\vec{x}$. This vector of length one and called a unit vector, which points in the same direction as $\vec{x}$.



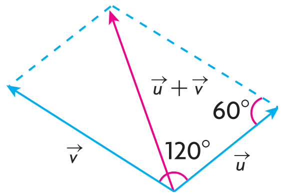
The concept of unit vector will prove to be very useful when we discuss applications of vectors.

EXAMPLE 4 **Using a scalar to create a unit vector**

Given that $|\vec{u}| = 4$ and $|\vec{v}| = 5$ and the angle between $\vec{u}$ and $\vec{v}$ is 120°, determine the unit vector in the same direction as $\vec{u} + \vec{v}$.

Solution

Draw a sketch and determine $|\vec{u} + \vec{v}|$.



Using the cosine law,

$$|\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$$
$$|\vec{u} + \vec{v}|^2 = 4^2 + 5^2 - 2(4)(5)\cos 60°$$
$$|\vec{u} + \vec{v}|^2 = 21$$
$$|\vec{u} + \vec{v}| = \sqrt{21}$$

To create a unit vector in the same direction as $\vec{u} + \vec{v}$, multiply by the scalar equal to $\frac{1}{|\vec{u} + \vec{v}|}$. In this case, the unit vector is $\frac{1}{\sqrt{21}}(\vec{u} + \vec{v}) = \frac{1}{\sqrt{21}}\vec{u} + \frac{1}{\sqrt{21}}\vec{v}$.

IN SUMMARY

Key Idea

- For the vector $k\vec{a}$ where k is a scalar and $\vec{a}$ is a nonzero vector:
  - If $k > 0$, then $k\vec{a}$ is in the same direction as $\vec{a}$ with magnitude $k|\vec{a}|$.
  - If $k < 0$, then $k\vec{a}$ is in the opposite direction as $\vec{a}$ with magnitude $|k||\vec{a}|$.

Need to Know

- If two or more vectors are nonzero scalar multiples of the same vector, then all these vectors are collinear.
- $\frac{1}{|\vec{x}|}\vec{x}$ is a vector of length one, called a unit vector, in the direction of the nonzero vector $\vec{x}$.
- $-\frac{1}{|\vec{x}|}\vec{x}$ is a unit vector in the opposite direction of the nonzero vector $\vec{x}$.

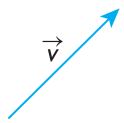
Exercise 6.3

PART A

1. Explain why the statement $\vec{a} = 2|\vec{b}|$ is not meaningful.

2. An airplane is flying at an airspeed of 300 km/h. Using a scale of 1 cm equivalent to 50 km/h, draw a velocity vector to represent each of the following:
   a. a speed of 150 km/h heading in the direction N45°E
   b. a speed of 450 km/h heading in the direction E15°S
   c. a speed of 100 km/h heading in an easterly direction
   d. a speed of 300 km/h heading on a bearing of 345°

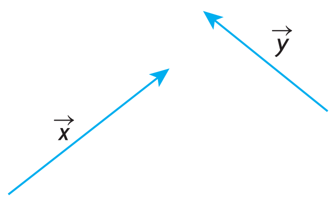
3. An airplane's direction is E25°N. Explain why this is the same as N65°E or a bearing of 65°.



4. The vector $\vec{v}$ has magnitude 2, i.e., $|\vec{v}| = 2$. Draw the following vectors and express each of them as a scalar multiple of $\vec{v}$.
   a. a vector in the same direction as $\vec{v}$ with twice its magnitude
   b. a vector in the same direction as $\vec{v}$ with one-half its magnitude
   c. a vector in the opposite direction as $\vec{v}$ with two-thirds its magnitude
   d. a vector in the opposite direction as $\vec{v}$ with twice its magnitude
   e. a unit vector in the same direction as $\vec{v}$

PART B

K 5. The vectors $\vec{x}$ and $\vec{y}$ are shown below. Draw a diagram for each of the following.



   a. $\vec{x} + 3\vec{y}$ b. $\vec{x} - 3\vec{y}$ c. $-2\vec{x} + \vec{y}$ d. $-2\vec{x} - \vec{y}$

6. Draw two vectors, $\vec{a}$ and $\vec{b}$, that do not have the same magnitude and are noncollinear. Using the vectors you drew, construct the following:
   a. $2\vec{a}$ b. $3\vec{b}$ c. $-3\vec{b}$ d. $2\vec{a} + 3\vec{b}$ e. $2\vec{a} - 3\vec{b}$

7. Three collinear vectors, $\vec{a}$, $\vec{b}$, and $\vec{c}$, are such that $\vec{a} = \frac{2}{3}\vec{b}$ and $\vec{a} = \frac{1}{2}\vec{c}$.
   a. Determine integer values for m and n such that $m\vec{c} + n\vec{b} = \vec{0}$. How many values are possible for m and n to make this statement true?
   b. Determine integer values for d, e, and f such that $d\vec{a} + e\vec{b} + f\vec{c} = \vec{0}$. Are these values unique?

8. The two vectors $\vec{a}$ and $\vec{b}$ are collinear and are chosen such that $|\vec{a}| = |\vec{b}|$. Draw a diagram showing different possible configurations for these two vectors.

9. The vectors $\vec{a}$ and $\vec{b}$ are perpendicular. Are the vectors $4\vec{a}$ and $-2\vec{b}$ also perpendicular? Illustrate your answer with a sketch.

10. If the vectors $\vec{a}$ and $\vec{b}$ are noncollinear, determine which of the following pairs of vectors are collinear and which are not.

    a. $2\vec{a}, -3\vec{a}$ b. $2\vec{a}, 3\vec{b}$ c. $5\vec{a}, -\frac{3}{2}\vec{b}$ d. $-\vec{b}, 2\vec{b}$

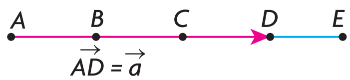
C 11. In the discussion, we defined $\frac{1}{|\vec{x}|}\vec{x}$. Using your own scale, draw your own vector to represent $\vec{x}$.

    a. Sketch $\frac{1}{|\vec{x}|}\vec{x}$ and describe this vector in your own words.

    b. Sketch $-\frac{1}{|\vec{x}|}\vec{x}$ and describe this vector in your own words.

12. Two vectors, $\vec{a}$ and $\vec{b}$, are such that $2\vec{a} = -3\vec{b}$. Draw a possible sketch of these two vectors. What is the value of m, if $|\vec{b}| = m|\vec{a}|$?

A 13. The points B, C, and D are drawn on line segment AE dividing it into four equal lengths. If $\overrightarrow{AD} = \vec{a}$, write each of the following in terms of $\vec{a}$ and $|\vec{a}|$.



    a. $\overrightarrow{EC}$ b. $\overrightarrow{BC}$ c. $|\overrightarrow{ED}|$ d. $|\overrightarrow{AC}|$ e. $\overrightarrow{AE}$

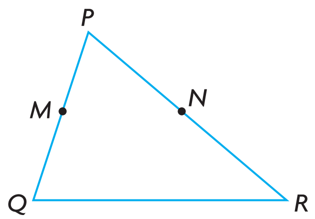
14. The vectors $\vec{x}$ and $\vec{y}$ are unit vectors that make an angle of 90° with each other. Calculate the value of $|2\vec{x} + \vec{y}|$ and the direction of $2\vec{x} + \vec{y}$.

15. The vectors $\vec{x}$ and $\vec{y}$ are unit vectors that make an angle of 30° with each other. Calculate the value of $|2\vec{x} + \vec{y}|$ and the direction of $2\vec{x} + \vec{y}$.

16. Prove that $\frac{1}{|\vec{a}|}\vec{a}$ is a unit vector pointing in the same direction as $\vec{a}$. (*Hint*: Let $\vec{b} = \frac{1}{|\vec{a}|}\vec{a}$ and then find the magnitude of each side of this equation.)

T 17. In $\triangle ABC$, a median is drawn from vertex A to the midpoint of BC, which is labelled D. If $\overrightarrow{AB} = \vec{b}$ and $\overrightarrow{AC} = \vec{c}$, prove that $\overrightarrow{AD} = \frac{1}{2}\vec{b} + \frac{1}{2}\vec{c}$.

18. Let PQR be a triangle in which M is the midpoint of PQ and N is the midpoint of PR. If $\overrightarrow{PM} = \vec{a}$ and $\overrightarrow{PN} = \vec{b}$, find vector expressions for $\overrightarrow{MN}$ and $\overrightarrow{QR}$ in terms of $\vec{a}$ and $\vec{b}$. What conclusions can be drawn about MN and QR? Explain.



19. Draw rhombus $ABCD$ where $AB = 3$ cm. For each of the following, name two vectors $\vec{u}$ and $\vec{v}$ in your diagram such that

a. $\vec{u} = \vec{v}$

b. $\vec{u} = 2\vec{v}$

c. $\vec{u} = -\vec{v}$

d. $\vec{u} = 0.5\vec{v}$

PART C

20. Two vectors, $\vec{x}$ and $\vec{y}$, are drawn such that $|\vec{x}| = 3|\vec{y}|$. Considering $m\vec{x} + n\vec{y} = \vec{0}$, determine all possible values for m and n such that

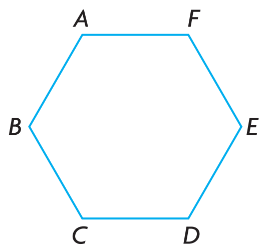
a. $\vec{x}$ and $\vec{y}$ are collinear

b. $\vec{x}$ and $\vec{y}$ are noncollinear

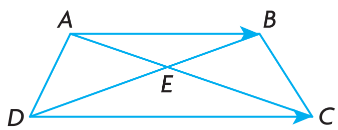
21. $ABCDEF$ is a regular hexagon such that $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$.

a. Express $\overrightarrow{CD}$ in terms of $\vec{a}$ and $\vec{b}$.

b. Prove that BE is parallel to CD and that $|\overrightarrow{BE}| = 2|\overrightarrow{CD}|$.



22. $ABCD$ is a trapezoid whose diagonals AC and BD intersect at the point E. If $\overrightarrow{AB} = \frac{2}{3}\overrightarrow{DC}$, prove that $\overrightarrow{AE} = \frac{3}{5}\overrightarrow{AB} + \frac{2}{5}\overrightarrow{AD}$.



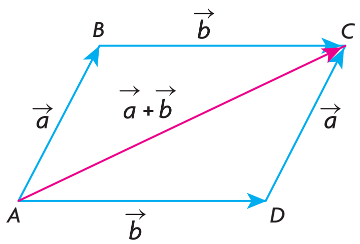
Section 6.4—Properties of Vectors

In previous sections, we developed procedures for adding and subtracting vectors and for multiplying a vector by a scalar. In carrying out these computations, certain assumptions were made about how to combine vectors without these rules being made explicit. Although these rules will seem apparent, they are important for understanding the basic structure underlying vectors, and for their use in computation. Initially, three specific rules for dealing with vectors will be discussed, and we will show that these rules are similar to those used in dealing with numbers and basic algebra. Later, we demonstrate an additional three rules.

Properties of Vector Addition

1. Commutative Property of Addition: When we are dealing with numbers, the order in which they are added does not affect the final answer. For example, if we wish to add 2 and 3, the answer is the same if it is written as $2 + 3$ or as $3 + 2$. In either case, the answer is 5. This property of being able to add numbers, in any chosen order, is called the commutative property of addition for real numbers. This property also works for algebra, because algebraic expressions are themselves numerical in nature. We make this assumption when simplifying in the following example: $2x + 3y + 3x = 2x + 3x + 3y = 5x + 3y$. Being able to switch the order like this allows us to carry out addition without concern for the order of the terms being added.

   This property that we have identified also holds for vectors, as can be seen in the following diagram:

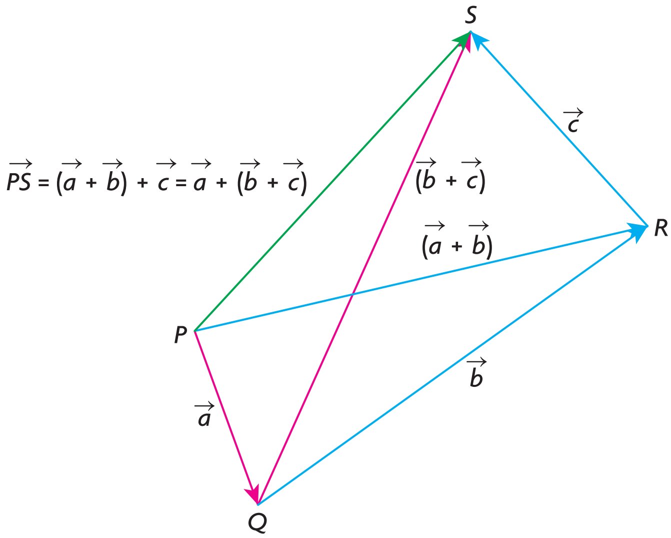
   From triangle ABC, using the vector addition rules,
   $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \vec{a} + \vec{b}$
   From triangle ADC,
   $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \vec{b} + \vec{a}$
   So, $\overrightarrow{AC} = \vec{a} + \vec{b} = \vec{b} + \vec{a}$.

   Although vector addition is commutative, certain types of vector operations are not always commutative. We will see this when dealing with cross products in Chapter 7.

2. Associative Property of Addition: When adding numbers, the associative property is used routinely. If we wish to add 3, 5 and 8, for example, we can do this as $(3 + 5) + 8$ or as $3 + (5 + 8)$. Doing it either way, we get the answer 16. In doing this calculation, we are free to associate the numbers however we choose. This property also holds when adding algebraic expressions, such as $2x + 3x - 7x = (2x + 3x) - 7x = 2x + (3x - 7x) = -2x$.

   In adding vectors, we are free to associate them in exactly the same way as we do for numbers or algebraic expressions. For vectors, this property is stated as $\vec{a} + \vec{b} + \vec{c} = (\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.

   We will use the following diagram and addition of vectors to demonstrate the associative property.

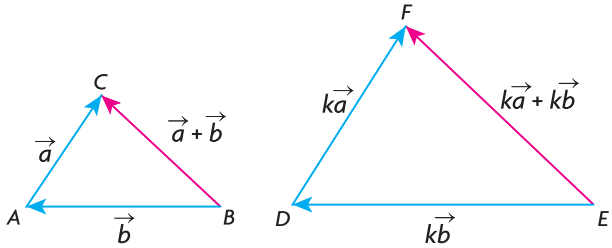


   In the diagram, $\overrightarrow{PQ} = \vec{a}$, $\overrightarrow{QR} = \vec{b}$, and $\overrightarrow{RS} = \vec{c}$. From triangle PRQ, $\overrightarrow{PR} = \vec{a} + \vec{b}$, and then from triangle PSR, $\overrightarrow{PS} = \overrightarrow{PR} + \overrightarrow{RS} = (\vec{a} + \vec{b}) + \vec{c}$. Similarly, from triangle SQR, $\overrightarrow{QS} = \vec{b} + \vec{c}$ and then from triangle PQS, $\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QS} = \vec{a} + (\vec{b} + \vec{c})$. So $\overrightarrow{PS} = (\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.

   It is interesting to note, just as we did with the commutative property, that the associative property holds for the addition of vectors but does not hold for certain kinds of multiplication.

3. Distributive Property of Addition: The distributive property is something we have used implicitly from the first day we thought about numbers or algebra. In calculating the perimeter of a rectangle with width w and length l, we write the perimeter as $P = 2(w + l) = 2w + 2l$. In this case, the 2 has been distributed across the brackets to give $2w$ and $2l$.

Demonstrating the distributive law for vectors depends on being able to multiply vectors by scalars and on the addition law for vectors.



In this diagram, we started with $\vec{a}$ and $\vec{b}$ and then multiplied each of them by k, a positive scalar, to give the vectors $k\vec{a}$ and $k\vec{b}$, respectively. In $\triangle ABC$, $\overrightarrow{BC} = \vec{a} + \vec{b}$, and in $\triangle DEF$, $\overrightarrow{EF} = k\vec{a} + k\vec{b}$. However, the two triangles are similar, so $\overrightarrow{EF} = k(\vec{a} + \vec{b})$. Since $\overrightarrow{EF} = k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$, we have shown that the distributive law is true for $k > 0$ and any pair of vectors.

Although we chose k to be a positive number, we could have chosen any real number for k.

Properties of Vector Addition

1. Commutative Property of Addition: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. Associative Property of Addition: $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
3. Distributive Property of Addition: $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$, $k \in \mathbf{R}$

EXAMPLE 1 Selecting appropriate vector properties to determine an equivalent vector

Simplify the following expression: $3(2\vec{a} + \vec{b} + \vec{c}) - (\vec{a} + 3\vec{b} - 2\vec{c})$.

Solution

$3(2\vec{a} + \vec{b} + \vec{c}) - (\vec{a} + 3\vec{b} - 2\vec{c})$

$= 6\vec{a} + 3\vec{b} + 3\vec{c} - \vec{a} - 3\vec{b} + 2\vec{c}$ (Distributive property)

$= 6\vec{a} - \vec{a} + 3\vec{b} - 3\vec{b} + 3\vec{c} + 2\vec{c}$ (Commutative property)

$= (6 - 1)\vec{a} + (3 - 3)\vec{b} + (3 + 2)\vec{c}$ (Distributive property for scalars)

$= 5\vec{a} + \vec{0} + 5\vec{c}$

$= 5\vec{a} + 5\vec{c}$

In doing the calculation in Example 1, assumptions were made that are implicit but that should be stated.

Further Laws of Vector Addition and Scalar Multiplication

1. Adding $\vec{0}$: $\vec{a} + \vec{0} = \vec{a}$
2. Associative Law for Scalars: $m(n\vec{a}) = (mn)\vec{a} = mn\vec{a}$
3. Distributive Law for Scalars: $(m + n)\vec{a} = m\vec{a} + n\vec{a}$

It is important to be aware of all these properties when calculating, but the properties can be assumed without having to refer to them for each simplification.

EXAMPLE 2 Selecting appropriate vector properties to create new vectors

If $\vec{x} = 3\vec{i} - 4\vec{j} + \vec{k}$, $\vec{y} = \vec{j} - 5\vec{k}$, and $\vec{z} = -\vec{i} - \vec{j} + 4\vec{k}$, determine each of the following in terms of $\vec{i}$, $\vec{j}$, and $\vec{k}$.

a. $\vec{x} + \vec{y}$ b. $\vec{x} - \vec{y}$ c. $\vec{x} - 2\vec{y} + 3\vec{z}$

Solution

a. $$\begin{aligned}\vec{x} + \vec{y} &= (3\vec{i} - 4\vec{j} + \vec{k}) + (\vec{j} - 5\vec{k})\\ &= 3\vec{i} - 4\vec{j} + \vec{j} + \vec{k} - 5\vec{k}\\ &= 3\vec{i} - 3\vec{j} - 4\vec{k}\end{aligned}$$

b. $$\begin{aligned}\vec{x} - \vec{y} &= (3\vec{i} - 4\vec{j} + \vec{k}) - (\vec{j} - 5\vec{k})\\ &= 3\vec{i} - 4\vec{j} - \vec{j} + \vec{k} + 5\vec{k}\\ &= 3\vec{i} - 5\vec{j} + 6\vec{k}\end{aligned}$$

c. $$\begin{aligned}\vec{x} - 2\vec{y} + 3\vec{z} &= (3\vec{i} - 4\vec{j} + \vec{k}) - 2(\vec{j} - 5\vec{k}) + 3(-\vec{i} - \vec{j} + 4\vec{k})\\ &= 3\vec{i} - 4\vec{j} + \vec{k} - 2\vec{j} + 10\vec{k} - 3\vec{i} - 3\vec{j} + 12\vec{k}\\ &= -9\vec{j} + 23\vec{k}\end{aligned}$$

As stated previously, it is not necessary to state the rules as we simplify, and furthermore, it is better to try to simplify without writing in every step.

The rules that were developed in this section will prove useful as we move ahead. They are necessary for our understanding of linear combinations, which will be dealt with later in this chapter.

IN SUMMARY

Key Idea

- Properties used to evaluate numerical expressions and simplify algebraic expressions also apply to vector addition and scalar multiplication.

Need to Know

- Commutative Property of Addition: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- Associative Property of Addition: $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
- Distributive Property of Addition: $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$, $k \in \mathbf{R}$
- Adding $\vec{0}$: $\vec{a} + \vec{0} = \vec{a}$
- Associative Law for Scalars: $m(n\vec{a}) = (mn)\vec{a} = mn\vec{a}$
- Distributive Law for Scalars: $(m + n)\vec{a} = m\vec{a} + n\vec{a}$

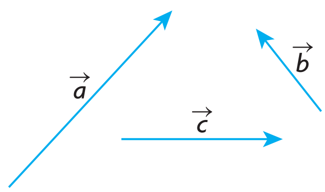
Exercise 6.4

PART A

1. If * is an operation on a set, S, the element x, such that $a * x = a$, is called the identity element for the operation *.
   a. For the addition of numbers, what is the identity element?
   b. For the multiplication of numbers, what is the identity element?
   c. For the addition of vectors, what is the identity element?
   d. For scalar multiplication, what is the identity element?

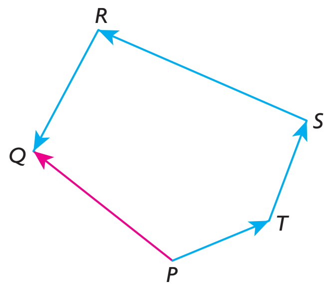
2. Illustrate the commutative law for two vectors that are perpendicular.

C 3. Redraw the following three vectors and illustrate the associative law.



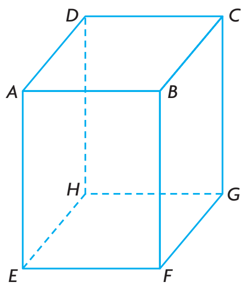
4. With the use of a diagram, show that the distributive law, $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$, holds where $k < 0$, $k \in \mathbf{R}$.

PART B



K 5. Using the given diagram, show that the following is true.

$$\begin{aligned}\overrightarrow{PQ} &= (\overrightarrow{RQ} + \overrightarrow{SR}) + \overrightarrow{TS} + \overrightarrow{PT}\\ &= \overrightarrow{RQ} + (\overrightarrow{SR} + \overrightarrow{TS}) + \overrightarrow{PT}\\ &= \overrightarrow{RQ} + \overrightarrow{SR} + (\overrightarrow{TS} + \overrightarrow{PT})\end{aligned}$$



6. *ABCDEFGH* is a rectangular prism.

a. Write a single vector that is equivalent to $\overrightarrow{EG} + \overrightarrow{GH} + \overrightarrow{HD} + \overrightarrow{DC}$.

b. Write a vector that is equivalent to $\overrightarrow{EG} + \overrightarrow{GD} + \overrightarrow{DE}$.

c. Is it true that $|\overrightarrow{HB}| = |\overrightarrow{GA}|$? Explain.

7. Write the following vector in simplified form:

$3(\vec{a} - 2\vec{b} - 5\vec{c}) - 3(2\vec{a} - 4\vec{b} + 2\vec{c}) - (\vec{a} - 3\vec{b} + 3\vec{c})$

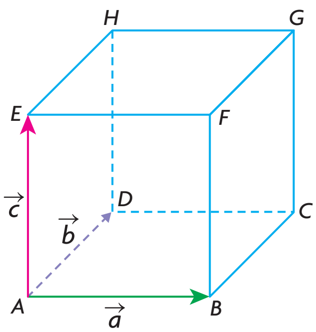
8. If $\vec{a} = 3\vec{i} - 4\vec{j} + \vec{k}$ and $\vec{b} = -2\vec{i} + 3\vec{j} - \vec{k}$, express each of the following in terms of $\vec{i}, \vec{j}$, and $\vec{k}$.

a. $2\vec{a} - 3\vec{b}$ b. $\vec{a} + 5\vec{b}$ c. $2(\vec{a} - 3\vec{b}) - 3(-2\vec{a} - 7\vec{b})$

T 9. If $2\vec{x} + 3\vec{y} = \vec{a}$ and $-\vec{x} + 5\vec{y} = 6\vec{b}$, express $\vec{x}$ and $\vec{y}$ in terms of $\vec{a}$ and $\vec{b}$.

10. If $\vec{x} = \frac{2}{3}\vec{y} + \frac{1}{3}\vec{z}$, $\vec{x} - \vec{y} = \vec{a}$, and $\vec{y} - \vec{z} = \vec{b}$, show that $\vec{a} = -\frac{1}{3}\vec{b}$.

A 11. A cube is constructed from the three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, as shown below.



a. Express each of the diagonals $\overrightarrow{AG}$, $\overrightarrow{BH}$, $\overrightarrow{CE}$, and $\overrightarrow{DF}$ in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$.

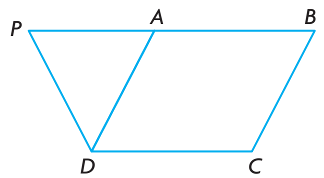
b. Is $|\overrightarrow{AG}| = |\overrightarrow{BH}|$? Explain.

PART C

12. In the trapezoid *TXYZ*, $\overrightarrow{TX} = 2\overrightarrow{ZY}$. If the diagonals meet at *O*, find an expression for $\overrightarrow{TO}$ in terms of $\overrightarrow{TX}$ and $\overrightarrow{TZ}$.

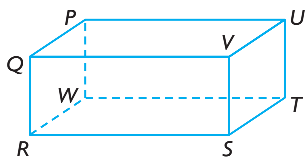
Mid-Chapter Review

1. $ABCD$ is a parallelogram, and $|\overrightarrow{PD}| = |\overrightarrow{DA}|$.



   a. Determine which vectors (if any) are equal to $\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overrightarrow{AD}$, $\overrightarrow{CB}$, and $\overrightarrow{AP}$.
   b. Explain why $|\overrightarrow{PD}| = |\overrightarrow{BC}|$.

2. The diagram below represents a rectangular prism. State a single vector equal to each of the following.



   a. $\overrightarrow{RQ} + \overrightarrow{RS}$
   b. $\overrightarrow{RQ} + \overrightarrow{QV}$
   c. $\overrightarrow{PW} + \overrightarrow{WS}$
   d. $(\overrightarrow{RQ} + \overrightarrow{RS}) + \overrightarrow{VU}$
   e. $\overrightarrow{PW} - \overrightarrow{VP}$
   f. $\overrightarrow{PW} + \overrightarrow{WR} + \overrightarrow{RQ}$

3. Two vectors, $\vec{a}$ and $\vec{b}$, have a common starting point with an angle of 120° between them. The vectors are such that $|\vec{a}| = 3$ and $|\vec{b}| = 4$.
   a. Calculate $|\vec{a} + \vec{b}|$.
   b. Calculate the angle between $\vec{a}$ and $\vec{a} + \vec{b}$.

4. Determine all possible values for t if the length of the vector $\vec{x} = t\vec{y}$ is $4|\vec{y}|$.

5. $PQRS$ is a quadrilateral where A, B, C, and D are the midpoints of SP, PQ, QR, and RS, respectively. Prove, using vector methods, that $ABCD$ is a parallelogram.

6. Given that $|\vec{u}| = 8$ and $|\vec{v}| = 10$ and the angle between vectors $\vec{u}$ and $\vec{v}$ is 60° determine:
   a. $|\vec{u} - \vec{v}|$
   b. the direction of $\vec{u} - \vec{v}$ relative to $\vec{u}$
   c. the unit vector in the direction of $\vec{u} + \vec{v}$
   d. $|5\vec{u} + 2\vec{v}|$

7. The vectors $\vec{p}$ and $\vec{q}$ are distinct unit vectors that are placed in a tail-to-tail position. If these two vectors have an angle of 60° between them, determine $|2\vec{p} - \vec{q}|$.

8. The vector $\vec{m}$ is collinear (parallel) to $\vec{n}$ but in the opposite direction. Express the magnitude of $\vec{m} + \vec{n}$ in terms of the magnitudes of $\vec{m}$ and $\vec{n}$.

9. $ABCD$ is a parallelogram. If $\overrightarrow{AB} = \vec{x}$ and $\overrightarrow{DA} = \vec{y}$, express $\overrightarrow{BC}$, $\overrightarrow{DC}$, $\overrightarrow{BD}$, and $\overrightarrow{AC}$ in terms of $\vec{x}$ and $\vec{y}$.

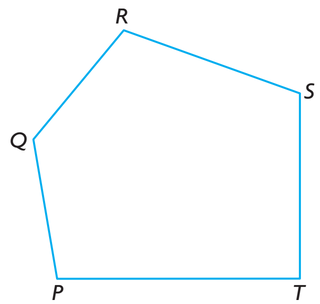
10. If A, B, and C are three collinear points with B at the midpoint of AC, and O is any point not on the line AC, prove that $\overrightarrow{OA} + \overrightarrow{OC} = 2\overrightarrow{OB}$. (*Hint*: $\overrightarrow{AB} = \overrightarrow{BC}$.)

11. $ABCD$ is a quadrilateral with $\overrightarrow{AB} = \vec{x}$, $\overrightarrow{CD} = 2\vec{y}$, and $\overrightarrow{AC} = 3\vec{x} - \vec{y}$. Express $\overrightarrow{BD}$ and $\overrightarrow{BC}$ in terms of $\vec{x}$ and $\vec{y}$.

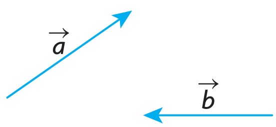
12. An airplane is heading due south at a speed of 500 km/h when it encounters a head wind from the south at 40 km/h. What is the resultant ground velocity of the airplane?

13. $PQRST$ is a pentagon. State a single vector that is equivalent to each of the following:

    a. $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RT}$ b. $\overrightarrow{PQ} + \overrightarrow{QR} - \overrightarrow{TR}$ c. $\overrightarrow{PR} - (\overrightarrow{PT} - \overrightarrow{ST})$



14. The vectors $\vec{a}$ and $\vec{b}$ are given below. Use these vectors to sketch each of the following.



    a. $\frac{1}{3}\vec{a} + \vec{b}$ b. $\frac{3}{2}\vec{a} - 2\vec{b}$ c. $-\vec{b} + \vec{a}$ d. $\frac{\vec{b} + \vec{a}}{2}$

15. $PQRS$ is a quadrilateral with $\overrightarrow{PQ} = 2\vec{a}$, $\overrightarrow{QR} = 3\vec{b}$, and $\overrightarrow{QS} = 3\vec{b} - 3\vec{a}$. Express $\overrightarrow{PS}$ and $\overrightarrow{RS}$ in terms of $\vec{a}$ and $\vec{b}$.

Section 6.5—Vectors in R^2 and R^3

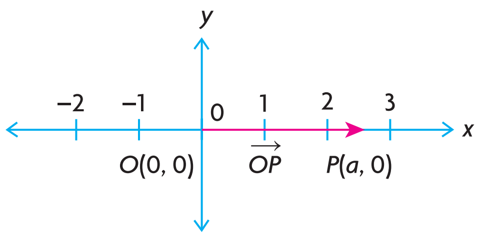
In the introduction to this chapter, we said vectors are important because of their application to a variety of different areas of study. In these areas, the value of using vectors is derived primarily from being able to consider them in coordinate form, or algebraic form, as it is sometimes described. Our experience with coordinate systems in mathematics thus far has been restricted to the *xy*-plane, but we will soon begin to see how ideas in two dimensions can be extended to higher dimensions and how this results in a greater range of applicability.

Introduction to Algebraic Vectors

Mathematicians started using coordinates to analyze physical situations in about the fourteenth century. However, a great deal of the credit for developing the methods used with coordinate systems should be given to the French mathematician Rene Descartes (1596–1650). Descartes was the first to realize that using a coordinate system would allow for the use of algebra in geometry. Since then, this idea has become important in the development of mathematical ideas in many areas. For our purposes, using algebra in this way leads us to the consideration of ideas involving vectors that otherwise would not be possible.

At the beginning of our study of algebraic vectors, there are a number of ideas that must be introduced and that form the foundation for what we are doing. After we start to work with vectors, these ideas are used implicitly without having to be restated each time.

One of the most important ideas that we must consider is that of the **unique** representation of vectors in the *xy*-plane. The unique representation of the vector $\overrightarrow{OP}$ is a matter of showing the unique representation of the point P because $\overrightarrow{OP}$ is determined by this point. The uniqueness of vector representation will be first considered for the **position vector** $\overrightarrow{OP}$, which has its head at the point $P(a, 0)$ and its tail at the origin $O(0, 0)$ shown on the x-axis below. The x-axis is the set of real numbers, **R**, which is made up of rational and irrational numbers.

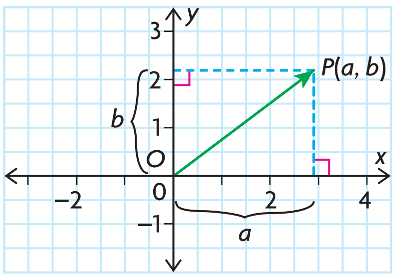


The point P is a distance of a units away from the origin and occupies exactly one position on the x-axis. Since each point P has a unique position on this axis, this implies that $\overrightarrow{OP}$ is also unique because this vector is determined by P.

The xy-plane is often referred to as R^2, which means that each of the x- and y-coordinates for any point on the plane is a real number. In technical terms, we would say that $R^2 = \{(x, y)$, where x and y are real numbers$\}$.

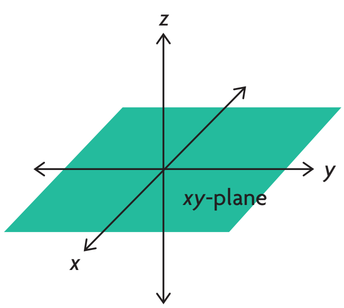
Points and Vectors in R^2

In the following diagram, $\overrightarrow{OP}$ can also be represented in component form by the vector defined as (a, b). This is a vector with its tail at $O(0, 0)$ and its head at $P(a, b)$. Perpendicular lines have been drawn from P to the two axes to help show the meaning of (a, b) in relation to $\overrightarrow{OP}$. a is called the x-component and b is the y-component of $\overrightarrow{OP}$. Again, each of the coordinates of this point is a unique real number and, because of this, the associated vector, $\overrightarrow{OP}$, has a unique location in the xy-plane.



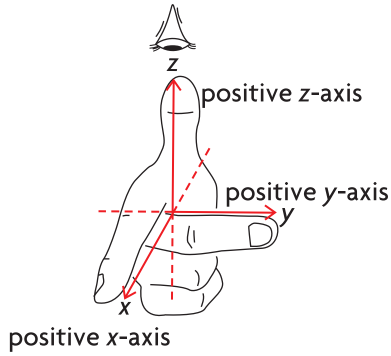
Points and Vectors in R^3

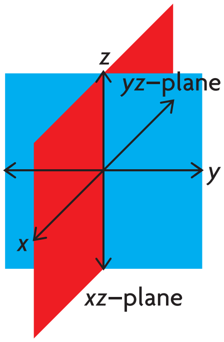
All planes in R^2 are flat surfaces that extend infinitely far in all directions and are said to be two-dimensional because each point is located using an x and a y or two coordinates. It is also useful to be able to represent points and vectors in three dimensions. The designation R^3 is used for three dimensions because each of the coordinates of a point $P(a, b, c)$, and its associated vector $\overrightarrow{OP} = (a, b, c)$, is a real number. Here, $O(0, 0, 0)$ is the origin in three dimensions. As in R^2, each point has a unique location in R^3, which again implies that each position vector $\overrightarrow{OP}$ is unique in R^3.



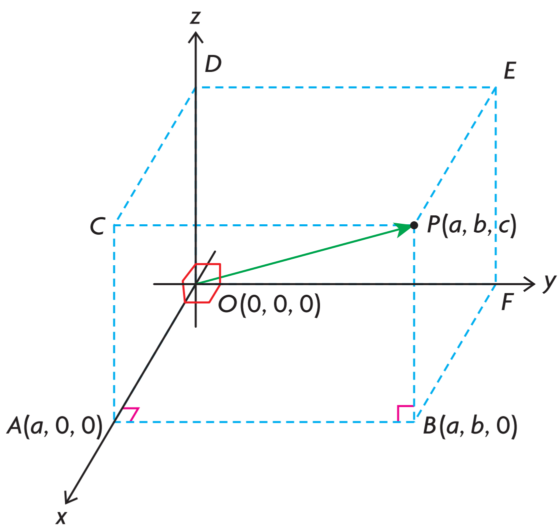
In placing points in R^3, we choose three axes called the x-, y-, and z-axis. Each pair of axes is perpendicular, and each axis is a copy of the real number line. There are several ways to choose the orientation of the positive axes, but we will use what is called a right-handed system. If we imagine ourselves looking down the positive z-axis onto the xy-plane so that, when the positive x-axis is rotated 90° *counterclockwise* it becomes coincident with the positive y-axis, then this is called a right-handed system. A right-handed system is normally what is used to represent R^3, and we will use this convention in this book.

Right-Handed System of Coordinates





Each pair of axes determines a plane. The xz-plane is determined by the x- and z-axes, and the yz-plane is determined by the y- and z-axes. Notice that, when we are discussing, for example, the xy-plane in R^3, this plane extends infinitely far in both the positive and negative directions. One way to visualize a right-handed system is to think of the y- and z-axes as lying in the plane of a book, determining the yz-plane, with the positive x-axis being perpendicular to the plane of the book and pointing directly toward you.



Each point $P(a, b, c)$ in R^3 has its location determined by an ordered triple. In the diagram above, the positive x-, y-, and z-axes are shown such that each pair of axes is perpendicular to the other and each axis represents a real number line. If we wish to locate $P(a, b, c)$, we move along the x-axis to $A(a, 0, 0)$, then in a direction perpendicular to the xz-plane, and parallel to the y-axis, to the point $B(a, b, 0)$. From there, we move in a direction perpendicular to the xy-plane and parallel to the z-axis to the point $P(a, b, c)$. This point is a vertex of a right rectangular prism.

Notice that the coordinates are signed, and so, for example, if we are locating the point $A(-2, 0, 0)$ we would proceed along the *negative* x-axis.

A source of confusion might be the meaning of $P(a, b, c)$ because it may be confused as either being a point or a vector. When referring to a vector, it will be stated explicitly that we are dealing with a vector and will be written as $\overrightarrow{OP} = (a, b, c)$, where a, b, and c are the x-, y-, and z-components respectively of the vector. In the diagram, this position vector is formed by joining the origin $O(0, 0, 0)$ to $P(a, b, c)$. When dealing with points, $P(a, b, c)$ will be named specifically as a point. In most situations, the distinction between the two should be evident from the context.

EXAMPLE 1 Reasoning about the coordinates of points in R^3

In the diagram on the previous page, determine the coordinates of C, D, E, and F.

Solution

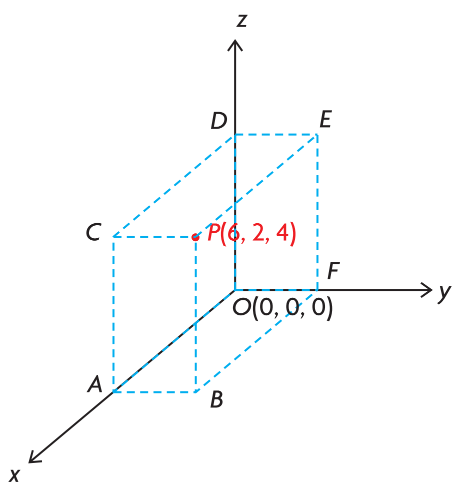
C is on the xz-plane and has coordinates $(a, 0, c)$, D is on the z-axis and has coordinates $(0, 0, c)$, E is on yz-plane and has coordinates $(0, b, c)$, and F is on the y-axis and has coordinates $(0, b, 0)$.

In the following example, we show how to locate points with the use of a rectangular box (prism) and line segments. It is useful, when we first start labelling points in R^3, to draw the box to gain familiarity with the coordinate system.

EXAMPLE 2 Connecting the coordinates of points and vector components in R^3

a. In the following diagram, the point $P(6, 2, 4)$ is located in R^3. What are the coordinates of A, B, C, D, E, and F?

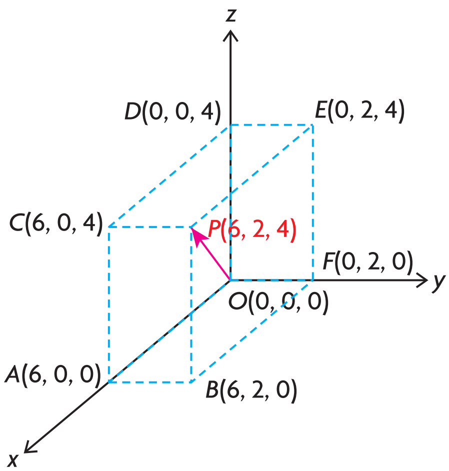
b. Draw the vector $\overrightarrow{OP}$.



Solution

a. $A(6, 0, 0)$ is a point on the positive x-axis, $B(6, 2, 0)$ is a point on the xy-plane, $C(6, 0, 4)$ is a point on the xz-plane, $D(0, 0, 4)$ is a point on the positive z-axis, $E(0, 2, 4)$ is a point on the yz-plane, and $F(0, 2, 0)$ is a point on the positive y-axis.

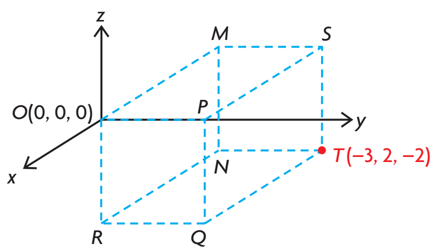
b. The vector $\overrightarrow{OP}$ is the vector associated with the point $P(a, b, c)$. It is the vector with its tail at the origin and its head at $P(6, 2, 4)$ and is named $\overrightarrow{OP} = (6, 2, 4)$.



EXAMPLE 3 Connecting the coordinates of points and vector components in R^3

a. In the following diagram, the point T is located in R^3. What are the coordinates of P, Q, R, M, N, and S?

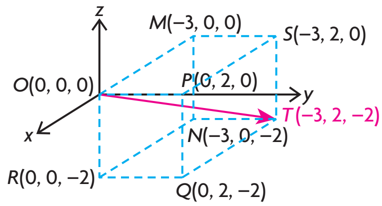
b. Draw the vector $\overrightarrow{OT}$.



Solution

a. The point $P(0, 2, 0)$ is a point on the positive y-axis. The point $Q(0, 2, -2)$ is on the yz-plane. The point $R(0, 0, -2)$ is on the negative z-axis. The point

$M(-3, 0, 0)$ is on the negative x-axis. The point $N(-3, 0, -2)$ is on the xz-plane. The point $S(-3, 2, 0)$ is on the xy-plane.



b. The vector $\overrightarrow{OT}$ is the vector associated with the point $T(-3, 2, -2)$ and is a vector with O as its tail and T as its head and is named $\overrightarrow{OT} = (-3, 2, -2)$.

When working with coordinate systems in R^3, it is possible to label planes using equations, which is demonstrated in the following example.

EXAMPLE 4 Representing planes in R^3 with equations

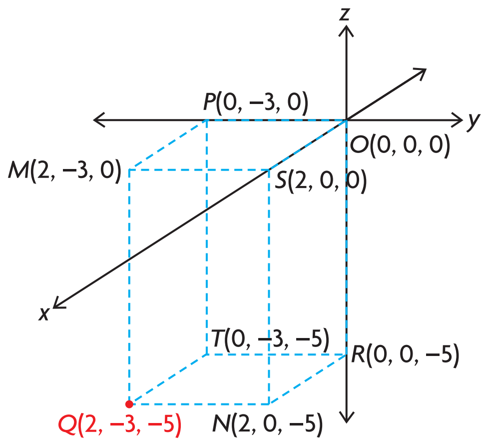
The point $Q(2, -3, -5)$ is shown in R^3.

a. Write an equation for the xy-plane.

b. Write an equation for the plane containing the points P, M, Q, and T.

c. Write a mathematical description of the set of points in rectangle $PMQT$.

d. What is the equation of the plane parallel to the xy-plane passing through $R(0, 0, -5)$?



Solution

a. Every point on the xy-plane has a z-component of 0, with every point on the plane having the form $(x, y, 0)$, where x and y are real numbers. The equation is $z = 0$.

b. Every point on this plane has a y-component equal to -3, with every point on the plane having the form $(x, -3, z)$, where x and z are real numbers. The equation is $y = -3$.

c. Every point in the rectangle has a y-component equal to -3, with every point in the rectangle having the form $(x, -3, z)$, where x and z are real numbers such that $0 \leq x \leq 2$ and $-5 \leq z \leq 0$.

d. Every point on this plane has a z-component equal to -5, with every point on the plane having the form $(x, y, -5)$, where x and y are real numbers. The equation is $z = -5$.

There is one further observation that should be made about placing points on coordinate axes. When using R^2 to describe the plane, which is two-dimensional, the exponent, n, in R^n is 2. Similarly, in three dimensions, the exponent is 3. The exponent in R^n corresponds to the number of dimensions of the coordinate system.

IN SUMMARY

Key Idea

- In R^2 or R^3, the location of every point is unique. As a result, every vector drawn with its tail at the origin and its head at a point is also unique. This type of vector is called a position vector.

Need to Know

- In R^2, $P(a, b)$ is a point that is a units from $O(0, 0)$ along the x-axis and b units parallel to the y-axis.
- The position vector $\overrightarrow{OP}$ has its tail located at $O(0, 0)$ and its head at $P(a, b)$. $\overrightarrow{OP} = (a, b)$
- In R^3, $P(a, b, c)$ is a point that is a units from $O(0, 0, 0)$ along the x-axis, b units parallel to the y-axis, and c units parallel to the z-axis. The position vector $\overrightarrow{OP}$ has its tail located at $O(0, 0, 0)$ and its head at $P(a, b, c)$. $\overrightarrow{OP} = (a, b, c)$
- In R^3, the three mutually perpendicular axes form a *right-handed* system.

Exercise 6.5

PART A

1. In R^3, is it possible to locate the point $P\left(\frac{1}{2}, \sqrt{-1}, 3\right)$? Explain.

2. a. Describe in your own words what it means for a point and its associated vector to be uniquely represented in R^3.
   b. Suppose that $\overrightarrow{OP} = (a, -3, c)$ and $\overrightarrow{OP} = (-4, b, -8)$. What are the corresponding values for a, b, and c? Why are we able to be certain that the determined values are correct?

3. a. The points $A(5, b, c)$ and $B(a, -3, 8)$ are located at the same point in R^3. What are the values of a, b, and c?
   b. Write the vector corresponding to $\overrightarrow{OA}$.

4. In R^3, each of the components for each point or vector is a real number. If we use the notation I^3, where I represents the set of integers, explain why $\overrightarrow{OP} = (-2, 4, -\sqrt{3})$ would not be an acceptable vector in I^3. Why is $\overrightarrow{OP}$ an acceptable vector in R^3?

5. Locate the points $A(4, -4, -2)$, $B(-4, 4, 2)$, and $C(4, 4, -2)$ using coordinate axes that you construct yourself. Draw the corresponding rectangular box (prism) for each, and label the coordinates of its vertices.

6. a. On what axis is $A(0, -1, 0)$ located? Name three other points on this axis.
   b. Name the vector $\overrightarrow{OA}$ associated with point A.

7. a. Name three vectors with their tails at the origin and their heads on the z-axis.
   b. Are the vectors you named in part a. collinear? Explain.
   c. How would you represent a general vector with its head on the z-axis and its tail at the origin?

K 8. Draw a set of x-, y-, and z-axes and plot the following points:

a. $A(1, 0, 0)$	c. $C(0, 0, -3)$	e. $E(2, 0, 3)$
b. $B(0, -2, 0)$	d. $D(2, 3, 0)$	f. $F(0, 2, 3)$

PART B

9. a. Draw a set of x-, y-, and z-axes and plot the following points: $A(3, 2, -4)$, $B(1, 1, -4)$, and $C(0, 1, -4)$.
   b. Determine the equation of the plane containing the points A, B, and C.

10. Plot the following points in R^3, using a rectangular prism to illustrate each coordinate.

a. $A(1, 2, 3)$	c. $C(1, -2, 1)$	e. $E(1, -1, 1)$
b. $B(-2, 1, 1)$	d. $D(1, 1, 1)$	f. $F(1, -1, -1)$

11. Name the vector associated with each point in question 10, express it in component form, and show the vectors associated with each of the points in the diagrams.

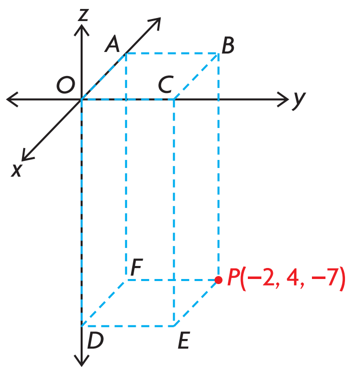
12. $P(2, a - c, a)$ and $Q(2, 6, 11)$ represent the same point in R^3.
    a. What are the values of a and c?
    b. Does $|\overrightarrow{OP}| = |\overrightarrow{OQ}|$? Explain.

13. Each of the points $P(x, y, 0)$, $Q(x, 0, z)$, and $R(0, y, z)$ represent general points on three different planes. Name the three planes to which each corresponds.

C 14. a. What is the equation of the plane that contains the points $M(1, 0, 3)$, $N(4, 0, 6)$, and $P(7, 0, 9)$? Explain your answer.

b. Explain why the plane that contains the points M, N, and P also contains the vectors $\overrightarrow{OM}$, $\overrightarrow{ON}$, and $\overrightarrow{OP}$.

A 15. The point $P(-2, 4, -7)$ is located in R^3 as shown on the coordinate axes below.



a. Determine the coordinates of points A, B, C, D, E, and F.

b. What are the vectors associated with each of the points in part a.?

c. How far below the xy-plane is the rectangle $DEPF$?

d. What is the equation of the plane containing the points B, C, E, and P?

e. Describe mathematically the set of points contained in rectangle $BCEP$.

16. Draw a diagram on the appropriate coordinate system for each of the following vectors:

a. $\overrightarrow{OP} = (4, -2)$ c. $\overrightarrow{OC} = (2, 4, 5)$ e. $\overrightarrow{OF} = (0, 0, 5)$

b. $\overrightarrow{OD} = (-3, 4)$ d. $\overrightarrow{OM} = (-1, 3, -2)$ f. $\overrightarrow{OJ} = (-2, -2, 0)$

PART C

T 17. Draw a diagram illustrating the set of points $\{(x, y, z) \in R^3 \mid 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$.

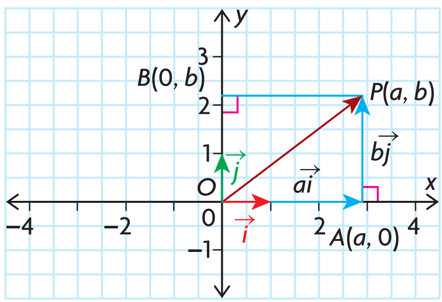
18. Show that if $\overrightarrow{OP} = (5, -10, -10)$, then $|\overrightarrow{OP}| = 15$.

19. If $\overrightarrow{OP} = (-2, 3, 6)$ and $B(4, -2, 8)$, determine the coordinates of point A such that $\overrightarrow{OP} = \overrightarrow{AB}$.

Section 6.6—Operations with Algebraic Vectors in R^2

In the previous section, we showed how to locate points and vectors in both two and three dimensions and then showed their connection to algebraic vectors. In R^2, we showed that $\overrightarrow{OP} = (a, b)$ was the vector formed when we joined the origin, $O(0, 0)$, to the point $P(a, b)$. We showed that the same meaning could be given to $\overrightarrow{OP} = (a, b, c)$, where the point $P(a, b, c)$ was in R^3 and $O(0, 0, 0)$ is the origin. In this section, we will deal with vectors in R^2 and show how a different representation of $\overrightarrow{OP} = (a, b)$ leads to many useful results.

Defining a Vector in R^2 in Terms of Unit Vectors



A second way of writing $\overrightarrow{OP} = (a, b)$ is with the use of the unit vectors $\vec{i}$ and $\vec{j}$.

The vectors $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$ have magnitude 1 and lie along the positive x- and y-axes, respectively, as shown on the graph.

Our objective is to show how $\overrightarrow{OP}$ can be written in terms of $\vec{i}$ and $\vec{j}$. In the diagram, $\overrightarrow{OA} = (a, 0)$ and, since $\overrightarrow{OA}$ is just a scalar multiple of $\vec{i}$, we can write $\overrightarrow{OA} = a\vec{i}$. In a similar way, $\overrightarrow{OB} = b\vec{j}$. Using the triangle law of addition, $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} = a\vec{i} + b\vec{j}$. Since $\overrightarrow{OP} = (a, b)$, it follows that $(a, b) = a\vec{i} + b\vec{j}$.

This means that $\overrightarrow{OP} = (-3, 8)$ can also be written as $\overrightarrow{OP} = -3\vec{i} + 8\vec{j}$. Notice that this result allows us to write *all* vectors in the plane in terms of $\vec{i}$ and $\vec{j}$ and, just as before, their representation is unique.

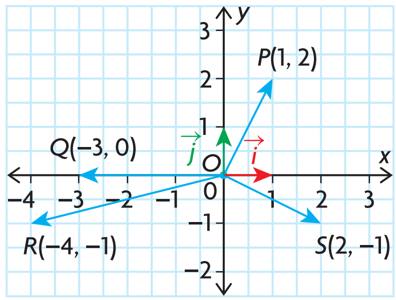
Representation of Vectors in R^2

The position vector $\overrightarrow{OP}$ can be represented as either $\overrightarrow{OP} = (a, b)$ or $\overrightarrow{OP} = a\vec{i} + b\vec{j}$, where $O(0, 0)$ is the origin, $P(a, b)$ is any point on the plane, and $\vec{i}$ and $\vec{j}$ are the standard unit vectors for R^2. Standard unit vectors, $\vec{i}$ and $\vec{j}$, are unit vectors that lie along the x- and y-axes, respectively, so $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$. Every vector in R^2, given in terms of its components, can also be written uniquely in terms of $\vec{i}$ and $\vec{j}$. For this reason, vectors $\vec{i}$ and $\vec{j}$ are also called the standard basis vectors in R^2.

EXAMPLE 1 Representing vectors in R^2 in two equivalent forms

a. Four position vectors, $\overrightarrow{OP} = (1, 2)$, $\overrightarrow{OQ} = (-3, 0)$, $\overrightarrow{OR} = (-4, -1)$, and $\overrightarrow{OS} = (2, -1)$, are shown. Write each of these vectors using the unit vectors $\vec{i}$ and $\vec{j}$.

b. The vectors $\overrightarrow{OA} = -\vec{i}$, $\overrightarrow{OB} = \vec{i} + 5\vec{j}$, $\overrightarrow{OC} = -5\vec{i} + 2\vec{j}$, and $\overrightarrow{OD} = \sqrt{2}\vec{i} - 4\vec{j}$ have been written using the unit vectors $\vec{i}$ and $\vec{j}$. Write them in component form (a, b).



Solution

a. $\overrightarrow{OP} = \vec{i} + 2\vec{j}$, $\overrightarrow{OQ} = -3\vec{i}$, $\overrightarrow{OR} = -4\vec{i} - \vec{j}$, $\overrightarrow{OS} = 2\vec{i} - \vec{j}$

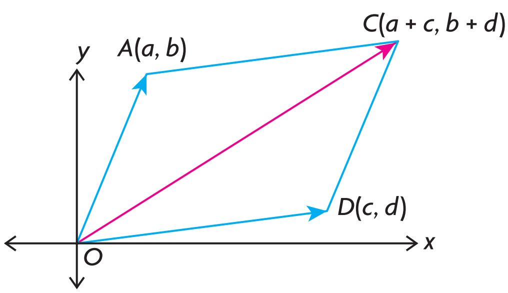
b. $\overrightarrow{OA} = (-1, 0)$, $\overrightarrow{OB} = (1, 5)$, $\overrightarrow{OC} = (-5, 2)$, and $\overrightarrow{OD} = (\sqrt{2}, -4)$

The ability to write vectors using $\vec{i}$ and $\vec{j}$ allows us to develop many of the same results with algebraic vectors that we developed with geometric vectors.

Addition of Two Vectors Using Component Form

We start by drawing the position vectors, $\overrightarrow{OA} = (a, b)$ and $\overrightarrow{OD} = (c, d)$, where A and D are any two points in R^2. For convenience, we choose these two points in the first quadrant. We rewrite each of the two position vectors, $\overrightarrow{OA} = (a, b) = a\vec{i} + b\vec{j}$ and $\overrightarrow{OD} = (c, d) = c\vec{i} + d\vec{j}$.

Adding these vectors gives
$$\begin{aligned}\overrightarrow{OA} + \overrightarrow{OD} &= a\vec{i} + b\vec{j} + c\vec{i} + d\vec{j}\\ &= a\vec{i} + c\vec{i} + b\vec{j} + d\vec{j}\\ &= (a + c)\vec{i} + (b + d)\vec{j}\\ &= (a + c, b + d)\\ &= \overrightarrow{OC}\end{aligned}$$



To find $\overrightarrow{OC}$, it was necessary to use the commutative and distributive properties of vector addition, along with the ability to write vectors in terms of the unit vectors $\vec{i}$ and $\vec{j}$.

To determine the sum of two vectors, $\overrightarrow{OA} = (a, b)$ and $\overrightarrow{OD} = (c, d)$, add their corresponding x- and y-components.
So, $\overrightarrow{OA} + \overrightarrow{OD} = (a, b) + (c, d) = (a + c, b + d) = \overrightarrow{OC}$

The process is similar for subtraction.
$\overrightarrow{OA} - \overrightarrow{OD} = (a, b) - (c, d) = (a - c, b - d)$

Scalar Multiplication of Vectors Using Components

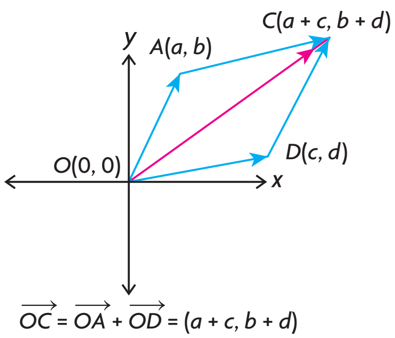
When dealing with geometric vectors, the meaning of multiplying a vector by a scalar was shown. The multiplication of a vector by a scalar in component form has the same meaning. In essence, if $\overrightarrow{OP} = (a, b)$, we wish to know how the coordinates of $m\overrightarrow{OP}$ are determined, where m is a real number. This can be determined by using various distributive properties for scalar multiplication of vectors along with the $\vec{i}$, $\vec{j}$ representation of a vector.

In algebraic form,
$$\begin{aligned} m\overrightarrow{OP} &= m(a, b) \\ &= m(a\vec{i} + b\vec{j}) \\ &= (ma)\vec{i} + (mb)\vec{j} \\ &= (ma, mb) \end{aligned}$$

To multiply an algebraic vector by a scalar, each component of the algebraic vector is multiplied by that scalar.

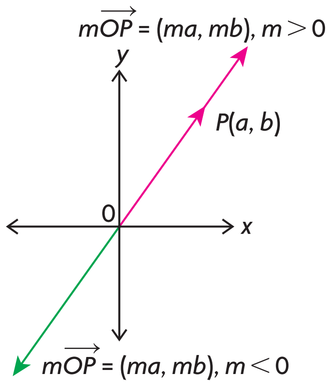
Adding Two Vectors in R^2

To determine the sum of two algebraic vectors, add their corresponding x- and y-components.



Multiplying a Vector by a Scalar in R^2

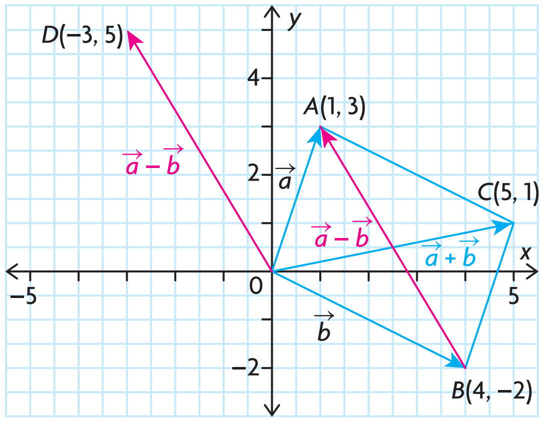
To multiply an algebraic vector by a scalar, multiply both x- and y-components by the scalar.



EXAMPLE 2 Representing the sum and difference of two algebraic vectors in R^2

Given $\vec{a} = \overrightarrow{OA} = (1, 3)$ and $\overrightarrow{OB} = \vec{b} = (4, -2)$, determine the components of $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, and illustrate each of these vectors on the graph.

Solution

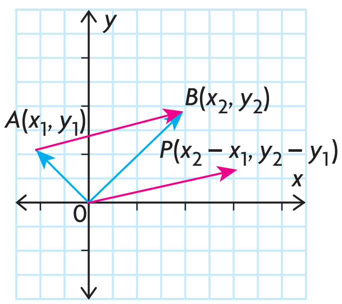


$$\vec{a} + \vec{b} = \overrightarrow{OA} + \overrightarrow{OB} = (1, 3) + (4, -2) = (1 + 4, 3 + (-2)) = (5, 1) = \overrightarrow{OC}$$
$$\vec{a} - \vec{b} = \overrightarrow{OA} - \overrightarrow{OB} = (1, 3) - (4, -2) = (1 - 4, 3 + 2) = (-3, 5) = \overrightarrow{OD}$$

From the diagram, we can see that $\vec{a} + \vec{b}$ and $\overrightarrow{BA}$ represent the diagonals of the parallelogram. It should be noted that the position vector, $\overrightarrow{OD}$, is a vector that is equivalent to diagonal $\overrightarrow{BA}$. The vector $\overrightarrow{OD} = \vec{a} - \vec{b}$ is described as a position vector because it has its tail at the origin and is equivalent to $\overrightarrow{BA}$, since their magnitudes are the same and they have the same direction.

Vectors in R^2 Defined by Two Points

In considering the vector $\overrightarrow{AB}$, determined by the points $A(x_1, y_1)$ and $B(x_2, y_2)$, an important consideration is to be able to find its related position vector and to calculate $|\overrightarrow{AB}|$. In order to do this, we use the triangle law of addition. From the diagram on the left, $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$, and $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2, y_2) - (x_1, y_1) = (x_2 - x_1, y_2 - y_1)$. Thus, the components of the algebraic vector are found by subtracting the coordinates of its tail from the coordinates of its head.



$\overrightarrow{OP}$ is the position vector for $\overrightarrow{AB}$.

To determine $|\overrightarrow{AB}|$, use the Pythagorean theorem.

$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The formula for determining $|\overrightarrow{AB}|$ is the same as the formula for finding the distance between two points.

Position Vectors and Magnitudes in R^2

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points, then the vector $\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1)$ is its related position vector $\overrightarrow{OP}$, and $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

EXAMPLE 3

Using algebraic vectors to solve a problem

$A(-3, 7)$, $B(5, 22)$, and $C(8, 18)$ are three points in R^2.

a. Calculate the value of $|\overrightarrow{AB}| + |\overrightarrow{BC}| + |\overrightarrow{CA}|$, the perimeter of triangle ABC.

b. Calculate the value of $|\overrightarrow{AB} + \overrightarrow{BC}|$.

Solution

a. Calculate a position vector for each of the three sides.

$\overrightarrow{AB} = (5 - (-3), 22 - 7) = (8, 15)$, $\overrightarrow{BC} = (8 - 5, 18 - 22) = (3, -4)$, and $\overrightarrow{CA} = (-3 - 8, 7 - 18) = (-11, -11)$

$|\overrightarrow{AB}| = \sqrt{8^2 + 15^2} = \sqrt{289} = 17$, $|\overrightarrow{BC}| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$, and $|\overrightarrow{CA}| = \sqrt{(-11)^2 + (-11)^2} = \sqrt{121 + 121} = \sqrt{242} \doteq 15.56$

The perimeter of the triangle is approximately 37.56.

b. Since $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$, $\overrightarrow{AC} = -\overrightarrow{CA} = (11, 11)$, and $|\overrightarrow{AC}| = \sqrt{11^2 + 11^2} = \sqrt{242}$, then $|\overrightarrow{AC}| \doteq 15.56$. Note that $|\overrightarrow{AC}| = |\overrightarrow{CA}| \doteq 15.56$.

EXAMPLE 4

Selecting a strategy to combine two vectors

For the vectors $\vec{x} = 2\vec{i} - 3\vec{j}$ and $\vec{y} = -4\vec{i} - 3\vec{j}$, determine $|\vec{x} + \vec{y}|$ and $|\vec{x} - \vec{y}|$.

Solution

Method 1: (Component Form)

Since $\vec{x} = 2\vec{i} - 3\vec{j}$, $\vec{x} = (2, -3)$. Similarly, $\vec{y} = (-4, -3)$.

The sum is $\vec{x} + \vec{y} = (2, -3) + (-4, -3) = (-2, -6)$.

The difference is $\vec{x} - \vec{y} = (2, -3) - (-4, -3) = (6, 0)$.

Method 2: (Standard Unit Vectors)

The sum is
$\vec{x} + \vec{y} = (2\vec{i} - 3\vec{j}) + (-4\vec{i} - 3\vec{j}) = (2 - 4)\vec{i} + (-3 - 3)\vec{j} = -2\vec{i} - 6\vec{j}$.

The difference is
$\vec{x} - \vec{y} = (2\vec{i} - 3\vec{j}) - (-4\vec{i} - 3\vec{j}) = (2 + 4)\vec{i} + (-3 + 3)\vec{j} = 6\vec{i}$.

Thus, $|\vec{x} + \vec{y}| = \sqrt{(-2)^2 + (-6)^2} = \sqrt{40} \doteq 6.32$ and
$|\vec{x} - \vec{y}| = \sqrt{6^2} = \sqrt{36} = 6$.

EXAMPLE 5 Calculating the magnitude of a vector in R^2

If $\vec{a} = (5, -6)$, $\vec{b} = (-7, 3)$, and $\vec{c} = (2, 8)$, calculate $\left|\vec{a} - 3\vec{b} - \frac{1}{2}\vec{c}\right|$.

Solution

$$\vec{a} - 3\vec{b} - \frac{1}{2}\vec{c} = (5, -6) - 3(-7, 3) - \frac{1}{2}(2, 8)$$
$$= (5, -6) + (21, -9) + (-1, -4) = (25, -19)$$

Thus, $\left|\vec{a} - 3\vec{b} - \frac{1}{2}\vec{c}\right| = \sqrt{25^2 + (-19)^2} = \sqrt{625 + 361} = \sqrt{986} \doteq 31.40$

IN SUMMARY

Key Ideas

- In R^2, $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$. Both are unit vectors on the x- and y-axes, respectively.
- $\overrightarrow{OP} = (a, b) = a\vec{i} + b\vec{j}$, $|\overrightarrow{OP}| = \sqrt{a^2 + b^2}$
- The vector between two points with its tail at $A(x_1, y_1)$ and head at $B(x_2, y_2)$ is determined as follows:

  $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2, y_2) - (x_1, y_1) = (x_2 - x_1, y_2 - y_1)$
- The vector $\overrightarrow{AB}$ is equivalent to the position vector $\overrightarrow{OP}$ since their directions and magnitude are the same: $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Need to Know

- If $\overrightarrow{OA} = (a, b) = a\vec{i} + b\vec{j}$ and $\overrightarrow{OD} = (c, d) = c\vec{i} + d\vec{j}$, then $\overrightarrow{OA} + \overrightarrow{OD} = (a + c, b + d)$.
- $m\overrightarrow{OP} = m(a, b) = (ma, mb)$

Exercise 6.6

PART A

1. For $A(-1, 3)$ and $B(2, 5)$, draw a coordinate plane and place the points on the graph.
   a. Draw vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$, and give vectors in component form equivalent to each of these vectors.
   b. Determine $|\overrightarrow{OA}|$ and $|\overrightarrow{OB}|$.
   c. Calculate $|\overrightarrow{AB}|$ and state the value of $|\overrightarrow{BA}|$.

2. Draw the vector $\overrightarrow{OA}$ on a graph, where point A has coordinates $(6, 10)$.

   a. Draw the vectors $m\overrightarrow{OA}$, where $m = \frac{1}{2}, \frac{-1}{2}, 2$, and -2.

   b. Which of these vectors have the same magnitude?

3. For the vector $\overrightarrow{OA} = 3\vec{i} - 4\vec{j}$, calculate $|\overrightarrow{OA}|$.

4. a. If $a\vec{i} + 5\vec{j} = (-3, b)$, determine the values of a and b.

   b. Calculate $|(-3, b)|$ after finding b.

5. If $\vec{a} = (-60, 11)$ and $\vec{b} = (-40, -9)$, calculate each of the following:

   a. $|\vec{a}|$ and $|\vec{b}|$ b. $|\vec{a} + \vec{b}|$ and $|\vec{a} - \vec{b}|$

PART B

6. Find a single vector equivalent to each of the following:

   a. $2(-2, 3) + (2, 1)$ b. $-3(4, -9) - 9(2, 3)$ c. $\frac{-1}{2}(6, -2) + \frac{2}{3}(6, 15)$

K

7. Given $\vec{x} = 2\vec{i} - \vec{j}$ and $\vec{y} = -\vec{i} + 5\vec{j}$, find a vector equivalent to each of the following:

   a. $3\vec{x} - \vec{y}$

   b. $-(\vec{x} + 2\vec{y}) + 3(-\vec{x} - 3\vec{y})$

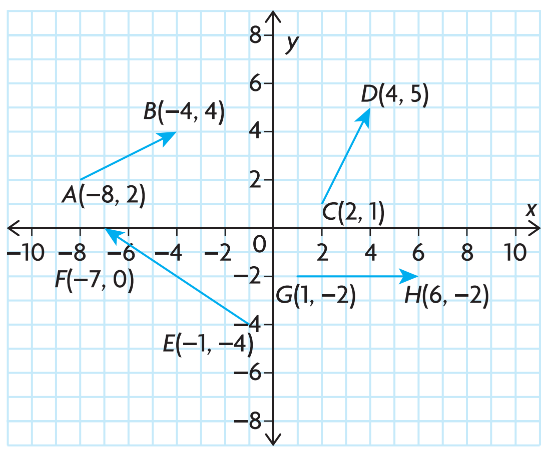
   c. $2(\vec{x} + 3\vec{y}) - 3(\vec{y} + 5\vec{x})$

8. Using $\vec{x}$ and $\vec{y}$ given in question 7, determine each of the following:

   a. $|\vec{x} + \vec{y}|$ b. $|\vec{x} - \vec{y}|$ c. $|2\vec{x} - 3\vec{y}|$ d. $|3\vec{y} - 2\vec{x}|$

9. a. For each of the vectors shown below, determine the components of the related position vector.

   b. Determine the magnitude of each vector.



A 10. Parallelogram $OBCA$ is determined by the vectors $\overrightarrow{OA} = (6, 3)$ and $\overrightarrow{OB} = (11, -6)$.

a. Determine $\overrightarrow{OC}$, $\overrightarrow{BA}$, and $\overrightarrow{BC}$.

b. Verify that $|\overrightarrow{OA}| = |\overrightarrow{BC}|$.

11. $\triangle ABC$ has vertices at $A(2, 3)$, $B(6, 6)$, and $C(-4, 11)$.

a. Sketch and label each of the points on a graph.

b. Calculate each of the lengths $|\overrightarrow{AB}|$, $|\overrightarrow{AC}|$, and $|\overrightarrow{CB}|$.

c. Verify that triangle ABC is a right triangle.

12. A parallelogram has three of its vertices at $A(-1, 2)$, $B(7, -2)$, and $C(2, 8)$.

a. Draw a grid and locate each of these points.

b. On your grid, draw the three locations for a fourth point that would make a parallelogram with points A, B, and C.

c. Determine all possible coordinates for the point described in part b.

13. Determine the value of x and y in each of the following:

a. $3(x, 1) - 5(2, 3y) = (11, 33)$

b. $-2(x, x + y) - 3(6, y) = (6, 4)$

C 14. Rectangle $ABCD$ has vertices at $A(2, 3)$, $B(-6, 9)$, $C(x, y)$, and $D(8, 11)$.

a. Draw a sketch of the points A, B, and D, and locate point C on your graph.

b. Explain how you can determine the coordinates of point C.

T 15. $A(5, 0)$ and $B(0, 2)$ are points on the x- and y-axes, respectively.

a. Find the coordinates of point $P(a, 0)$ on the x-axis such that $|\overrightarrow{PA}| = |\overrightarrow{PB}|$.

b. Find the coordinates of a point on the y-axis such that $|\overrightarrow{QB}| = |\overrightarrow{QA}|$.

PART C

16. Find the components of the unit vector in the direction opposite to $\overrightarrow{PQ}$, where $\overrightarrow{OP} = (11, 19)$ and $\overrightarrow{OQ} = (2, -21)$.

17. Parallelogram $OPQR$ is such that $\overrightarrow{OP} = (-7, 24)$ and $\overrightarrow{OR} = (-8, -1)$.

a. Determine the angle between the vectors $\overrightarrow{OR}$ and $\overrightarrow{OP}$.

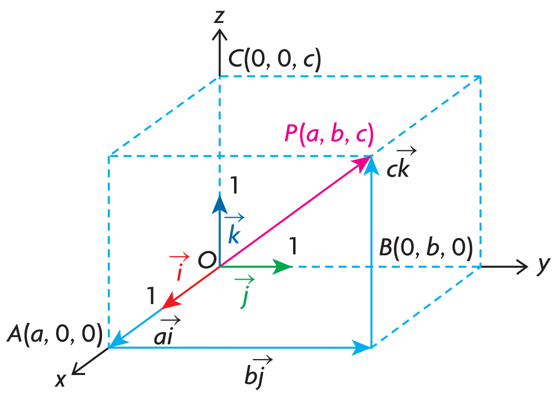
b. Determine the acute angle between the diagonals $\overrightarrow{OQ}$ and $\overrightarrow{RP}$.

Section 6.7—Operations With Vectors in R^3

The most important applications of vectors occur in R^3. In this section, results will be developed that will allow us to begin to apply ideas in R^3.

Defining a Vector in R^3 in Terms of Unit Vectors

In R^2, the vectors $\vec{i}$ and $\vec{j}$ were chosen as basis vectors. In R^3, the vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ were chosen as basis vectors. These are vectors that each have magnitude 1, but now we introduce $\vec{k}$ as a vector that lies along the positive z-axis. If we use the same reasoning applied for two dimensions, then it can be seen that each vector $\overrightarrow{OP} = (a, b, c)$ can be written as $\overrightarrow{OP} = a\vec{i} + b\vec{j} + c\vec{k}$. Each of the vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ are shown below, as well as $\overrightarrow{OP} = (a, b, c)$.



From the diagram, $\overrightarrow{OA} = a\vec{i}$, $\overrightarrow{OB} = b\vec{j}$, and $\overrightarrow{OC} = c\vec{k}$. Using the triangle law of addition, $\overrightarrow{OP} = a\vec{i} + b\vec{j} + c\vec{k}$. Since $\overrightarrow{OP} = (a, b, c)$, we conclude that $\overrightarrow{OP} = a\vec{i} + b\vec{j} + c\vec{k} = (a, b, c)$. This result is analogous to the result derived for R^2.

Representation of Vectors in R^3

The position vector, $\overrightarrow{OP}$, whose tail is at the origin and whose head is located at point P, can be represented as either $\overrightarrow{OP} = (a, b, c)$ or $\overrightarrow{OP} = a\vec{i} + b\vec{j} + c\vec{k}$, where $O(0, 0, 0)$ is the origin, $P(a, b, c)$ is a point in R^3, and $\vec{i}$, $\vec{j}$, and $\vec{k}$ are the standard unit vectors along the x-, y- and z- axes, respectively. This means that $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$, and $\vec{k} = (0, 0, 1)$. Every vector in R^3 can be expressed uniquely in terms of $\vec{i}$, $\vec{j}$, and $\vec{k}$.

EXAMPLE 1 Representing vectors in R^3 in two equivalent forms

a. Write each of the vectors $\vec{OP} = (2, 1, -3)$, $\vec{OQ} = (-3, 1, -5)$, $\vec{OR} = (0, -2, 0)$, and $\vec{OS} = (3, 0, 0)$ using the standard unit vectors.

b. Express each of the following vectors in component form: $\vec{OP} = \vec{i} - 2\vec{j} - \vec{k}$, $\vec{OS} = 3\vec{k}$, $\vec{OM} = 2\vec{i} - 6\vec{k}$, and $\vec{ON} = \vec{i} - \vec{j} - 7\vec{k}$.

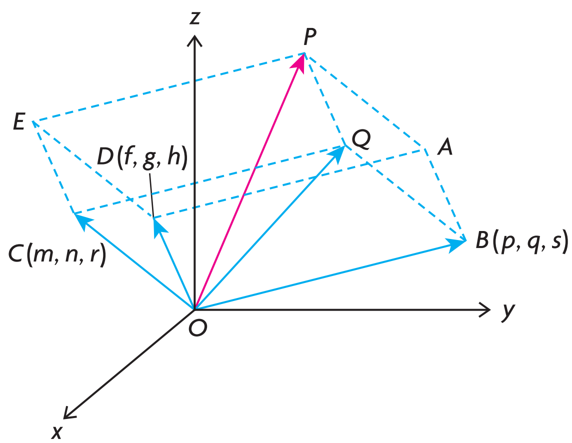
Solution

a. $\vec{OP} = 2\vec{i} + \vec{j} - 3\vec{k}$, $\vec{OQ} = -3\vec{i} + \vec{j} - 5\vec{k}$, $\vec{OR} = -2\vec{j}$, and $\vec{OS} = 3\vec{i}$

b. $\vec{OP} = (1, -2, -1)$, $\vec{OS} = (0, 0, 3)$, $\vec{OM} = (2, 0, -6)$, and $\vec{ON} = (1, -1, -7)$

In R^2, we showed how to add two algebraic vectors. The result in R^3 is analogous to this result.

Addition of Three Vectors in R^3



Writing each of the three given position vectors in terms of the standard basis vectors, $\vec{OB} = p\vec{i} + q\vec{j} + s\vec{k}$, $\vec{OC} = m\vec{i} + n\vec{j} + r\vec{k}$, and $\vec{OD} = f\vec{i} + g\vec{j} + h\vec{k}$.

Using the parallelogram law of addition, $\vec{OP} = \vec{OD} + \vec{OQ}$ and $\vec{OQ} = \vec{OB} + \vec{OC}$.

Substituting, $\vec{OP} = \vec{OD} + (\vec{OB} + \vec{OC})$.

Therefore,

$$\begin{aligned}
\vec{OP} &= (f\vec{i} + g\vec{j} + h\vec{k}) + ((p\vec{i} + q\vec{j} + s\vec{k}) + (m\vec{i} + n\vec{j} + r\vec{k})) \\
&= (f\vec{i} + p\vec{i} + m\vec{i}) + (g\vec{j} + q\vec{j} + n\vec{j}) + (h\vec{k} + s\vec{k} + r\vec{k}) \\
&= (f + p + m)\vec{i} + (g + q + n)\vec{j} + (h + s + r)\vec{k} \\
&= (f + p + m, g + q + n, h + s + r)
\end{aligned}$$

(Commutative and associative properties of vector addition)

(Distributive property of scalars)

This result demonstrates that the method for adding algebraic vectors in R^3 is the same as in R^2. Adding two vectors means adding their respective components. It should also be noted that the result for the subtraction of vectors in R^3 is analogous to the result in R^2. If $\overrightarrow{OA} = (a_1, a_2, a_3)$ and $\overrightarrow{OB} = (b_1, b_2, b_3)$, then $\overrightarrow{OA} - \overrightarrow{OB} = (a_1, a_2, a_3) - (b_1, b_2, b_3) = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$.

In R^3, the shape that was used to generate the result for the addition of three vectors was not a parallelogram but a *parallelepiped*, which is a box-like shape with pairs of opposite faces being identical parallelograms. From our diagram, it can be seen that parallelograms *ODAB* and *CEPQ* are copies of each other. It is also interesting to note that the parallelepiped is completely determined by the components of the three position vectors $\overrightarrow{OB}$, $\overrightarrow{OC}$, and $\overrightarrow{OD}$. That is to say, the coordinates of *all* the vertices of the parallelepiped can be determined by the repeated application of the Triangle Law of Addition.

For vectors in R^2, we showed that the multiplication of an algebraic vector by a scalar was produced by multiplying each component of the vector by the scalar. In R^3, this result also holds, i.e., $m\overrightarrow{OP} = m(a, b, c) = (ma, mb, mc), m \in \mathbf{R}$.

EXAMPLE 2 Selecting a strategy to determine a combination of vectors in R^3

Given $\vec{a} = -\vec{i} + 2\vec{j} + \vec{k}$, $\vec{b} = 2\vec{j} - 3\vec{k}$, and $\vec{c} = \vec{i} - 3\vec{j} + 2\vec{k}$, determine each of the following:

a. $2\vec{a} - \vec{b} + \vec{c}$ b. $\vec{a} + \vec{b} + \vec{c}$

Solution

a. *Method 1* (Standard Unit Vectors)

$$\begin{aligned}2\vec{a} - \vec{b} + \vec{c} &= 2(-\vec{i} + 2\vec{j} + \vec{k}) - (2\vec{j} - 3\vec{k}) + (\vec{i} - 3\vec{j} + 2\vec{k})\\ &= -2\vec{i} + 4\vec{j} + 2\vec{k} - 2\vec{j} + 3\vec{k} + \vec{i} - 3\vec{j} + 2\vec{k}\\ &= -2\vec{i} + \vec{i} + 4\vec{j} - 2\vec{j} - 3\vec{j} + 2\vec{k} + 3\vec{k} + 2\vec{k}\\ &= -\vec{i} - \vec{j} + 7\vec{k}\\ &= (-1, -1, 7)\end{aligned}$$

Method 2 (Components)

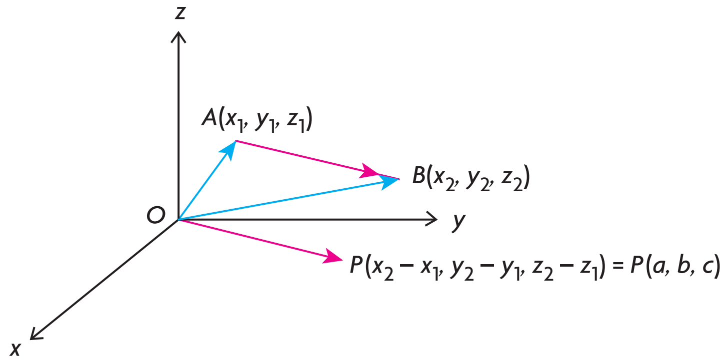
Converting to component form, we have $\vec{a} = (-1, 2, 1)$, $\vec{b} = (0, 2, -3)$, and $\vec{c} = (1, -3, 2)$.

$$\begin{aligned}\text{Therefore, } 2\vec{a} - \vec{b} + \vec{c} &= 2(-1, 2, 1) - (0, 2, -3) + (1, -3, 2)\\ &= (-2, 4, 2) + (0, -2, 3) + (1, -3, 2)\\ &= (-2 + 0 + 1, 4 - 2 - 3, 2 + 3 + 2)\\ &= (-1, -1, 7)\\ &= -\vec{i} - \vec{j} + 7\vec{k}\end{aligned}$$

b. Using components, $\vec{a} + \vec{b} + \vec{c} = (-1, 2, 1) + (0, 2, -3) + (1, -3, 2)$
$$= (-1 + 0 + 1, 2 + 2 + (-3), 1 + (-3) + 2)$$
$$= (0, 1, 0) = \vec{j}$$

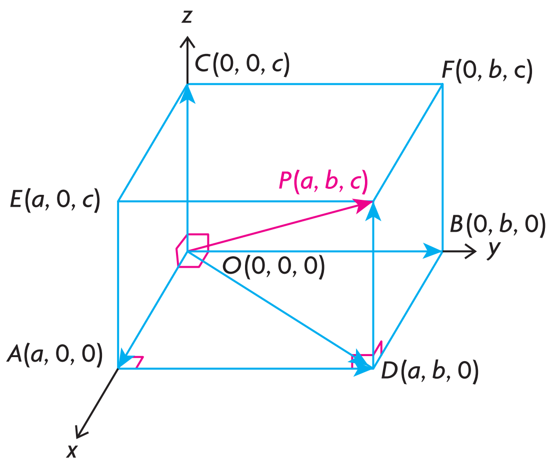
Vectors in R^3 Defined by Two Points

Position vectors and their magnitude in R^3 are calculated in a manner similar to R^2.



To determine the components of $\overrightarrow{AB}$, the same method is used in R^3 as was used in R^2, i.e., $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$, which implies that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$, or, in component form, $\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$. Thus, the components of the algebraic vector $\overrightarrow{AB}$ can be found by subtracting the coordinates of point A from the coordinates of point B.

If P has coordinates (a, b, c), we can calculate the magnitude of $\overrightarrow{OP}$.



From the diagram, we first note that $|\overrightarrow{OA}| = |a|$, $|\overrightarrow{OB}| = |b|$, and $|\overrightarrow{OC}| = |c|$. We also observe, using the Pythagorean theorem, that $|\overrightarrow{OP}|^2 = |\overrightarrow{OD}|^2 + |\overrightarrow{OC}|^2$ and, since $|\overrightarrow{OD}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2$, substitution gives $|\overrightarrow{OP}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2 + |\overrightarrow{OC}|^2$.

Writing this expression in its more familiar coordinate form, we get $|\overrightarrow{OP}|^2 = |a|^2 + |b|^2 + |c|^2$ or $|\overrightarrow{OP}| = \sqrt{|a|^2 + |b|^2 + |c|^2}$. The use of the absolute value signs in the formula guarantees that the components are positive before they are squared. Because squaring components guarantees the result will be positive, it would have been just as easy to write the formula as $|\overrightarrow{OP}| = \sqrt{a^2 + b^2 + c^2}$, which gives an identical result.

Position Vectors and Magnitude in R^3

If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two points, then the vector $|\overrightarrow{AB}| = (x_2 - x_1, y_2 - y_1, z_2 - z_1) = (a, b, c)$ is equivalent to the related position vector, $\overrightarrow{OP}$, and
$|\overrightarrow{AB}| = |\overrightarrow{OP}| = \sqrt{a^2 + b^2 + c^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

EXAMPLE 3 Connecting vectors in R^3 with their components

If $A(7, -11, 13)$ and $B(4, -7, 25)$ are two points in R^3, determine each of the following:

a. $|\overrightarrow{OA}|$ b. $|\overrightarrow{OB}|$ c. $\overrightarrow{AB}$ d. $|\overrightarrow{AB}|$

Solution

a. $|\overrightarrow{OA}| = \sqrt{7^2 + (-11)^2 + 13^2} = \sqrt{339} \doteq 18.41$

b. $|\overrightarrow{OB}| = \sqrt{4^2 + (-7)^2 + 25^2} = \sqrt{690} \doteq 26.27$

c. $\overrightarrow{AB} = (4 - 7, -7 - (-11), 25 - 13) = (-3, 4, 12)$

d. $|\overrightarrow{AB}| = \sqrt{(-3)^2 + 4^2 + 12^2} = \sqrt{169} = 13$

In this section, we developed further properties of algebraic vectors. In the next section, we will demonstrate how these properties can be used to understand the geometry of R^3.

IN SUMMARY

Key Ideas

- In R^3, $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$, $\vec{k} = (0, 0, 1)$. All are unit vectors along the x-, y- and z-axes, respectively.
- $\overrightarrow{OP} = (a, b, c) = a\vec{i} + b\vec{j} + c\vec{k}$, $|\overrightarrow{OP}| = \sqrt{a^2 + b^2 + c^2}$
- The vector between two points with its tail at $A(x_1, y_1, z_1)$ and head at $B(x_1, y_1, z_1)$ is determined as follows:
  $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2, y_2, z_2) - (x_1, y_1, z_1) = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$
- The vector $\overrightarrow{AB}$ is equivalent to the position vector $\overrightarrow{OP}$ since their directions and magnitude are the same.
  $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Need to Know

- If $\overrightarrow{OA} = (a, b, c)$ and $\overrightarrow{OD} = (d, e, f)$, then $\overrightarrow{OA} + \overrightarrow{OD} = (a + d, b + e, c + f)$.
- $m\overrightarrow{OP} = m(a, b, c) = (ma, mb, mc)$, $m \in \mathbf{R}$

Exercises 6.7

PART A

1. a. Write the vector $\overrightarrow{OA} = (-1, 2, 4)$ using the standard unit vectors.
   b. Determine $|\overrightarrow{OA}|$.

2. Write the vector $\overrightarrow{OB} = 3\vec{i} + 4\vec{j} - 4\vec{k}$ in component form and calculate its magnitude.

3. If $\vec{a} = (1, 3, -3)$, $\vec{b} = (-3, 6, 12)$, and $\vec{c} = (0, 8, 1)$, determine $\left|\vec{a} + \frac{1}{3}\vec{b} - \vec{c}\right|$.

4. For the vectors $\overrightarrow{OA} = (-3, 4, 12)$ and $\overrightarrow{OB} = (2, 2, -1)$, determine the following:
   a. the components of vector $\overrightarrow{OP}$, where $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB}$
   b. $|\overrightarrow{OA}|$, $|\overrightarrow{OB}|$, and $|\overrightarrow{OP}|$
   c. $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$. What does $\overrightarrow{AB}$ represent?

PART B

K 5. Given $\vec{x} = (1, 4, -1)$, $\vec{y} = (1, 3, -2)$, and $\vec{z} = (-2, 1, 0)$, determine a vector equivalent to each of the following:

a. $\vec{x} - 2\vec{y} - \vec{z}$

b. $-2\vec{x} - 3\vec{y} + \vec{z}$

c. $\frac{1}{2}\vec{x} - \vec{y} + 3\vec{z}$

d. $3\vec{x} + 5\vec{y} + 3\vec{z}$

6. Given $\vec{p} = 2\vec{i} - \vec{j} + \vec{k}$ and $\vec{q} = -\vec{i} - \vec{j} + \vec{k}$, determine the following in terms of the standard unit vectors.

a. $\vec{p} + \vec{q}$ b. $\vec{p} - \vec{q}$ c. $2\vec{p} - 5\vec{q}$ d. $-2\vec{p} + 5\vec{q}$

7. If $\vec{m} = 2\vec{i} - \vec{k}$ and $\vec{n} = -2\vec{i} + \vec{j} + 2\vec{k}$, calculate each of the following:

a. $|\vec{m} - \vec{n}|$ b. $|\vec{m} + \vec{n}|$ c. $|2\vec{m} + 3\vec{n}|$ d. $|-5\vec{m}|$

8. Given $\vec{x} + \vec{y} = -\vec{i} + 2\vec{j} + 5\vec{k}$ and $\vec{x} - \vec{y} = 3\vec{i} + 6\vec{j} - 7\vec{k}$, determine $\vec{x}$ and $\vec{y}$.

C 9. Three vectors, $\overrightarrow{OA} = (a, b, 0)$, $\overrightarrow{OB} = (a, 0, c)$, and $\overrightarrow{OC} = (0, b, c)$, are given.

a. In a sentence, describe what each vector represents.

b. Write each of the given vectors using the standard unit vectors.

c. Determine a formula for each of $|\overrightarrow{OA}|$, $|\overrightarrow{OB}|$, and $|\overrightarrow{OC}|$.

d. Determine $\overrightarrow{AB}$. What does $\overrightarrow{AB}$ represent?

10. Given the points $A(-2, -6, 3)$ and $B(3, -4, 12)$, determine each of the following:

a. $|\overrightarrow{OA}|$

b. $|\overrightarrow{OB}|$

c. $\overrightarrow{AB}$

d. $|\overrightarrow{AB}|$

e. $\overrightarrow{BA}$

f. $|\overrightarrow{BA}|$

11. The vertices of quadrilateral $ABCD$ are given as $A(0, 3, 5)$, $B(3, -1, 17)$, $C(7, -3, 15)$, and $D(4, 1, 3)$. Prove that $ABCD$ is a parallelogram.

12. Given $2\vec{x} + \vec{y} - 2\vec{z} = \vec{0}$, $\vec{x} = (-1, b, c)$, $\vec{y} = (a, -2, c)$, and $\vec{z} = (-a, 6, c)$, determine the value of the unknowns.

A 13. A parallelepiped is determined by the vectors $\overrightarrow{OA} = (-2, 2, 5)$, $\overrightarrow{OB} = (0, 4, 1)$, and $\overrightarrow{OC} = (0, 5, -1)$.

a. Draw a sketch of the parallelepiped formed by these vectors.

b. Determine the coordinates of all of the vertices for the parallelepiped.

T 14. Given the points $A(-2, 1, 3)$ and $B(4, -1, 3)$, determine the coordinates of the point on the x-axis that is equidistant from these two points.

PART C

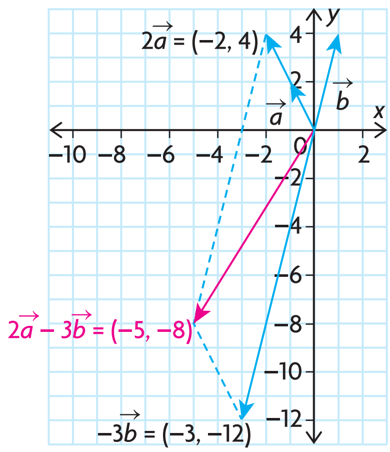
15. Given $|\vec{a}| = 3$, $|\vec{b}| = 5$, and $|\vec{a} + \vec{b}| = 7$, determine $|\vec{a} - \vec{b}|$.

Section 6.8—Linear Combinations and Spanning Sets

We have discussed concepts involving geometric and algebraic vectors in some detail. In this section, we are going to use these ideas as a basis for understanding the notion of a **linear combination**, an important idea for understanding the geometry of three dimensions.

Examining Linear Combinations of Vectors in R^2

We'll begin by considering linear combinations in R^2. If we consider the vectors $\vec{a} = (-1, 2)$ and $\vec{b} = (1, 4)$ and write $2(-1, 2) - 3(1, 4) = (-5, -8)$, then the expression on the left side of this equation is called a linear combination. In this case, the linear combination produces the vector $(-5, -8)$. Whenever vectors are multiplied by scalars and then added, the result is a new vector that is a linear combination of the vectors. If we take the two vectors $\vec{a} = (-1, 2)$ and $\vec{b} = (1, 4)$, then $2\vec{a} - 3\vec{b}$ is a vector on the xy-plane and is the diagonal of the parallelogram formed by the vectors $2\vec{a}$ and $-3\vec{b}$, as shown in the diagram.



Linear Combination of Vectors

For noncollinear vectors, $\vec{u}$ and $\vec{v}$, a linear combination of these vectors is $a\vec{u} + b\vec{v}$, where a and b are scalars (real numbers). The vector $a\vec{u} + b\vec{v}$ is the diagonal of the parallelogram formed by the vectors $a\vec{u}$ and $b\vec{v}$.

It was shown that every vector in the xy-plane can be written uniquely in terms of the unit vectors $\vec{i}$ and $\vec{j}$. $\overrightarrow{OP} = (a, b) = a\vec{i} + b\vec{j}$, where $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$. This can be done in only one way. Writing $\overrightarrow{OP}$ in this way is really just writing this vector as a linear combination of $\vec{i}$ and $\vec{j}$. Because every vector in R^2 can be written as a linear combination of these two vectors, we say that $\vec{i}$ and $\vec{j}$ span R^2. Another way of stating this is to say that the set of vectors $\{\vec{i}, \vec{j}\}$ forms a **spanning set** for R^2.

Spanning Set for R^2

The set of vectors $\{\vec{i}, \vec{j}\}$ is said to form a spanning set for R^2. Every vector in R^2 can be written uniquely as a linear combination of these two vectors.

What is interesting about spanning sets is that there is not just one set of vectors that can be used to span R^2. There is an infinite number of sets, each set containing a minimum of exactly two vectors, that would serve the same purpose. The concepts of span and spanning set will prove significant for the geometry of planes studied in Chapter 8.

EXAMPLE 1 Representing a vector as a linear combination of two other vectors

Show that $\vec{x} = (4, 23)$ can be written as a linear combination of either set of vectors, $\{(-1, 4), (2, 5)\}$ or $\{(1, 0), (-2, 1)\}$.

Solution

In each case, the procedure is the same, and so we will show the details for just one set of calculations. We are looking for solutions to the following separate equations: $a(-1, 4) + b(2, 5) = (4, 23)$ and $c(1, 0) + d(-2, 1) = (4, 23)$.

Multiplying, $(-a, 4a) + (2b, 5b) = (4, 23)$ (Properties of scalar multiplication)

$(-a + 2b, 4a + 5b) = (4, 23)$ (Properties of vector addition)

Since the vector on the left side is equal to that on the right side, we can write

① $-a + 2b = 4$

② $4a + 5b = 23$

This forms a linear system that can be solved using the method of elimination.

③ $-4a + 8b = 16$, after multiplying equation ① by 4.

Adding equation ② and equation ③ gives, $13b = 39$, so $b = 3$ and, by substitution, $a = 2$.

Therefore, $2(-1, 4) + 3(2, 5) = (4, 23)$.
The calculations for the second linear combination are done in the same way as the first, and so $c - 2d = 4$ and $d = 23$. Substituting gives $c = 50$ and $d = 23$. Therefore, $50(1, 0) + 23(-2, 1) = (4, 23)$. You should verify the calculations on your own.

In R^2, it is possible to take any pair of noncollinear (non-parallel) vectors as a spanning set, provided that $(0, 0)$ is not one of the two vectors.

EXAMPLE 2

Reasoning about spanning sets in R^2

Show that the set of vectors $\{(2, 3), (4, 6)\}$ does not span R^2.

Solution

Since these vectors are scalar multiples of each other, i.e., $(4, 6) = 2(2, 3)$, they cannot span R^2. All linear combinations of these two vectors produce only vectors that are scalar multiples of (2, 3). This is shown by the following calculation:

$$a(2, 3) + b(4, 6) = (2a + 4b, 3a + 6b) = (2(a + 2b), 3(a + 2b))$$
$$= (a + 2b)(2, 3)$$

This result means that we cannot use linear combinations of the set of vectors $\{(2, 3), (4, 6)\}$ to obtain anything but a multiple of $(2, 3)$. As a result, the only vectors that can be created are ones in either the same or opposite direction of (2, 3). There is no linear combination of these vectors that would allow us to obtain, for example, the vector $(3, 4)$.

When we say that a set of vectors spans R^2, we are saying that every vector in the plane can be written as a linear combination of the two given vectors. In Example 1, we did not prove that either set of vectors was a spanning set. All that we showed was that the given vector could be written as a linear combination of a set of vectors. It is true in this case, however, that both sets do span R^2. In the following example, it will be shown how a set of vectors in R^2 can be proven to be a spanning set.

EXAMPLE 3

Proving that a given set of vectors spans R^2

Show that the set of vectors $\{(2, 1), (-3, -1)\}$ is a spanning set for R^2.

Solution

In order to show that the set spans R^2, we write the linear combination $a(2, 1) + b(-3, -1) = (x, y)$, where (x, y) represents *any* vector in R^2. Carrying out the same procedure as in the previous example, we obtain

① $2a - 3b = x$

② $a - b = y$

Again the process of elimination will be used to solve this system of equations.

① $2a - 3b = x$

③ $2a - 2b = 2y$, after multiplying equation ② by 2

Subtracting eliminates a, $-3b - (-2b) = x - 2y$

Therefore, $-b = x - 2y$ or $b = -x + 2y$. By substituting this value of b into equation ②, we find $a = -x + 3y$. Therefore, the solution to this system of equations is $a = -x + 3y$ and $b = -x + 2y$.

This means that, whenever we are given the components of any vector, we can find the corresponding values of a and b by substituting into the formula. Since the values of x and y are unique, the corresponding values of a and b are also unique. Using this formula to write $(-3, 7)$ as a linear combination of the two given vectors, we would say $x = -3,\ y = 7$ and solve for a and b to obtain

$$a = -(-3) + 3(7) = 24$$

and

$$b = -(-3) + 2(7) = 17$$

$24(2, 1) + 17(-3, -1) = (-3, 7)$

So the vector $(-3, 7)$ can be written as a linear combination of $(2, 1)$ and $(-3, -1)$. Therefore, the set of vectors $\{(2, 1), (-3, -1)\}$ spans R^2.

Examining Linear Combinations of Vectors in R^3

In the previous section, the set of unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ was introduced as unit vectors lying along the positive x-, y-, and z-axes, respectively. This set of vectors is referred to as the standard basis for R^3, meaning that every vector in R^3 can be written uniquely as a linear combination of these three vectors. (It should be pointed out that there is an infinite number of sets containing three vectors that could also be used as a basis for R^3.)

EXAMPLE 4 Representing linear combinations in R^3

Show that the vector $(2, 3, -5)$ can be written as a linear combination of $\vec{i}$, $\vec{j}$, and $\vec{k}$ and illustrate this geometrically.

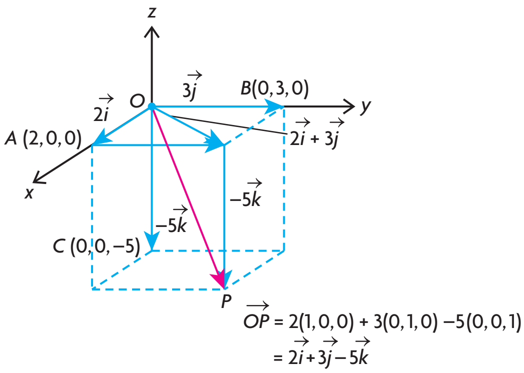
Solution

Writing the given vector as a linear combination

$$\begin{aligned}(2, 3, -5) &= 2(1, 0, 0) + 3(0, 1, 0) - 5(0, 0, 1)\\ &= 2\vec{i} + 3\vec{j} - 5\vec{k}\end{aligned}$$

This is exactly what we would expect based on the work in the previous section.

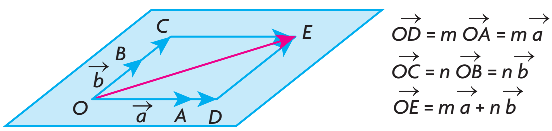
Geometrically, the linear combination of the vectors can be visualized in the following way.



The vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ are basis vectors in R^3. This has the same meaning for R^3 that it has for R^2. As before, every vector in R^3 can be uniquely written as a linear combination of $\vec{i}$, $\vec{j}$, and $\vec{k}$. Stated simply, $\overrightarrow{OP} = (a, b, c)$ can be written as $\overrightarrow{OP} = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = a\vec{i} + b\vec{j} + c\vec{k}$.

The methods for working in R^3 are similar to methods we have already seen at the beginning of this section. A natural question to ask is, "Suppose you are given two noncollinear (non-parallel) vectors in R^3; what do these vectors span?"

In the diagram, the large parallelogram is meant to represent an infinite plane extending in all directions. On this parallelogram are drawn the nonzero vectors $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$ with their tails at the origin, O. When we write the linear combination $m\vec{a} + n\vec{b}$, $\overrightarrow{OE}$ is the resulting vector and is the diagonal of the smaller parallelogram, $ODEC$. Since each of the scalars, m and n, can be any real number, an infinite number of vectors, each unique, will be generated from this linear combination. All of these vectors lie on the plane determined by $\vec{a}$ and $\vec{b}$. It should be noted that if we say $\overrightarrow{OE}$ lies on the plane, the point E also lies on the plane so that when we say $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OE}$ lie on the plane, this is effectively saying that the points A, B, and E also lie on the plane. When two or more points or vectors lie on the same plane they are said to be **coplanar**.



Spanning Sets

1. Any pair of nonzero, noncollinear vectors will span R^2.
2. Any pair of nonzero, noncollinear vectors will span a plane in R^3.

EXAMPLE 5 Selecting a linear combination strategy to determine if vectors lie on the same plane

a. Given the two vectors $\vec{a} = (-1, -2, 1)$ and $\vec{b} = (3, -1, 1)$, does the vector $\vec{c} = (-9, -4, 1)$ lie on the plane determined by $\vec{a}$ and $\vec{b}$? Explain.

b. Does the vector $(-9, -5, 1)$ lie in the plane determined by the first two vectors?

Solution

a. This question is asking whether $\vec{c}$ lies in the span of $\vec{a}$ and $\vec{b}$. Stated algebraically, are there values of m and n for which $m(-1, -2, 1) + n(3, -1, 1) = (-9, -4, 1)$?

Multiplying, $(-m, -2m, m) + (3n, -n, n) = (-9, -4, 1)$
or $(-m + 3n, -2m - n, m + n) = (-9, -4, 1)$

Equating components leads to

① $-m + 3n = -9$
② $2m + n = 4$
③ $m + n = 1$

The easiest way of dealing with these equations is to work with equations ① and ③. If we add these equations, m is eliminated and $4n = -8$, so $n = -2$. Substituting into equation ③ gives $m = 3$. We must verify that these values give a consistent answer in the remaining equation. Checking in equation ②: $2(3) + (-2) = 4$.

Since $(-9, -4, 1)$ can be written as a linear combination of $(-1, -2, 1)$ and $(3, -1, 1)$, i.e., $(-9, -4, 1) = 3(-1, -2, 1) - 2(3, -1, 1)$, it lies in the plane determined by the two given vectors.

b. If we carry out calculations identical to those in the solution for part a., the only difference would be that the second equation would now be $2m + n = 5$, and substituting $m = 3$ and $n = -2$ would give $2(3) + (-2) = 4 \neq 5$. Since we have an inconsistent result, this implies that the vector $(-9, -5, 1)$ does not lie on the same plane as $\vec{a}$ and $\vec{b}$.

In general, when we are trying to determine whether a vector lies in the plane determined by two other nonzero, noncollinear vectors, it is sufficient to solve any pair of equations and look for consistency in the third equation. If the result is consistent, the vector lies in the plane, and if not, the vector does not lie in the plane.

IN SUMMARY

Key Ideas

- In R^2, $\overrightarrow{OP} = (a, b) = a(1, 0) + b(0, 1) = a\vec{i} + b\vec{j}$. $\vec{i}$ and $\vec{j}$ *span* R^2. Every vector in R^2 can be written uniquely as a linear combination of these two vectors.
- In R^3, $\overrightarrow{OP} = (a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = a\vec{i} + b\vec{j} + c\vec{k}$. $\vec{i}$, $\vec{j}$, and $\vec{k}$ span R^3 since every vector in R^3 can be written uniquely as a linear combination of these three vectors.

Need to Know

- Any pair of nonzero, noncollinear vectors will span R^2.
- Any pair of nonzero, noncollinear vectors will span a plane in R^3. This means that every vector in the plane can be expressed as a linear combination involving this pair of vectors.

Exercise 6.8

PART A

1. A student writes $2(1, 0) + 4(-1, 0) = (-2, 0)$ and then concludes that $(1, 0)$ and $(-1, 0)$ span R^2. What is wrong with this conclusion?
2. It is claimed that $\{(1, 0, 0), (0, 1, 0), (0, 0, 0)\}$ is a set of vectors spanning R^3. Explain why it is not possible for these vectors to span R^3.
3. Describe the set of vectors spanned by $(0, 1)$. Say why this is the same set as that spanned by $(0, -1)$.
4. In R^3, the vector $\vec{i} = (1, 0, 0)$ spans a set. Describe the set spanned by this vector. Name two other vectors that would also span the same set.
5. It is proposed that the set $\{(0, 0), (1, 0)\}$ could be used to span R^2. Explain why this is not possible.
6. The following is a spanning set for R^2:
   $\{(-1, 2), (2, -4), (-1, 1), (-3, 6), (1, 0)\}$.
   Remove three of the vectors and write down a spanning set that can be used to span R^2.

PART B

7. Simplify each of the following linear combinations and write your answer in component form: $\vec{a} = \vec{i} - 2\vec{j}$, $\vec{b} = \vec{j} - 3\vec{k}$, and $\vec{c} = \vec{i} - 3\vec{j} + 2\vec{k}$
   a. $2(2\vec{a} - 3\vec{b} + \vec{c}) - 4(-\vec{a} + \vec{b} - \vec{c}) + (\vec{a} - \vec{c})$
   b. $\frac{1}{2}(2\vec{a} - 4\vec{b} - 8\vec{c}) - \frac{1}{3}(3\vec{a} - 6\vec{b} + 9\vec{c})$

K 8. Name two sets of vectors that could be used to span the xy-plane in R^3. Show how the vectors $(-1, 2, 0)$ and $(3, 4, 0)$ could each be written as a linear combination of the vectors you have chosen.

C 9. a. The set of vectors $\{(1, 0, 0), (0, 1, 0)\}$ spans a set in R^3. Describe this set.
   b. Write the vector $(-2, 4, 0)$ as a linear combination of these vectors.
   c. Explain why it is not possible to write $(3, 5, 8)$ as a linear combination of these vectors.
   d. If the vector $(1, 1, 0)$ were added to this set, what would these three vectors span in R^3?

10. Solve for a, b, and c in the following equation:
    $2(a, 3, c) + 3(c, 7, c) = (5, b + c, 15)$

11. Write the vector $(-10, -34)$ as a linear combination of the vectors $(-1, 3)$ and $(1, 5)$.

12. In Example 3, it was shown how to find a formula for the coefficients a and b whenever we are given a general vector (x, y).
    a. Repeat this procedure for $\{(2, -1), (-1, 1)\}$.
    b. Write each of the following vectors as a linear combination of the set given in part a.: $(2, -3)$, $(124, -5)$, and $(4, -11)$.

A 13. a. Show that the vectors $(-1, 2, 3)$, $(4, 1, -2)$, and $(-14, -1, 16)$ do not lie on the same plane.
    b. Show that the vectors $(-1, 3, 4)$, $(0, -1, 1)$, and $(-3, 14, 7)$ lie on the same plane, and show how one of the vectors can be written as a linear combination of the other two.

14. Determine the value for x such that the points $A(-1, 3, 4)$, $B(-2, 3, -1)$, and $C(-5, 6, x)$ all lie on a plane that contains the origin.

T 15. The vectors $\vec{a}$ and $\vec{b}$ span R^2. What values of m and n will make the following statement true: $(m - 2)\vec{a} = (n + 3)\vec{b}$? Explain your reasoning.

PART C

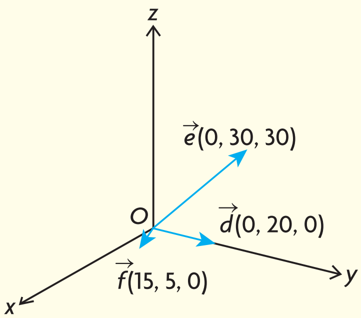
16. The vectors $(4, 1, 7)$, $(-1, 1, 6)$, and $(p, q, 5)$ are coplanar. Determine three sets of values for p and q for which this is true.

17. The vectors $\vec{a}$ and $\vec{b}$ span R^2. For what values of m is it true that $(m^2 + 2m - 3)\vec{a} + (m^2 + m - 6)\vec{b} = \vec{0}$? Explain your reasoning.

CAREER LINK WRAP-UP Investigate and Apply

CHAPTER 6: FIGURE SKATING

A figure skater is attempting to perform a quadruple spin jump. He sets up his jump with an initial skate along vector $\vec{d}$. He then plants his foot and applies vertical force at an angle according to vector $\vec{e}$. This causes him to leap into the air and spin. After landing, his momentum will carry him into the wall if he does not apply force to stop himself. So he applies force along vector $\vec{f}$ to slow himself down and change direction.



a. Add vectors $\vec{d}$ and $\vec{e}$ to find the resulting vector ($\vec{a}$) for the skater's jump. The angle between $\vec{d}$ and $\vec{e}$ is 25°. If the *xy*-plane represents the ice surface, calculate the angle the skater will take with respect to the ice surface on this jump.

b. Discuss why the skater will return to the ground even though the vector that represents his leap carries him in an upward direction.

c. Rewrite the resulting vector $\vec{a}$ without the vertical coordinate. For example, if the vector has components (20, 30, 15), rewrite as (20, 30). Explain the significance of this vector.

d. Add vectors $\vec{a}$ and $\vec{f}$ to find the resulting vector ($\vec{b}$) as the skater applies force to slow himself and change direction. Explain the significance of this vector.

Key Concepts Review

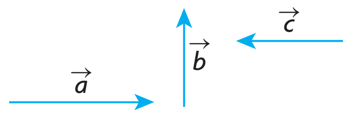
In Chapter 6, you were introduced to vectors: quantities that are described in terms of both magnitude and direction. You should be familiar with the difference between a geometric vector and an algebraic vector. Consider the following summary of key concepts:

- Scalar quantities have only magnitude, while vector quantities have both magnitude and direction.
- Two vectors are equal if they have the same magnitude and direction.
- Two vectors are opposite if they have the same magnitude and opposite directions.
- When vectors are drawn tail-to-tail, their sum or resultant is the diagonal of the parallelogram formed by the vectors.
- When vectors are drawn head-to-tail, their sum or resultant is the vector drawn from the tail of the first to the head of the second.
- Multiplying a vector by a nonzero scalar results in a new vector in the same or opposite direction of the original vector with a greater or lesser magnitude compared to the original. The set of vectors formed are described as collinear (parallel vectors).
- The vector $\overrightarrow{OP}$ is called a position vector and is drawn on a coordinate axis with its tail at the origin and its head located at point P.
- In R^2, $\overrightarrow{OP} = (a, b) = a\vec{i} + b\vec{j}$, $|\overrightarrow{OP}| = \sqrt{a^2 + b^2}$ where $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$.
- In R^3, $\overrightarrow{OP} = (a, b, c) = a\vec{i} + b\vec{j} + c\vec{k}$, $|\overrightarrow{OP}| = \sqrt{a^2 + b^2 + c^2}$ where $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$ and $\vec{k} = (0, 0, 1)$.
- In R^2, the vector between two points with its tail at $A(x_1, y_1)$ and head at $B(x_2, y_2)$ is determined as follows:
  $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2, y_2) - (x_1, y_1) = (x_2 - x_1, y_2 - y_1)$
  $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- In R^3, the vector between two points with its tail at $A(x_1, y_1, z_1)$ and head at $B(x_2, y_2, z_2)$ is determined as follows:
  $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2, y_2, z_2) - (x_1, y_1, z_1) = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$
  $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- Any pair of nonzero, noncollinear vectors will span R^2.
- Any pair of nonzero, noncollinear vectors will span a plane in R^3.

Review Exercise

1. Determine whether each of the following statements is true or false. Provide a brief explanation for each answer.
   a. $|\vec{a} + \vec{b}| \geq |\vec{a}|$
   b. $|\vec{a} + \vec{b}| = |\vec{a} + \vec{c}|$ implies $|\vec{b}| = |\vec{c}|$
   c. $\vec{a} + \vec{b} = \vec{a} + \vec{c}$ implies $\vec{b} = \vec{c}$
   d. $\overrightarrow{RF} = \overrightarrow{SW}$ implies $\overrightarrow{RS} = \overrightarrow{FW}$
   e. $m\vec{a} + n\vec{a} = (m + n)\vec{a}$
   f. If $|\vec{a}| = |\vec{b}|$ and $|\vec{c}| = |\vec{d}|$, then $|\vec{a} + \vec{b}| = |\vec{c} + \vec{d}|$.
2. If $\vec{x} = 2\vec{a} - 3\vec{b} - 4\vec{c}$, $\vec{y} = -2\vec{a} + 3\vec{b} + 3\vec{c}$, and $\vec{z} = 2\vec{a} - 3\vec{b} + 5\vec{c}$, determine simplified expressions for each of the following:
   a. $2\vec{x} - 3\vec{y} + 5\vec{z}$
   b. $3(-2\vec{x} - 4\vec{y} + \vec{z}) - (2\vec{x} - \vec{y} + \vec{z}) - 2(-4\vec{x} - 5\vec{y} + \vec{z})$
3. If $X(-2, 1, 2)$ and $Y(-4, 4, 8)$ are two points in R^3, determine the following:
   a. $\overrightarrow{XY}$ and $|\overrightarrow{XY}|$
   b. The coordinates of a unit vector in the same direction as $\overrightarrow{XY}$.
4. $X(-1, 2, 6)$ and $Y(5, 5, 12)$ are two points in R^3.
   a. Determine the components of a position vector equivalent to $\overrightarrow{YX}$.
   b. Determine the components of a *unit* vector that is in the same direction as $\overrightarrow{YX}$.
5. Find the components of the unit vector with the opposite direction to that of the vector from $M(2, 3, 5)$ to $N(8, 1, 2)$.
6. A parallelogram has its sides determined by the vectors $\overrightarrow{OA} = (3, 2, -6)$ and $\overrightarrow{OB} = (-6, 6, -2)$.
   a. Determine the components of the vectors representing the diagonals.
   b. Determine the angles between the sides of the parallelogram.
7. The points $A(-1, 1, 1)$, $B(2, 0, 3)$, and $C(3, 3, -4)$ are vertices of a triangle.
   a. Show that this triangle is a right triangle.
   b. Calculate the area of triangle ABC.
   c. Calculate the perimeter of triangle ABC.
   d. Calculate the coordinates of the fourth vertex D that completes the rectangle of which A, B, and C are the other three vertices.

8. The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are as shown.



a. Construct the vector $\vec{a} - \vec{b} + \vec{c}$.

b. If the vectors $\vec{a}$ and $\vec{b}$ are perpendicular, and if $|\vec{a}| = 4$ and $|\vec{b}| = 3$, determine $|\vec{a} + \vec{b}|$.

9. Given $\vec{p} = (-11, 7)$, $\vec{q} = (-3, 1)$, and $\vec{r} = (-1, 2)$, express each vector as a linear combination of the other two.

10. a. Find an equation to describe the set of points equidistant from $A(2, -1, 3)$ and $B(1, 2, -3)$.

b. Find the coordinates of two points that are equidistant from A and B.

11. Calculate the values of a, b, and c in each of the following:

a. $2(a, b, 4) + \frac{1}{2}(6, 8, c) - 3(7, c, -4) = (-24, 3, 25)$

b. $2\left(a, a, \frac{1}{2}a\right) + (3b, 0, -5c) + 2\left(c, \frac{3}{2}c, 0\right) = (3, -22, 54)$

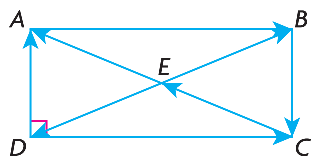
12. a. Determine whether the points $A(1, -1, 1)$, $B(2, 2, 2)$, and $C(4, -2, 1)$ represent the vertices of a right triangle.

b. Determine whether the points $P(1, 2, 3)$, $Q(2, 4, 6)$, and $R(-1, -2, -3)$ are collinear.

13. a. Show that the points $A(3, 0, 4)$, $B(1, 2, 5)$, and $C(2, 1, 3)$ represent the vertices of a right triangle.

b. Determine $\cos \angle ABC$.

14. In the following rectangle, vectors are indicated by the direction of the arrows.

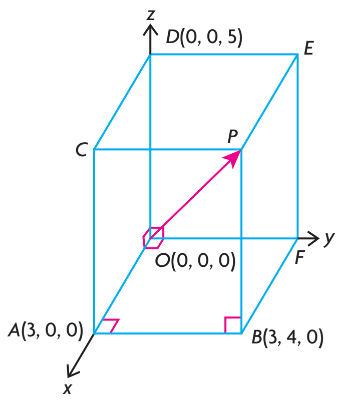


a. Name two pairs of vectors that are opposites.

b. Name two pairs of identical vectors.

c. Explain why $|\overrightarrow{AD}|^2 + |\overrightarrow{DC}|^2 = |\overrightarrow{DB}|^2$.

15. A rectangular prism measuring 3 by 4 by 5 is drawn on a coordinate axis as shown in the diagram.



a. Determine the coordinates of points C, P, E, and F.

b. Determine position vectors for $\overrightarrow{DB}$ and $\overrightarrow{CF}$.

c. By drawing the rectangle containing $\overrightarrow{DB}$ and $\overrightarrow{OP}$, determine the acute angle between these vectors.

d. Determine the angle between $\overrightarrow{OP}$ and $\overrightarrow{AE}$.

16. The vectors $\vec{d}$ and $\vec{e}$ are such that $|\vec{d}| = 3$ and $|\vec{e}| = 5$, and the angle between them is 30°. Determine each of the following:

a. $|\vec{d} + \vec{e}|$ b. $|\vec{d} - \vec{e}|$ c. $|\vec{e} - \vec{d}|$

17. An airplane is headed south at speed 400 km/h. The airplane encounters a wind from the east blowing at 100 km/h.

a. How far will the airplane travel in 3 h?

b. What is the direction of the airplane?

18. a. Explain why the set of vectors: $\{(2, 3), (3, 5)\}$ spans R^2.

b. Find m and n in the following: $m(2, 3) + n(3, 5) = (323, 795)$.

19. a. Show that the vector $\vec{a} = (5, 9, 14)$ can be written as a linear combination of the vectors $\vec{b}$ and $\vec{c}$, where $\vec{b} = (-2, 3, 1)$ and $\vec{c} = (3, 1, 4)$. Explain why $\vec{a}$ lies in the plane determined by $\vec{b}$ and $\vec{c}$.

b. Is the vector $\vec{a} = (-13, 36, 23)$ in the span of $\vec{b} = (-2, 3, 1)$ and $\vec{c} = (3, 1, 4)$? Explain your answer.

20. A cube is placed so that it has three of its edges located along the positive x-, y-, and z-axes (one edge along each axis) and one of its vertices at the origin.

a. If the cube has a side length of 4, draw a sketch of this cube and write the coordinates of its vertices on your sketch.

b. Write the coordinates of the vector with its head at the origin and its tail at the opposite vertex.

c. Write the coordinates of a vector that starts at $(4, 4, 4)$ and is a diagonal in the plane parallel to the xz-plane.

d. What vector starts at the origin and is a diagonal in the xy-plane?

21. If $\vec{a} = \vec{i} + \vec{j} - \vec{k}$, $\vec{b} = 2\vec{i} - \vec{j} + 3\vec{k}$, and $\vec{c} = 2\vec{i} + 13\vec{k}$, determine $\left|2\left(\vec{a} + \vec{b} - \vec{c}\right) - \left(\vec{a} + 2\vec{b}\right) + 3\left(\vec{a} - \vec{b} + \vec{c}\right)\right|$.

22. The three points $A(-3, 4)$, $B(3, -4)$, and $C(5, 0)$ are on a circle with radius 5 and centre at the origin. Points A and B are the endpoints of a diameter, and point C is on the circle.

a. Calculate $\left|\overrightarrow{AB}\right|$, $\left|\overrightarrow{AC}\right|$, and $\left|\overrightarrow{BC}\right|$.

b. Show that A, B, and C are the vertices of a right triangle.

23. In terms of $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{0}$, find a vector expression for each of the following:

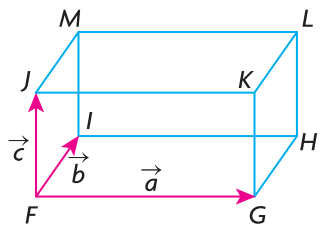
a. $\overrightarrow{FL}$

b. $\overrightarrow{MK}$

c. $\overrightarrow{HJ}$

d. $\overrightarrow{IH} + \overrightarrow{KJ}$

e. $\overrightarrow{IK} - \overrightarrow{IH}$



24. Draw a diagram showing the vectors $\vec{a}$ and $\vec{b}$, where $|\vec{a}| = 2|\vec{b}|$ and $|\vec{b}| = |\vec{a} + \vec{b}|$ are both true. (Make sure to indicate the direction of the vectors.)

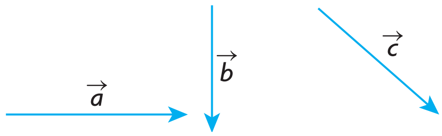
25. If the vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other, express each of the following in terms of $|\vec{a}|$ and $|\vec{b}|$:

a. $|\vec{a} + \vec{b}|$ b. $|\vec{a} - \vec{b}|$ c. $|2\vec{a} + 3\vec{b}|$

26. Show that if $\vec{a}$ is perpendicular to each of the vectors $\vec{b}$ and $\vec{c}$, then $\vec{a}$ is perpendicular to $2\vec{b} + 4\vec{c}$.

Chapter 6 Test

1. The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are shown.



Using these three vectors, demonstrate that $\vec{a} + \left(\vec{b} + \vec{c}\right) = \left(\vec{a} + \vec{b}\right) + \vec{c}$. Name this property and explain how your answer shows this to be true.

2. $A(-2, 3, -5)$ and $B(6, 7, 3)$ are two points in R^3. Determine each of the following:

   a. $\overrightarrow{AB}$ b. $\left|\overrightarrow{AB}\right|$ c. a unit vector in the direction of $\overrightarrow{BA}$

3. The vectors $\vec{x}$ and $\vec{y}$ are each of length 3 units, i.e., $|\vec{x}| = |\vec{y}| = 3$. If $|\vec{x} + \vec{y}| = \sqrt{17}$, determine $|\vec{x} - \vec{y}|$.

4. a. If $3\vec{x} - 2\vec{y} = \vec{a}$ and $5\vec{x} - 3\vec{y} = \vec{b}$, express the vectors $\vec{x}$ and $\vec{y}$ in terms of $\vec{a}$ and $\vec{b}$.

   b. Solve for a, b, and c: $(2, -1, c) + (a, b, 1) - 3(2, a, 4) = (-3, 1, 2c)$.

5. a. Explain why the vectors $\vec{a} = (-2, 3)$ and $\vec{b} = (3, -1)$ span R^2.

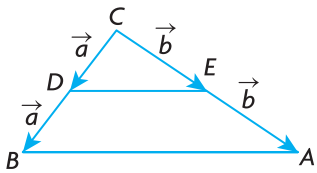
   b. Determine the values of p and q in $p(-2, 3) + q(3, -1) = (13, -9)$.

6. a. Show that the vector $\vec{a} = (1, 12, -29)$ can be written as a linear combination of $\vec{b} = (3, 1, 4)$ and $\vec{c} = (1, 2, -3)$.

   b. Determine whether $\vec{r} = (16, 11, -24)$ can be written as a linear combination of $\vec{p} = (-2, 3, 4)$ and $\vec{q} = (4, 1, -6)$. Explain the significance of your result geometrically.

7. $\vec{x}$ and $\vec{y}$ are vectors of magnitude 1 and 2, respectively, with an angle of 120° between them. Determine $|3\vec{x} + 2\vec{y}|$ and the direction of $3\vec{x} + 2\vec{y}$.

8. In triangle ABC, point D is the midpoint of $\overrightarrow{BC}$ and point E is the midpoint of $\overrightarrow{AC}$. Vectors are marked as shown. Use vectors to prove that $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BA}$.



Chapter 7

APPLICATIONS OF VECTORS

In Chapter 6, we discussed some of the basic ideas about vectors. In this chapter, we will use vectors in both mathematical and physical situations to calculate quantities that would otherwise be difficult to determine. You will discover how vectors enable calculations in situations involving the velocity at which a plane flies under windy conditions and the force at which two other people must pull to balance the force created by two others in a game of tug-o-war. In addition, we will introduce the concept of vector multiplication and show how vectors can be applied in a variety of contexts.

CHAPTER EXPECTATIONS

In this chapter, you will

- use vectors to model and solve problems arising from real-world applications involving velocity and force, **Sections 7.1, 7.2**
- perform the operation of the dot product on two vectors, **Sections 7.3, 7.4**
- determine properties of the dot product, **Sections 7.3, 7.4**
- determine the scalar and vector projections of a vector, **Section 7.5**
- perform the operation of cross product on two algebraic vectors in three-dimensional space, **Section 7.6**
- determine properties of the cross product, **Section 7.6**
- solve problems involving the dot product and cross product, **Section 7.7**

Review of Prerequisite Skills

In this chapter, you will use vectors in applications involving elementary force and velocity problems. As well, you will be introduced to the study of scalar and vector products. You will find it helpful to be able to

- find the magnitude and the direction of vectors using trigonometry
- plot points and find coordinates of points in two- and three-dimensional systems

Exercise

1. The velocity of an airplane is 800 km/h north. A wind is blowing due east at 100 km/h. Determine the velocity of the airplane relative to the ground.

2. A particle is displaced 5 units to the west and then displaced 12 units in a direction $N45°W$. Find the magnitude and direction of the resultant displacement.

3. Draw the x-axis, y-axis, and z-axis, and plot the following points:

a. $A(0, 1, 0)$
b. $B(-3, 2, 0)$
c. $C(-2, 0, 1)$
d. $D(0, 2, -3)$

4. Express each of the following vectors in component form (a, b, c). Then determine its magnitude.

a. $3\vec{i} - 2\vec{j} + 7\vec{k}$
b. $-9\vec{i} + 3\vec{j} + 14\vec{k}$
c. $\vec{i} + \vec{j}$
d. $2\vec{i} - 9\vec{k}$

5. Describe where the following general points are located.

a. $A(x, y, 0)$
b. $B(x, 0, z)$
c. $C(0, y, z)$

6. Find a single vector that is equivalent to each linear combination.

a. $(-6, 0) + 7(1, -1)$
b. $(4, -1, 3) - (-2, 1, 3)$
c. $2(-1, 1, 3) + 3(-2, 3, -1)$
d. $-\frac{1}{2}(4, -6, 8) + \frac{3}{2}(4, -6, 8)$

7. If $\vec{a} = 3\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{b} = -2\vec{i} + \vec{j}$, determine a single vector that is equivalent to each linear combination.

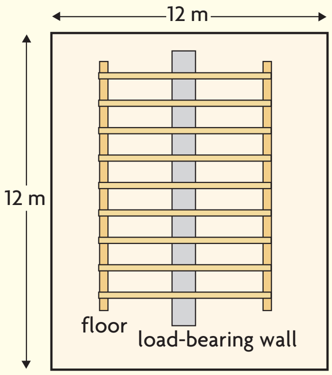
a. $\vec{a} + \vec{b}$
b. $\vec{a} - \vec{b}$
c. $2\vec{a} - 3\vec{b}$

CAREER LINK Investigate

CHAPTER 7: FORCES IN ARCHITECTURE: STRUCTURAL ENGINEERING

Structural engineers are a specific kind of architect: they help in the design of large-scale structures, such as bridges and skyscrapers. The role of structural engineers is to make sure that the structure being built will be stable and not collapse. To do this, they need to calculate all the forces acting on the structure, including the weight of building materials, occupants, and any furniture or items that might be stored in the building. They also need to account for the forces of wind, water, and seismic activity, including hurricanes and earthquakes. To design a safe structure effectively, the composition of forces must be calculated to find the resultant force. The strength and structure of the materials must exert a force greater than the equilibrant force. A simple example of this is a load-bearing wall inside a house. A structural engineer must calculate the total weight of the floor above, which is considered a dead load. Then the engineer has to factor in the probable weight of the occupants and their furniture, which is considered a live load. The load-bearing wall must be built to exert an opposing force that is greater than the force created by the live load.

Type of Load	Load (kg/m^2)
live	90
dead	150



Case Study—Replacing a Load-Bearing Wall with a Steel Support Beam

The table at the left shows the normal loads created by a timber floor and a non-load-bearing wall above. A homeowner wants to make one large room out of two. This will require removing a wall that is bearing the load of the floor above and replacing the wall with a steel support beam. The horizontal and vertical yellow segments represent the framing for the area of the upper floor that is currently being supported by the wall, without help from the walls at the edges of the rooms. Complete the discussion questions to determine what size of beam will be required to bear the load.

DISCUSSION QUESTIONS

1. Find the area of the floor that is currently being supported by the load-bearing wall. Use the information in the table to calculate the live and dead loads for this area.
2. Find the resultant downward force created by the weight of the floor above including an estimate for the expected weight of four occupants and their furniture.
3. Determine the equilibrant force required by a steel support beam that would support the force you calculated in question 2. Explain why, for safety reasons, a beam that supports a greater force is used.

Section 7.1—Vectors as Forces

The concept of **force** is something that everyone is familiar with. When we think of force, we usually think of it associated with effort or muscular exertion. This is experienced when an object is moved from one place to another. Examples of activities that involve forces are pulling a toboggan, lifting a book, shooting a basketball, or pedalling a bicycle. Each of these activities involves the use of muscular action that exerts a force. There are, however, many other examples of force in which muscular action is not present. For example, the attraction of the Moon to Earth, the attraction of a piece of metal to a magnet, the thrust exerted by an engine when gasoline combusts in its cylinders, or the force exerted by shock absorbers in cars to reduce vibration.

Force as a Vector Quantity

Force can be considered something that either pushes or pulls an object. When a large enough force is applied to an object at rest, the object tends to move. When a push or pull is applied to a body that is already in motion, the motion of the body tends to change. Generally speaking, force can be defined as that which changes, or tends to change, the state of rest, or uniform motion of a body.

When describing certain physical quantities, there is little value in describing them with magnitude alone. For example, if we are describing the velocity of wind, it is not very practical to say that the wind has a speed of 30 km/h without specifying the direction of the wind. It makes more sense to say that a wind has a speed of 30 km/h travelling south. Similarly, the description of a force without specifying its magnitude and direction has little practical value. Because force is described by both magnitude and direction, it is a vector. The rules that apply to vectors also apply to forces.

Before we consider situations involving the calculation of force, it is necessary to describe the unit in which force is measured. On Earth, force is defined as the product between the mass of an object and the acceleration due to gravity (9.8 m/s^2). So a 1 kg mass exerts a downward force of $1 \text{ kg} \times 9.8 \text{ m/s}^2$ or $9.8 \text{ kg} \cdot \text{m/s}^2$. This unit of measure is called a newton and is abbreviated as N. Because of Earth's gravitational field, which acts downward, we say that a 1 kg mass exerts a force of 9.8 N. Thus, the force exerted by a 2 kg mass at Earth's surface is about 19.6 N. A person having a mass of 60 kg would exert approximately 60×9.8, or 588 N, on the surface of Earth. So weight, expressed in newtons, is a force acting with a downward direction.

In problems involving forces, it is often the case that two or more forces act simultaneously on an object. To better understand the effect of these forces, it is useful to be able to find the single force that would produce exactly the same effect as all the forces acting together produced. This single force is called the **resultant**, or **sum**, of all the forces working on the object. It is important that we be able to determine the direction and magnitude of this single force. When we find the resultant of several forces, this resultant may be substituted for the individual forces, and the separate forces need not be considered further. The process of finding the resultant of all the forces acting on an object is called the **composition of forces**.

Resultant and Composition of Forces

The resultant of several forces is the single force that can be used to represent the combined effect of all the forces. The individual forces that make up the resultant are referred to as the components of the resultant.

If several forces are acting on an object, it is often advantageous to find a single force which, when applied to the object, would prevent any further motion that these original forces tended to produce. This single force is called the **equilibrant** because it would keep the object in a state of equilibrium.

Equilibrant of Several Forces

The equilibrant of a number of forces is the single force that opposes the resultant of the forces acting on an object. When the equilibrant is applied to the object, this force maintains the object in a state of equilibrium.

In the first example, we will consider collinear forces and demonstrate how to calculate their resultant and equilibrant. Collinear forces are those forces that act along the same straight line (in the same or opposite direction).

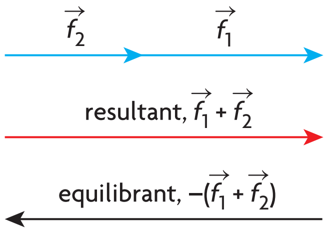
EXAMPLE 1 Representing force using vectors

Two children, James and Fred, are pushing on a rock. James pushes with a force of 80 N in an easterly direction, and Fred pushes with a force of 60 N in the same direction. Determine the resultant and equilibrant of these two forces.

Solution

To visualize the first force, we represent it with a horizontal line segment measuring 8 cm, pointing east. We represent the 60 N force with a line segment of 6 cm, also pointing east. The vectors used to represent forces are proportionate in length to the magnitude of the forces they represent.

Resultant and Equilibrant of the Two Collinear Force Vectors

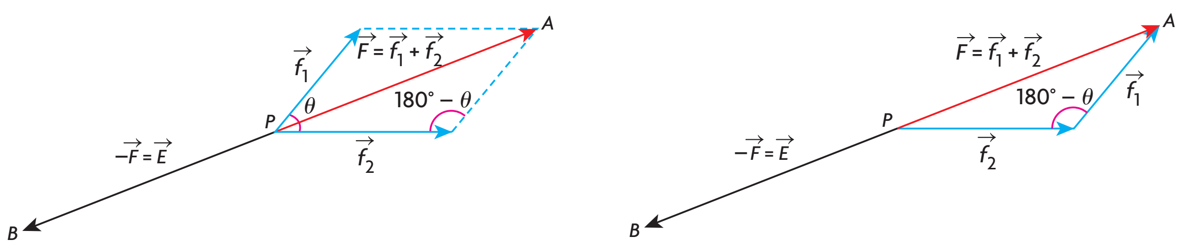


The resultant of these forces, $\vec{f_1} + \vec{f_2}$, is the single vector pointing east with a magnitude of 140 N. The combined effort of James and Fred working together exerts a force on the rock of magnitude 140 N in an easterly direction. The equilibrant of these forces is the vector, $-(\vec{f_1} + \vec{f_2})$, which has a magnitude of 140 N pointing in the opposite direction, west. In general, the resultant and equilibrant are two vectors having the same magnitude but pointing in opposite directions.

It is not typical that forces acting on an object are collinear. In the following diagram, the two noncollinear forces, $\vec{f_1}$ and $\vec{f_2}$, are applied at the point P and could be thought of as two forces applied to an object in an effort to move it.

The natural question is, how do we determine the resultant of these two forces? Since forces are vectors, it follows from our work in the previous chapter that the resultant of two noncollinear forces is represented by either the diagonal of the parallelogram determined by these two vectors when placed tail to tail or the third side of the triangle formed when the vectors are placed head to tail. In the following diagrams, vector $\overrightarrow{PA} = \vec{F}$ is the resultant of $\vec{f_1}$ and $\vec{f_2}$, while the vector $\overrightarrow{PB} = \vec{E}$ is the equilibrant of $\vec{f_1}$ and $\vec{f_2}$.

Resultant and Equilibrant of the Force Vectors $\vec{f_1}$ and $\vec{f_2}$



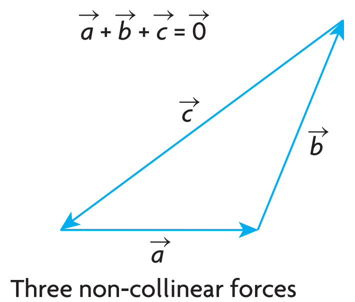
The resultant vector $\vec{F}$ and the equilibrant vector $\vec{E}$ are examples of two vectors that are in a state of equilibrium. When both these forces are applied to an object at point P, the object does not move. Since these vectors have the same magnitude but opposite directions it follows that $\vec{F} + \vec{E} = \vec{F} + (-\vec{F}) = \vec{0}$.

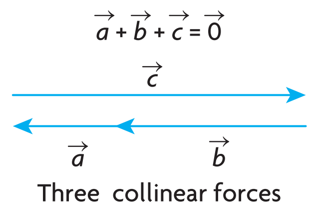
Vectors in a State of Equilibrium

When three noncollinear vectors are in a state of equilibrium, these vectors will always lie in the same plane and form a linear combination. When the three vectors are arranged head to tail, the result is a triangle because the resultant of two of the forces is opposed by the third force. This means that if three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are in equilibrium, such that $\vec{c}$ is the equilibrant of $\vec{a}$ and $\vec{b}$, then $-\vec{c} = \vec{a} + \vec{b}$ or $\vec{a} + \vec{b} + \vec{c} = (-\vec{c}) + (\vec{c}) = \vec{0}$.

It is important to note that it also is possible for three vectors to be in equilibrium when the three forces are collinear. As with noncollinear vectors, one of the three forces is balanced by the resultant of the two other forces. In this case, the three forces do not form a triangle in the traditional sense. Instead, the sides of the "triangle" lie along the same straight line.

Forces in Equilibrium





In the following example, the resultant of two noncollinear forces is calculated.

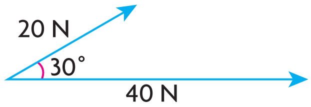
EXAMPLE 2

Connecting the resultant force to vector addition

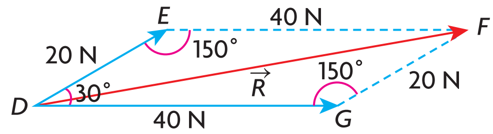
Two forces of 20 N and 40 N act at an angle of 30° to each other. Determine the resultant of these two forces.

Solution

We start the solution to this problem by drawing both a position diagram and a vector diagram. A position diagram indicates the actual position of the given vectors, and a vector diagram takes the information given in the position diagram and puts it in a form that allows for the determination of the resultant vector using either the triangle or parallelogram law. As before, the position diagram is drawn approximately to scale, and the side lengths of the parallelogram are labelled. The resultant of the two given vectors is $\overrightarrow{DF} = \vec{R}$, and the supplement of $\angle EDG$ is $\angle FED$, which measures 150°.



Position diagram



Vector diagram

In $\triangle DEF$, $|\vec{R}|^2 = 20^2 + 40^2 - 2(20)(40)\cos 150°$ (Cosine law)

$$|\vec{R}|^2 = 20^2 + 40^2 - 2(20)(40)\left(\frac{-\sqrt{3}}{2}\right)$$

$$|\vec{R}|^2 = 2000 + 800\sqrt{3}$$

$$|\vec{R}|^2 \doteq 3385.64$$

Therefore, $|\vec{R}| \doteq 58.19$ N. If we let $\vec{E}$ represent the equilibrant, then $|\vec{E}| \doteq 58.19$ N.

Since we are asked to calculate the resultant and equilibrant of the two forces, we must also calculate angles so that we can state each of their relative positions. To do this, we use the sine law.

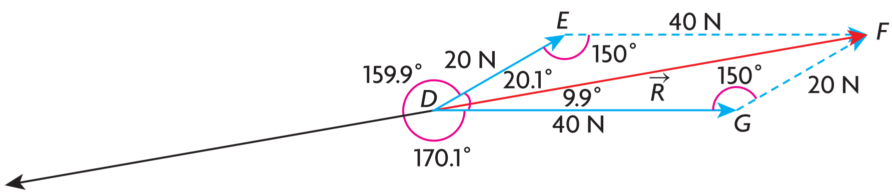
In $\triangle DEF$, $\dfrac{\sin \angle DEF}{|\vec{R}|} = \dfrac{\sin \angle EDF}{|\overrightarrow{EF}|}$ (Sine law)

$$\frac{\sin 150°}{58.19} \doteq \frac{\sin \angle EDF}{40}$$

$$\sin \angle EDF \doteq \frac{40(\sin 150°)}{58.19}$$

$$\sin \angle EDF \doteq 0.3437$$

Thus, $\angle EDF \doteq 20.1°$



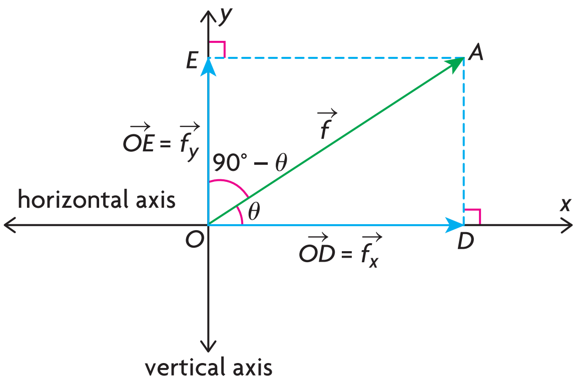
The resultant and equilibrant are forces, each having a magnitude of approximately 58.19 N. The resultant makes an angle of 20.1° with the 20 N force and 9.9° with the 40 N force. The equilibrant makes an angle of 159.9° with the 20 N force and an angle of 170.1° with the 40 N force.

We have shown that if we take any two forces that act at the same point, acting at an angle of θ to each other, the forces may be composed to obtain the resultant of these two forces. Furthermore, the resultant of any two forces is unique because there is only one parallelogram that can be formed with these two forces.

Resolving a Vector into Its Components

In many situations involving forces, we are interested in a process that is the opposite of composition. This process is called **resolution**, which means taking a single force and decomposing it into two components. When we resolve a force into two components, it is possible to do this in an infinite number of ways because there are infinitely many parallelograms having a particular single force as the diagonal. However, the most useful and important way to resolve a force vector occurs when this vector is resolved into two components that are at right angles to each other. These components are usually referred to as the horizontal and vertical components.

In the following diagram, we demonstrate how to resolve the force vector $\vec{f}$ into its horizontal and vertical components.



The vector resolved into components is the vector $\overrightarrow{OA}$, or vector $\vec{f}$. From A, the head of the vector, perpendicular lines are drawn to meet the x-axis and y-axis at points D and E, respectively. The vectors $\overrightarrow{OD}$ and $\overrightarrow{OE}$ are called the horizontal and vertical components of the vector $\overrightarrow{OA}$, where the angle between $\vec{f}$ and the x-axis is labelled θ.

To calculate $|\overrightarrow{OD}|$, we use the cosine ratio in the right triangle OAD.

$$\text{In } \triangle OAD,\ \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{|\overrightarrow{OD}|}{|\overrightarrow{OA}|}$$

$$\text{Therefore, } |\overrightarrow{OD}| = |\overrightarrow{OA}|\cos\theta$$

This means that the vector $\overrightarrow{OD}$, the horizontal component of $\overrightarrow{OA}$, has magnitude $|\overrightarrow{OA}|\cos\theta$.

The magnitude of the vertical component of $\overrightarrow{OA}$ is calculated in the same way using $\triangle OEA$.

$$\text{In } \triangle OEA,\ \cos(90° - \theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{|\overrightarrow{OE}|}{|\overrightarrow{OA}|}$$

$$\text{Therefore, } |\overrightarrow{OE}| = |\overrightarrow{OA}|\cos(90° - \theta)$$

$$\text{Since } \sin\theta = \cos(90° - \theta),$$

$$|\overrightarrow{OE}| = |\overrightarrow{OA}|\sin\theta$$

What we have shown is that $|\overrightarrow{OD}| = |\overrightarrow{OA}|\cos\theta$ and $|\overrightarrow{OE}| = |\overrightarrow{OA}|\sin\theta$. If we replace $\overrightarrow{OA}$ with $\vec{f}$, this would imply that $|\vec{f_x}| = |\vec{f}|\cos\theta$ and $|\vec{f_y}| = |\vec{f}|\sin\theta$, where $\vec{f_x}$ and $\vec{f_y}$ represent the horizontal and vertical components of $\vec{f}$, respectively.

Resolution of a Vector into Horizontal and Vertical Components

If the vector $\vec{f}$ is resolved into its respective horizontal and vertical components, $\vec{f_x}$ and $\vec{f_y}$, then $|\vec{f_x}| = |\vec{f}|\cos\theta$ and $|\vec{f_y}| = |\vec{f}|\sin\theta$, where θ is the angle that $\vec{f}$ makes with the x-axis.

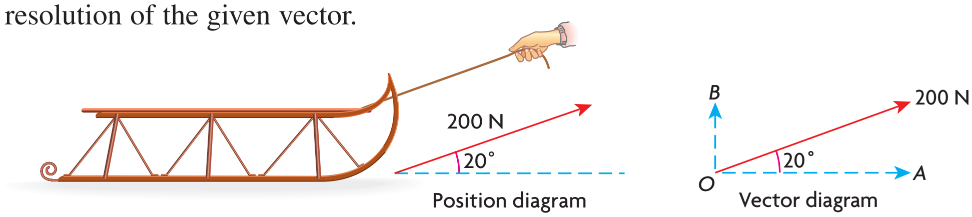
EXAMPLE 3 Connecting forces to the components of a given vector

Kayla pulls on a rope attached to her sleigh with a force of 200 N. If the rope makes an angle of 20° with the horizontal, determine:

a. the force that pulls the sleigh forward
b. the force that tends to lift the sleigh

Solution

In this problem, we are asked to resolve the force vector into its two rectangular components. We start by drawing a position diagram and, beside it, show the resolution of the given vector.



From the diagram, the vector $\overrightarrow{OA}$ is the horizontal component of the given force vector that pulls the sleigh forward. The vector $\overrightarrow{OB}$ is the vertical component of the given force vector that tends to lift the sleigh. To calculate their magnitudes, we directly apply the formulas developed.

$$|\vec{OA}| = 200(\cos 20°) \doteq 200(0.9397) \doteq 187.94\ \text{N}$$

and

$$|\vec{OB}| = 200(\sin 20°) \doteq 200(0.3420) \doteq 68.40\ \text{N}$$

The sleigh is pulled forward with a force of approximately 187.94 N, and the force that tends to lift it is approximately 68.40 N.

In the following example, we will use two different methods to solve the problem. In the first solution, a triangle of forces will be used. In the second solution, the concept of resolution of forces will be used.

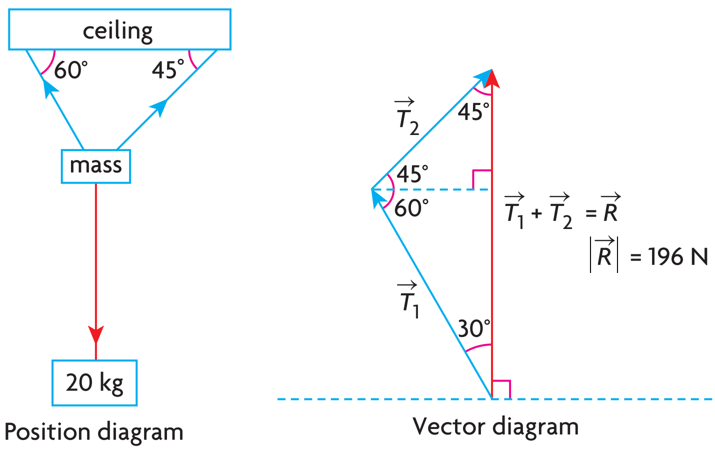
EXAMPLE 4 Selecting a strategy to solve a problem involving several forces

A mass of 20 kg is suspended from a ceiling by two lengths of rope that make angles of 60° and 45° with the ceiling. Determine the tension in each of the ropes.

Solution

Method 1 Triangle of forces

First, recall that the downward force exerted per kilogram is 9.8 N. So the 20 kg mass exerts a downward force of 196 N. Draw a position diagram and a vector diagram. Let the tension vectors for the two pieces of rope be $\vec{T_1}$ and $\vec{T_2}$, and let their resultant be $\vec{R}$. The magnitude of the resultant force created by the tensions in the ropes must equal the magnitude of the downward force on the mass caused by gravity since the system is in a state of equilibrium.



To calculate the required tensions, it is necessary to use the sine law in the vector diagram.

Thus, $\dfrac{|\vec{T_1}|}{\sin 45°} = \dfrac{|\vec{T_2}|}{\sin 30°} = \dfrac{196}{\sin 105°}$

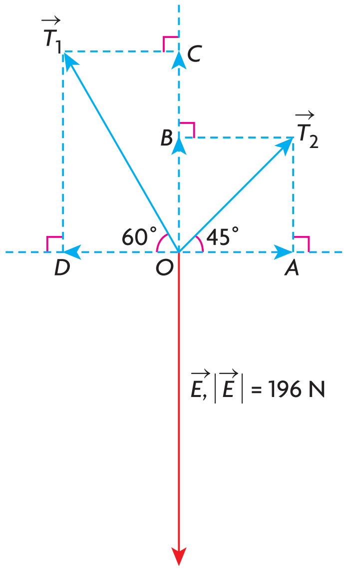
$|\vec{T_1}|\sin 105° = 196(\sin 45°)$ and $|\vec{T_2}|\sin 105° = 196(\sin 30°)$

$|\vec{T_1}| = \dfrac{196(0.7071)}{0.9659} \doteq 143.48\text{ N}$ and $|\vec{T_2}| = \dfrac{196(0.5)}{0.9659} \doteq 101.46\text{ N}$

Therefore, the tensions in the two ropes are approximately 143.48 N and 101.46 N.

Method 2 Resolution of Forces

We start by drawing a diagram showing the tension vectors, $\vec{T_1}$ and $\vec{T_2}$, and the equilibrant, $\vec{E}$. The tension vectors are shown in their resolved form.



For the tension vectors, the magnitudes of their components are calculated.

Horizontal components:

$|\vec{OA}| = \cos 45°|\vec{T_2}| \doteq 0.7071|\vec{T_2}|$ and $|\vec{OB}| \doteq 0.7071|\vec{T_2}|$;

Vertical components:

$|\vec{OC}| = \sin 60°|\vec{T_1}| \doteq 0.8660|\vec{T_1}|$ and $|\vec{OD}| = 0.5|\vec{T_1}|$

For the system to be in equilibrium, the magnitudes of the horizontal and vertical components must balance each other.

Horizontal components: $|\vec{OA}| = |\vec{OD}|$ or $0.7071|\vec{T_2}| \doteq 0.5|\vec{T_1}|$

Vertical components: $|\vec{OB}| + |\vec{OC}| = |\vec{E}|$ or $0.7071|\vec{T_2}| + 0.8660|\vec{T_1}| \doteq 196$

This gives the following system of two equations in two unknowns.

① $0.7071|\vec{T_2}| \doteq 0.5|\vec{T_1}|$

② $0.7071|\vec{T_2}| + 0.8660|\vec{T_1}| \doteq 196$

In equation ①, $|\vec{T_1}| \doteq \dfrac{0.7071|\vec{T_2}|}{0.5}$ or $|\vec{T_1}| \doteq 1.4142|\vec{T_2}|$.

If we substitute this into equation ②, we obtain

$$0.7071|\vec{T_2}| + 0.8660[(1.4142)(|\vec{T_2}|)] \doteq 196$$

$$1.9318|\vec{T_2}| \doteq 196$$

$$|\vec{T_2}| \doteq \frac{196}{1.9318}$$

$$\doteq 101.46 \text{ N}$$

Since $|\vec{T_1}| \doteq 1.4142|\vec{T_2}|$,

$$|\vec{T_1}| \doteq 1.4142(101.46)$$

$$\doteq 143.48 \text{ N}$$

Therefore, the tensions in the ropes are 143.48 N and 101.46 N, as before.

EXAMPLE 5 Reasoning about equilibrium in a system involving three forces

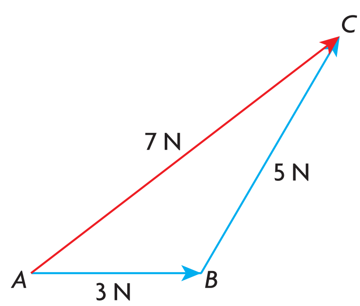
a. Is it possible for three forces of 15 N, 18 N, and 38 N to keep a system in a state of equilibrium?

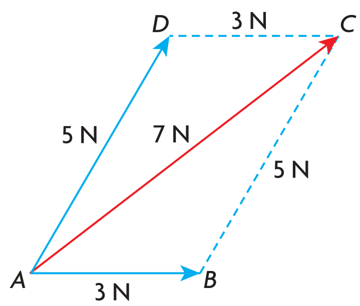
b. Three forces having magnitudes 3 N, 5 N, and 7 N are in a state of equilibrium. Calculate the angle between the two smaller forces.

Solution

a. For a system to be in equilibrium, it is necessary that a triangle be formed having lengths proportional to 15, 18, and 38. Since $15 + 18 < 38$, a triangle cannot be formed because the triangle inequality states that for a triangle to be formed, the sum of any two sides must be greater than or equal to the third side. Therefore, three forces of 15 N, 18 N, and 38 N cannot keep a system in a state of equilibrium.

b. We start by drawing the triangle of forces and the related parallelogram.





Using $\triangle ABC$,

$$|\vec{AC}|^2 = |\vec{AB}|^2 + |\vec{BC}|^2 - 2|\vec{AB}||\vec{BC}|\cos \angle CBA \quad \text{(Cosine law)}$$

$$7^2 = 3^2 + 5^2 - 2(3)(5)\cos \angle CBA$$

$$
\begin{aligned}
49 &= 34 - 30 \cos \angle CBA \\
\frac{-1}{2} &= \cos \angle CBA \\
120° &= \cos^{-1}(-0.5) \angle CBA
\end{aligned}
$$

The angle that is required is $\angle DAB$, the supplement of $\angle CBA$. $\angle DAB = 60°$, and the angle between the 3 N and 5 N force is 60°.

IN SUMMARY

Key Ideas

- Problems involving forces can be solved using strategies involving vectors.
- When two or more forces are applied to an object, the net effect of the forces can be represented by the resultant vector determined by adding the vectors that represent each of the forces.
- A system is in a state of equilibrium when the net effect of all the forces acting on an object causes no movement of the object.

Need to Know

- $\vec{F} = \vec{F_1} + \vec{F_2}$ is the resultant of $\vec{F_1}$ and $\vec{F_2}$.
- $-\vec{F} = -(\vec{F_1} + \vec{F_2})$ is the equilibrant of $\vec{F_1}$ and $\vec{F_2}$.
- If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then $\vec{a}$, $\vec{b}$, and $\vec{c}$ are in a state of equilibrium.

Exercise 7.1

PART A

1. a. Name some common household items that have approximate weights of 10 N, 50 N, and 100 N.

   b. What is your weight in newtons?

2. Three forces of 10 N, 20 N, and 30 N are in a state of equilibrium.

   a. Draw a sketch of these three forces.

   b. What is the angle between the equilibrant and each of the smaller forces?

3. Two forces of 10 N and 20 N are acting on an object. How should these forces be arranged to produce the largest possible resultant?

4. Explain in your own words why three forces must lie in the same plane if they are acting on an object in equilibrium.

K

PART B

5. Determine the resultant and equilibrant of each pair of forces acting on an object.
   a. $\vec{f_1}$ has a magnitude of 5 N acting due east, and $\vec{f_2}$ has a magnitude of 12 N acting due north.
   b. $\vec{f_1}$ has a magnitude of 9 N acting due west, and $\vec{f_2}$ has a magnitude of 12 N acting due south.

6. Which of the following sets of forces acting on an object could produce equilibrium?
   a. 2 N, 3 N, 4 N
   b. 9 N, 40 N, 41 N
   c. $\sqrt{5}$ N, 6 N, 9 N
   d. 9 N, 10 N, 19 N

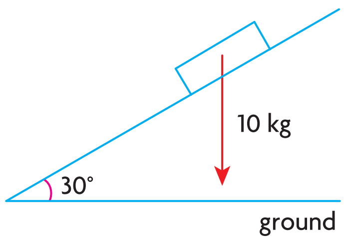
7. Using a vector diagram, explain why it is easier to do chin-ups when your hands are 30 cm apart instead of 90 cm apart. (Assume that the force exerted by your arms is the same in both cases.)

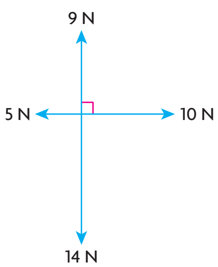
8. A force, $\vec{f_1}$, of magnitude 6 N acts on particle P. A second force, $\vec{f_2}$, of magnitude 8 N acts at 60° to $\vec{f_1}$. Determine the resultant and equilibrant of $\vec{f_1}$ and $\vec{f_2}$.

9. Resolve a force of 10 N into two forces perpendicular to each other, such that one component force makes an angle of 15° with the 10 N force.

A

10. A 10 kg block lies on a smooth ramp that is inclined at 30°. What force, parallel to the ramp, would prevent the block from moving? (Assume that 1 kg exerts a force of 9.8 N.)





11. Three forces, with magnitudes 13 N, 7 N, and 8 N, are in a state of equilibrium.
    a. Draw a sketch of these three forces.
    b. Determine the angle between the two smallest forces.

12. Four forces of magnitude 5 N, 9 N, 10 N, and 14 N are arranged as shown in the diagram at the left. Determine the resultant of these forces.

13. Two forces, $\vec{f_1}$ and $\vec{f_2}$, act at right angles to each other. The magnitude of the resultant of these two forces is 25 N, and $|\vec{f_1}| = 24$ N.

a. Determine $|\vec{f_2}|$.

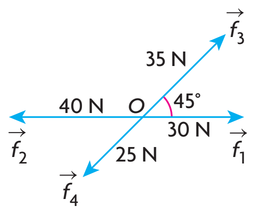
b. Determine the angle between $\vec{f_1}$ and the resultant, and the angle between $\vec{f_1}$ and the equilibrant.

C 14. Three forces, each having a magnitude of 1 N, are arranged to produce equilibrium.

a. Draw a sketch showing an arrangement of these forces, and demonstrate that the angle between the resultant and each of the other two forces is 60°.

b. Explain how to determine the angle between the equilibrant and the other two vectors.

T 15. Four forces, $\vec{f_1}$, $\vec{f_2}$, $\vec{f_3}$, and $\vec{f_4}$, are acting on an object and lie in the same plane, as shown. The forces $\vec{f_1}$ and $\vec{f_2}$ act in an opposite direction to each other, with $|\vec{f_1}| = 30$ N and $|\vec{f_2}| = 40$ N. The forces $\vec{f_3}$ and $\vec{f_4}$ also act in opposite directions, with $|\vec{f_3}| = 35$ N and $|\vec{f_4}| = 25$ N. If the angle between $\vec{f_1}$ and $\vec{f_3}$ is 45°, determine the resultant of these four forces.



16. A mass of 20 kg is suspended from a ceiling by two lengths of rope that make angles of 30° and 45° with the ceiling. Determine the tension in each of the ropes.

17. A mass of 5 kg is suspended by two strings, 24 cm and 32 cm long, from two points that are 40 cm apart and at the same level. Determine the tension in each of the strings.

PART C

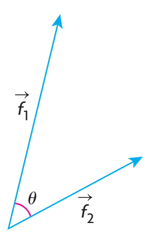
18. Two tugs are towing a ship. The smaller tug is 15° off the port bow, and the larger tug is 20° off the starboard bow. The larger tug pulls twice as hard as the smaller tug. In what direction will the ship move?

19. Three forces of 5 N, 8 N, and 10 N act from the corner of a rectangular solid along its three edges.

a. Calculate the magnitude of the equilibrant of these three forces.

b. Determine the angle that the equilibrant makes with each of the three forces.

20. Two forces, $\vec{f_1}$ and $\vec{f_2}$, make an angle θ with each other when they are placed tail to tail, as shown. Prove that $|\vec{f_1} + \vec{f_2}| = \sqrt{|\vec{f_1}|^2 + |\vec{f_2}|^2 + 2|\vec{f_1}||\vec{f_2}|\cos\theta}$.



Section 7.2—Velocity

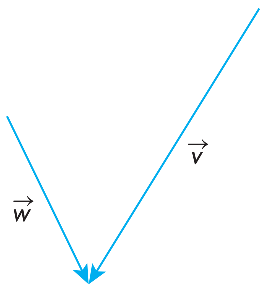
In the previous chapter, we showed that velocity is a vector because it had both magnitude (speed) and direction. In this section, we will demonstrate how two velocities can be combined to determine their resultant velocity.

EXAMPLE 1

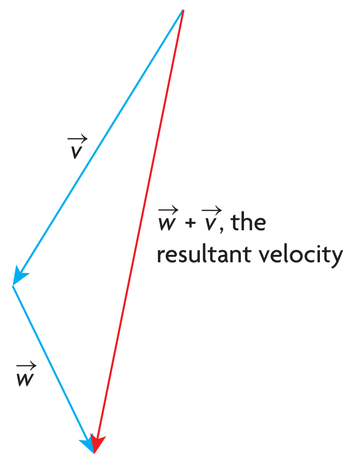
Representing velocity with diagrams

An airplane has a velocity of $\vec{v}$ (relative to the air) when it encounters a wind having a velocity of $\vec{w}$ (relative to the ground). Draw a diagram showing the possible positions of the velocities and another diagram showing the resultant velocity.

Solution



Position diagram



Vector diagram

The resultant velocity of any two velocities is their sum. In all calculations involving resultant velocities, it is necessary to draw a triangle showing the velocities so there is a clear recognition of the resultant and its relationship to the other two velocities. When the velocity of the airplane is mentioned, it is understood that we are referring to its air speed. When the velocity of the wind is mentioned, we are referring to its velocity relative to a fixed point, the ground. The resultant velocity of the airplane is the velocity of the airplane relative to the ground and is called the ground velocity of the airplane.

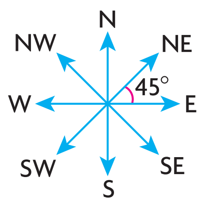
EXAMPLE 2

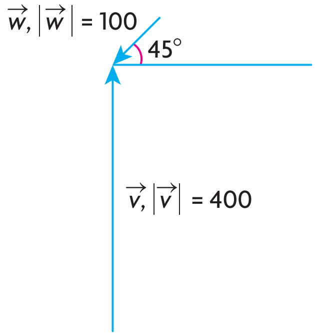
Selecting a vector strategy to determine ground velocity

A plane is heading due north with an air speed of 400 km/h when it is blown off course by a wind of 100 km/h from the northeast. Determine the resultant ground velocity of the airplane.

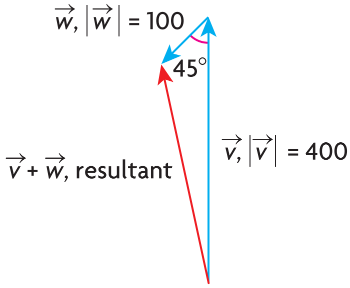
Solution

We start by drawing position and vector diagrams where $\vec{w}$ represents the velocity of the wind and $\vec{v}$ represents the velocity of the airplane in kilometres per hour.





Position diagram



Vector diagram

Use the cosine law to determine the magnitude of the resultant velocity.

$$|\vec{v} + \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}||\vec{w}|\cos\theta,\ \theta = 45°,\ |\vec{w}| = 100,\ |\vec{v}| = 400$$

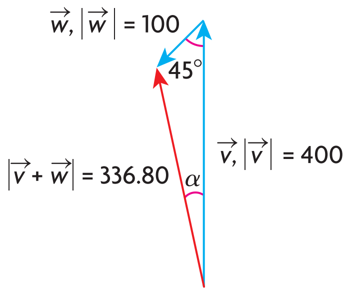
$$|\vec{v} + \vec{w}|^2 = 400^2 + 100^2 - 2(100)(400)\cos 45°$$

$$|\vec{v} + \vec{w}|^2 = 160\,000 + 10\,000 - 80\,000\left(\frac{1}{\sqrt{2}}\right)$$

$$|\vec{v} + \vec{w}|^2 = 170\,000 - \frac{80\,000}{\sqrt{2}}$$

$$|\vec{v} + \vec{w}| \doteq 336.80$$

To state the required velocity, the direction of the resultant vector is needed. Use the sine law to calculate α, the angle between the velocity vector of the plane and the resultant vector.



$$\frac{\sin\alpha}{100} \doteq \frac{\sin 45°}{336.80}$$

$$\sin\alpha \doteq \frac{100\sin 45°}{336.80} \doteq 0.2099$$

$$\alpha \doteq 12.1°$$

Therefore, the resultant velocity is approximately 336.80 km/h, $N12.1°W$ (or $W77.9°N$).

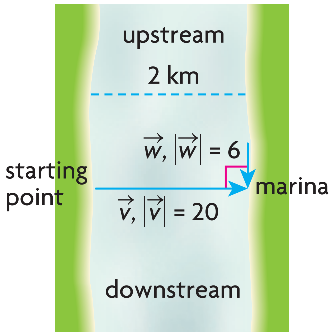
EXAMPLE 3 Using vectors to represent velocities

A river is 2 km wide and flows at 6 km/h. Anna is driving a motorboat, which has a speed of 20 km/h in still water and she heads out from one bank in a direction perpendicular to the current. A marina lies directly across the river from the starting point on the opposite bank.

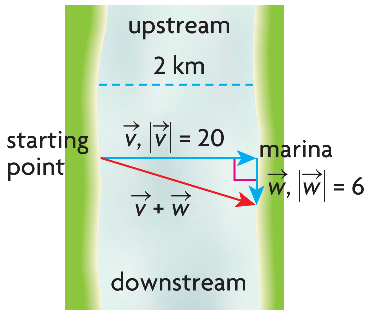
a. How far downstream from the marina will the current push the boat?
b. How long will it take for the boat to cross the river?
c. If Anna decides that she wants to end up directly across the river at the marina, in what direction should she head? What is the resultant velocity of the boat?

Solution

a. As before, we construct a vector and position diagram, where $\vec{w}$ and $\vec{v}$ represent the velocity of the river and the boat, respectively, in kilometres per hour.

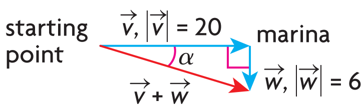


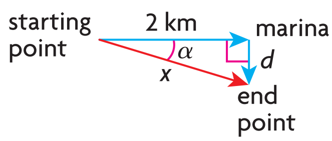
Position diagram



Vector diagram

The distance downstream that the boat lands can be calculated in a variety of ways, but the easiest way is to redraw the velocity triangle from the vector diagram, keeping in mind that the velocity triangle is *similar* to the distance triangle. This is because the distance travelled is directly proportional to the velocity.





Using similar triangles, $\frac{6}{20} = \frac{d}{2}$, $d = 0.6$.

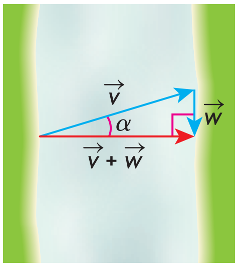
The boat will touch the opposite bank 0.6 km downstream.

b. To calculate the actual distance between the starting and end points, the Pythagorean theorem is used for the distance triangle, with x being the required distance. Thus, $x^2 = 2^2 + (0.6)^2 = 4.36$ and $x \doteq 2.09$, which means that the actual distance the boat travelled was approximately 2.09 km.

To calculate the length of time it took to make the trip, it is necessary to calculate the speed at which this distance was travelled. Again, using similar triangles, $\frac{20}{2} \doteq \frac{|\vec{v} + \vec{w}|}{2.09}$. Solving this proportion, $|\vec{v} + \vec{w}| \doteq 20.9$, so the actual speed of the boat crossing the river was about 20.9 km/h. The actual time taken to cross the river is $t = \frac{d}{v} \doteq \frac{2.09}{20.9} \doteq 0.1$ h, or about 6 min.

Therefore, the boat landed 0.6 km downstream, and it took approximately 6 min to make the crossing.

c. To determine the velocity with which she must travel to reach the marina, we will draw the related vector diagram.



We are given $|\vec{w}| = 6$ and $|\vec{v}| = 20$. To determine the direction in which the boat must travel, let α represent the angle upstream at which the boat heads out.

$$\sin \alpha = \frac{6}{20} \text{ or } \sin^{-1}\left(\frac{6}{20}\right) = \alpha$$

$$\alpha \doteq 17.5°$$

To calculate the magnitude of the resultant velocity, use the Pythagorean theorem.
$|\vec{v}|^2 = |\vec{w}|^2 + |\vec{v} + \vec{w}|^2$ where $|\vec{v}| = 20$ and $|\vec{w}| = 6$

Thus, $20^2 = 6^2 + |\vec{v} + \vec{w}|^2$

$$|\vec{v} + \vec{w}|^2 = 400 - 36$$

$$|\vec{v} + \vec{w}| \doteq 19.08$$

This implies that if Anna wants to travel directly across the river, she will have to travel upstream 17.5° with a speed of approximately 19.08 km/h. The nose of the boat will be headed upstream at 17.5°, but the boat will actually be moving directly across the river at a water speed of 19.08 km/h.

IN SUMMARY

Key Idea

- Problems involving velocities can be solved using strategies involving vectors.

Need to Know

- The velocity of an object is stated relative to a frame of reference. The frame of reference used influences the stated velocity of the object.
- Air speed/water speed is the speed of a plane/boat relative to a person on board. Ground speed is the speed of a plane or boat relative to a person on the ground and includes the effect of wind or current.
- The resultant velocity $\vec{v_r} = \vec{v_1} + \vec{v_2}$.

Exercise 7.2

PART A

1. A woman walks at 4 km/h down the corridor of a train that is travelling at 80 km/h on a straight track.
   a. What is her resultant velocity in relation to the ground if she is walking in the same direction as the train?
   b. If she walks in the opposite direction as the train, what is her resultant velocity?
2. An airplane heading north has an air speed of 600 km/h.
   a. If the airplane encounters a wind from the north at 100 km/h, what is the resultant ground velocity of the plane?
   b. If there is a wind from the south at 100 km/h, what is the resultant ground velocity of the plane?

PART B

3. An airplane has an air speed of 300 km/h and is heading due west. If it encounters a wind blowing south at 50 km/h, what is the resultant ground velocity of the plane?
4. K Adam can swim at the rate of 2 km/h in still water. At what angle to the bank of a river must he head if he wants to swim directly across the river and the current in the river moves at the rate of 1 km/h?
5. A child, sitting in the backseat of a car travelling at 20 m/s, throws a ball at 2 m/s to her brother who is sitting in the front seat.
   a. What is the velocity of the ball relative to the children?
   b. What is the velocity of the ball relative to the road?
6. A boat heads 15° west of north with a water speed of 12 m/s. Determine its resultant velocity, relative to the ground, when it encounters a 5 m/s current from 15° north of east.
7. An airplane is heading due north at 800 km/h when it encounters a wind from the northeast at 100 km/h.
   a. What is the resultant velocity of the airplane?
   b. How far will the plane travel in 1 h?
8. An airplane is headed north with a constant velocity of 450 km/h. The plane encounters a wind from the west at 100 km/h.
   a. In 3 h, how far will the plane travel?
   b. In what direction will the plane travel?

A 9. A small airplane has an air speed of 244 km/h. The pilot wishes to fly to a destination that is 480 km due west from the plane's present location. There is a 44 km/h wind from the south.

a. In what direction should the pilot fly in order to reach the destination?

b. How long will it take to reach the destination?

10. Judy and her friend Helen live on opposite sides of a river that is 1 km wide. Helen lives 2 km downstream from Judy on the opposite side of the river. Judy can swim at a rate of 3 km/h, and the river's current has a speed of 4 km/h. Judy swims from her cottage directly across the river.

a. What is Judy's resultant velocity?

b. How far away from Helen's cottage will Judy be when she reaches the other side?

c. How long will it take Judy to reach the other side?

T 11. An airplane is travelling $N60°E$ with a resultant ground speed of 205 km/h. The nose of the plane is actually pointing east with an airspeed of 212 km/h.

a. What is the wind direction?

b. What is the wind speed?

C 12. Barbara can swim at 4 km/h in still water. She wishes to swim across a river to a point directly opposite from where she is standing. The river is moving at a rate of 5 km/h. Explain, with the use of a diagram, why this is not possible.

PART C

13. Mary leaves a dock, paddling her canoe at 3 m/s. She heads downstream at an angle of 30° to the current, which is flowing at 4 m/s.

a. How far downstream does Mary travel in 10 s?

b. What is the length of time required to cross the river if its width is 150 m?

14. Dave wants to cross a 200 m wide river whose current flows at 5.5 m/s. The marina he wants to visit is located at an angle of S45°W from his starting position. Dave can paddle his canoe at 4 m/s in still water.

a. In which direction should he head to get to the marina?

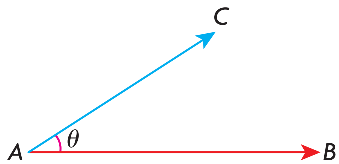
b. How long will the trip take?

15. A steamboat covers the distance between town A and town B (located downstream) in 5 h without making any stops. Moving upstream from B to A, the boat covers the same distance in 7 h (again making no stops). How many hours does it take a raft moving with the speed of the river current to get from A to B?

Section 7.3—The Dot Product of Two Geometric Vectors

In Chapter 6, the concept of multiplying a vector by a scalar was discussed. In this section, we introduce the dot product of two vectors and deal specifically with geometric vectors. When we refer to geometric vectors, we are referring to vectors that do not have a coordinate system associated with them. The dot product for any two vectors is defined as the product of their magnitudes multiplied by the cosine of the angle between the two vectors when the two vectors are placed in a tail-to-tail position.

Dot Product of Two Vectors



$\overrightarrow{AC} \cdot \overrightarrow{AB} = |\overrightarrow{AC}||\overrightarrow{AB}|\cos\theta, 0 \leq \theta \leq 180°$

Observations about the Dot Product

There are some elementary but important observations that can be made about this calculation. First, the result of the dot product is always a scalar. Each of the quantities on the right side of the formula above is a scalar quantity, and so their product must be a scalar. For this reason, the dot product is also known as the **scalar product**. Second, the dot product can be positive, zero, or negative, depending upon the size of the angle between the two vectors.

Sign of the Dot Product

For the vectors $\vec{a}$ and $\vec{b}$, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta, 0 \leq \theta \leq 180°$:

- for $0 \leq \theta < 90°$, $\cos\theta > 0$, so $\vec{a} \cdot \vec{b} > 0$
- for $\theta = 90°$, $\cos\theta = 0$, so $\vec{a} \cdot \vec{b} = 0$
- for $90° < \theta \leq 180°$, $\cos\theta < 0$, so $\vec{a} \cdot \vec{b} < 0$

The dot product is only calculated for vectors when the angle θ between the vectors is to 0° to 180°, inclusive. (For convenience in calculating, the angle between the vectors is usually expressed in degrees, but radian measure is also correct.)

Perhaps the most important observation to be made about the dot product is that when two nonzero vectors are perpendicular, their dot product is always 0. This will have many important applications in Chapter 8, when we discuss lines and planes.

EXAMPLE 1 Calculating the dot product of two geometric vectors

Two vectors, $\vec{a}$ and $\vec{b}$, are placed tail to tail and have magnitudes 3 and 5, respectively. There is an angle of 120° between the vectors. Calculate $\vec{a} \cdot \vec{b}$.

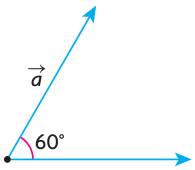
Solution

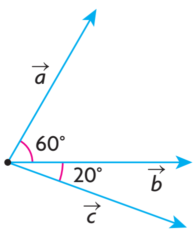
Since $|\vec{a}| = 3$, $|\vec{b}| = 5$, and $\cos 120° = -0.5$,

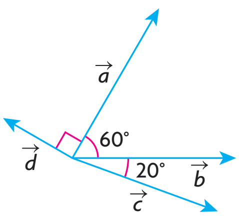
$$\begin{aligned}\vec{a} \cdot \vec{b} &= (3)(5)(-0.5)\\ &= -7.5\end{aligned}$$

Notice that, in this example, it is stated that the vectors are tail to tail when taking the dot product. This is the convention that is always used, since this is the way of defining the angle between any two vectors.

INVESTIGATION







A. Given vectors $\vec{a}$ and $\vec{b}$ where $|\vec{a}| = 5$, $|\vec{b}| = 8$ and the angle between the vectors is 60°, calculate $\vec{a} \cdot \vec{b}$.

B. For the vectors given in part A calculate $\vec{b} \cdot \vec{a}$. What do you notice? Will this relationship always hold regardless of the two vectors used and the measure of the angle between them? Explain.

C. For the vectors given in part A calculate $\vec{a} \cdot \vec{a}$ and $\vec{b} \cdot \vec{b}$. Based on your observations, what can you conclude about $\vec{u} \cdot \vec{u}$ for any vector $\vec{u}$?

D. Using the vectors given in part A and a third vector $\vec{c}$, $|\vec{c}| = 4$, as shown in the diagram, calculate each of the following without rounding:

   i. $|\vec{b} + \vec{c}|$

   ii. the angle between $\vec{b} + \vec{c}$ and $\vec{a}$.

   iii. $\vec{a} \cdot (\vec{b} + \vec{c})$

   iv. $\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

E. Compare your results from part iii and iv in part D. What property does this demonstrate? Write an equivalent expression for $\vec{c} \cdot (\vec{a} + \vec{b})$ and confirm it using the appropriate calculations with the vectors given in part D.

F. Using the 3 vectors given above, explain why $(\vec{a} \cdot \vec{b}) \cdot \vec{c} \neq \vec{a} \cdot (\vec{b} \cdot \vec{c})$.

G. A fourth vector $\vec{d}$, $|\vec{d}| = 3$, is given as shown in the diagram. Explain why $\vec{a} \cdot \vec{d} = \vec{a} \cdot (-\vec{d}) = (-\vec{a}) \cdot \vec{d} = (-\vec{a}) \cdot (-\vec{d})$

H. Using the vectors given, calculate $\vec{a} \cdot \vec{0}$, $\vec{b} \cdot \vec{0}$ and $\vec{c} \cdot \vec{0}$. What does this imply?

Properties of the Dot Product

It should also be noted that the dot product can be calculated in whichever order we choose. In other words, $\vec{p} \cdot \vec{q} = |\vec{p}||\vec{q}|\cos\theta = |\vec{q}||\vec{p}|\cos\theta = \vec{q} \cdot \vec{p}$. We can change the order in the multiplication because the quantities in the formula are just scalars (that is, numbers) and the order of multiplication does not affect the final answer. This latter property is known as the *commutative* property for the dot product.

Another property that proves to be quite important for both computation and theoretical purposes is the dot product between a vector $\vec{p}$ and itself. The angle between $\vec{p}$ and itself is 0°, so $\vec{p} \cdot \vec{p} = |\vec{p}||\vec{p}|(1) = |\vec{p}|^2$ since $\cos(0°) = 1$.

EXAMPLE 2 **Calculating the dot product between a vector and itself**

a. If $|\vec{a}| = \sqrt{7}$, calculate $\vec{a} \cdot \vec{a}$.

b. Calculate $\vec{i} \cdot \vec{i}$.

Solution

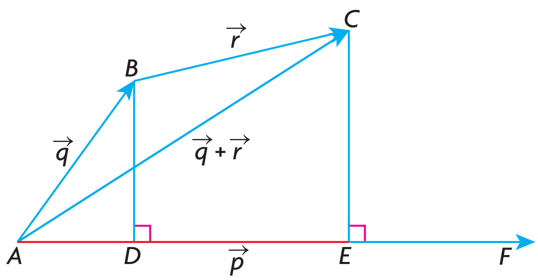
a. This is an application of the property just shown. So, $\vec{a} \cdot \vec{a} = (\sqrt{7})(\sqrt{7}) = 7$.

b. Since we know that $\vec{i}$ is a unit vector (along the positive x-axis), $\vec{i} \cdot \vec{i} = (1)(1) = 1$. In general, for any vector $\vec{x}$ of unit length, $\vec{x} \cdot \vec{x} = |\vec{x}|^2 = 1$. Thus, $\vec{j} \cdot \vec{j} = 1$ and $\vec{k} \cdot \vec{k} = 1$, where $\vec{j}$ and $\vec{k}$ are the unit vectors along the positive y- and z-axes, respectively.

Another important property that the dot product follows is the *distributive* property. In elementary algebra, the distributive property states that $p(q + r) = pq + pr$. We will prove that the distributive property also holds for the dot product. We will prove this geometrically below and algebraically in the next section.

Theorem: For the vectors $\vec{p}$, $\vec{q}$, and $\vec{r}$, $\vec{p}(\vec{q} + \vec{r}) = \vec{p} \cdot \vec{q} + \vec{p} \cdot \vec{r}$.

Proof: The vectors $\vec{p}$, $\vec{q}$, and $\vec{r}$, are drawn, and the diagram is labelled as shown with $\overrightarrow{AC} = \vec{q} + \vec{r}$. To help visualize the dot products, lines from B and C have been drawn perpendicular to $\vec{p}$ (which is $\overrightarrow{AF}$).

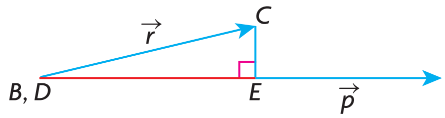


Using the definition of a dot product, we write $\vec{q} \cdot \vec{p} = |\vec{q}||\vec{p}|\cos BAF$.

If we look at the right-angled triangle ABD and use the cosine ratio, we note that $\cos BAD = \frac{AD}{|\vec{q}|}$ or $AD = |\vec{q}|\cos BAD$. The two angles BAD and BAF are identical, and so $AD = |\vec{q}|\cos BAF$.

Rewriting the formula $\vec{q}\cdot\vec{p} = |\vec{q}||\vec{p}|\cos BAF$ as $\vec{q}\cdot\vec{p} = (|\vec{q}|\cos BAF)|\vec{p}|$, and substituting $AD = |\vec{q}|\cos BAF$, we obtain, $\vec{q}\cdot\vec{p} = AD|\vec{p}|$.

We also consider the vectors $\vec{r}$ and $\vec{p}$. We translate the vector $\overrightarrow{BC}$ so that point B is moved to be coincident with D. (The vector $\overrightarrow{BC}$ maintains the same direction and size under this translation.)



Writing the dot product for $\vec{r}$ and $\vec{p}$, we obtain $\vec{r}\cdot\vec{p} = |\vec{r}||\vec{p}|\cos CDE$. If we use trigonometric ratios in the right triangle, $\cos CDE = \frac{DE}{|\vec{r}|}$ or $DE = |\vec{r}|\cos CDE$.

Substituting $DE = |\vec{r}|\cos CDE$ into $\vec{r}\cdot\vec{p} = |\vec{r}||\vec{p}|\cos CDE$, we obtain $\vec{r}\cdot\vec{p} = DE|\vec{p}|$. If we use the formula for the dot product of $\vec{q} + \vec{r}$ and $\vec{p}$, we get the following: $(\vec{q} + \vec{r})\cdot\vec{p} = |\vec{q} + \vec{r}||\vec{p}|\cos CAE$. Using the same reasoning as before, $\cos CAE = \frac{AE}{|\vec{q} + \vec{r}|}$ and $AE = (\cos CAE)|\vec{q} + \vec{r}|$, and then, by substitution, $(\vec{q} + \vec{r})\cdot\vec{p} = |\vec{p}|AE$.

Adding the two quantities $\vec{q}\cdot\vec{p}$ and $\vec{r}\cdot\vec{p}$,

$$\begin{aligned}\vec{q}\cdot\vec{p} + \vec{r}\cdot\vec{p} &= AD|\vec{p}| + DE|\vec{p}| \\ &= |\vec{p}|(AD + DE) && \text{(Factoring)} \\ &= |\vec{p}|AE \\ &= (\vec{q} + \vec{r})\cdot\vec{p}\end{aligned}$$

Thus, $\vec{q}\cdot\vec{p} + \vec{r}\cdot\vec{p} = (\vec{q} + \vec{r})\cdot\vec{p}$, or, written in the more usual way,

$$\vec{p}\cdot(\vec{q} + \vec{r}) = \vec{p}\cdot\vec{q} + \vec{p}\cdot\vec{r}$$

We list some of the properties of the dot product below.
This final property has not been proven, but it comes directly from the definition of the dot product and proves most useful in computation.

Properties of the Dot Product

Commutative Property: $\vec{p} \cdot \vec{q} = \vec{q} \cdot \vec{p}$,

Distributive Property: $\vec{p} \cdot (\vec{q} + \vec{r}) = \vec{p} \cdot \vec{q} + \vec{p} \cdot \vec{r}$,

Magnitudes Property: $\vec{p} \cdot \vec{p} = |\vec{p}|^2$,

Associative Property with a scalar K: $(k\vec{p}) \cdot \vec{q} = \vec{p} \cdot (k\vec{q}) = k(\vec{p} \cdot \vec{q})$

EXAMPLE 3 Selecting a strategy to determine the angle between two geometric vectors

If the vectors $\vec{a} + 3\vec{b}$ and $4\vec{a} - \vec{b}$ are perpendicular, and $|\vec{a}| = 2|\vec{b}|$, determine the angle (to the nearest degree) between the nonzero vectors $\vec{a}$ and $\vec{b}$.

Solution

Since the two given vectors are perpendicular, $(\vec{a} + 3\vec{b}) \cdot (4\vec{a} - \vec{b}) = 0$.

Multiplying, $\vec{a} \cdot (4\vec{a} - \vec{b}) + 3\vec{b} \cdot (4\vec{a} - \vec{b}) = 0$

$4\vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} + 12\vec{b} \cdot \vec{a} - 3\vec{b} \cdot \vec{b} = 0$ (Distributive property)

Simplifying, $4|\vec{a}|^2 + 11\vec{a} \cdot \vec{b} - 3|\vec{b}|^2 = 0$ (Commutative property)

Since $|\vec{a}| = 2|\vec{b}|$, $|\vec{a}|^2 = (2|\vec{b}|)^2 = 4|\vec{b}|^2$ (Squaring both sides)

Substituting, $4(4|\vec{b}|^2) + 11((2|\vec{b}|)(|\vec{b}|)\cos\theta) - 3|\vec{b}|^2 = 0$

Solving for $\cos\theta$,

$$\cos\theta = \frac{-13|\vec{b}|^2}{22|\vec{b}|^2}$$

$$\cos\theta = \frac{-13}{22}, \ |\vec{b}|^2 \neq 0$$

Thus, $\cos^{-1}\left(\frac{-13}{22}\right) = \theta, \theta \doteq 126.2°$

Therefore, the angle between the two vectors is approximately 126.2°.

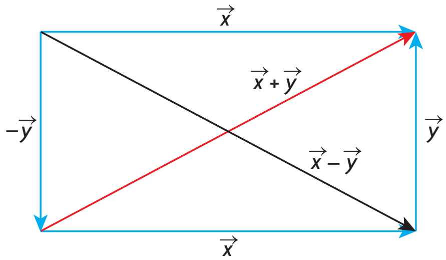
It is often necessary to square the magnitude of a vector expression. This is illustrated in the following example.

EXAMPLE 4 Proving that two vectors are perpendicular using the dot product

If $|\vec{x} + \vec{y}| = |\vec{x} - \vec{y}|$, prove that the nonzero vectors, $\vec{x}$ and $\vec{y}$, are perpendicular.

Solution

Consider the following diagram.



Since $|\vec{x} + \vec{y}| = |\vec{x} - \vec{y}|$, $|\vec{x} + \vec{y}|^2 = |\vec{x} - \vec{y}|^2$ (Squaring both sides)

$|\vec{x} + \vec{y}|^2 = (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y})$ and $|\vec{x} - \vec{y}|^2 = (\vec{x} - \vec{y}) \cdot (\vec{x} - \vec{y})$ (Magnitudes property)

Therefore, $(\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) = (\vec{x} - \vec{y}) \cdot (\vec{x} - \vec{y})$

$$|\vec{x}|^2 + 2\vec{x} \cdot \vec{y} + |\vec{y}|^2 = |\vec{x}|^2 - 2\vec{x} \cdot \vec{y} + |\vec{y}|^2$$ (Multiplying out)

So, $4\vec{x} \cdot \vec{y} = 0$ and $\vec{x} \cdot \vec{y} = 0$

Thus, the two required vectors are shown to be perpendicular. (Geometrically, this means that if diagonals in a parallelogram are equal in length, then the sides must be perpendicular. In actuality, the parallelogram is a rectangle.)

In this section, we dealt with the dot product and its geometric properties. In the next section, we will illustrate these same ideas with algebraic vectors.

IN SUMMARY

Key Idea

- The dot product between two geometric vectors $\vec{a}$ and $\vec{b}$ is a scalar quantity defined as $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between the two vectors.

Need to Know

- If $0° \le \theta < 90°$, then $\vec{a} \cdot \vec{b} > 0$
- If $\theta = 90°$, then $\vec{a} \cdot \vec{b} = 0$
- If $90° < \theta \le 180°$, then $\vec{a} \cdot \vec{b} < 0$
- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
- $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
- $\vec{i} \cdot \vec{i} = 1$, $\vec{j} \cdot \vec{j} = 1$, and $\vec{k} \cdot \vec{k} = 1$
- $(k\vec{a}) \cdot \vec{b} = \vec{a} \cdot (k\vec{b}) = k(\vec{a} \cdot \vec{b})$

Exercise 7.3

PART A

1. If $\vec{a} \cdot \vec{b} = 0$, why can we not necessarily conclude that the given vectors are perpendicular? (In other words, what restrictions must be placed on the vectors to make this statement true?)

C 2. Explain why the calculation $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$ is not meaningful.

3. A student writes the statement $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c}$ and then concludes that $\vec{a} = \vec{c}$. Construct a simple numerical example to show that this is not correct if the given vectors are all nonzero.

4. Why is it correct to say that if $\vec{a} = \vec{c}$, then $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c}$?

5. If two vectors $\vec{a}$ and $\vec{b}$ are unit vectors pointing in opposite directions, what is the value of $\vec{a} \cdot \vec{b}$?

PART B

6. If θ is the angle (in degrees) between the two given vectors, calculate the dot product of the vectors.

a. $|\vec{p}| = 4, |\vec{q}| = 8, \theta = 60°$

b. $|\vec{x}| = 2, |\vec{y}| = 4, \theta = 150°$

c. $|\vec{a}| = 0, |\vec{b}| = 8, \theta = 100°$

d. $|\vec{p}| = 1, |\vec{q}| = 1, \theta = 180°$

e. $|\vec{m}| = 2, |\vec{n}| = 5, \theta = 90°$

f. $|\vec{u}| = 4, |\vec{v}| = 8, \theta = 145°$

7. Calculate, to the nearest degree, the angle between the given vectors.

a. $|\vec{x}| = 8, |\vec{y}| = 3, \vec{x} \cdot \vec{y} = 12\sqrt{3}$

b. $|\vec{m}| = 6, |\vec{n}| = 6, \vec{m} \cdot \vec{n} = 6$

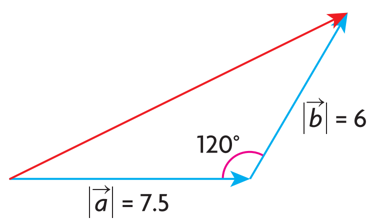
c. $|\vec{p}| = 1, |\vec{q}| = 5, \vec{p} \cdot \vec{q} = 3$

d. $|\vec{p}| = 1, |\vec{q}| = 5, \vec{p} \cdot \vec{q} = -3$

e. $|\vec{a}| = 7, |\vec{b}| = 3, \vec{a} \cdot \vec{b} = 10.5$

f. $|\vec{u}| = 10, |\vec{v}| = 10, \vec{u} \cdot \vec{v} = -50$

K 8. For the two vectors $\vec{a}$ and $\vec{b}$ whose magnitudes are shown in the diagram below, calculate the dot product.



9. Use the properties of the dot product to simplify each of the following expressions:

a. $(\vec{a} + 5\vec{b}) \cdot (2\vec{a} - 3\vec{b})$

b. $3\vec{x} \cdot (\vec{x} - 3\vec{y}) - (\vec{x} - 3\vec{y}) \cdot (-3\vec{x} + \vec{y})$

10. What is the value of the dot product between $\vec{0}$ and any nonzero vector? Explain.

A 11. The vectors $\vec{a} - 5\vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular. If $\vec{a}$ and $\vec{b}$ are unit vectors, then determine $\vec{a} \cdot \vec{b}$.

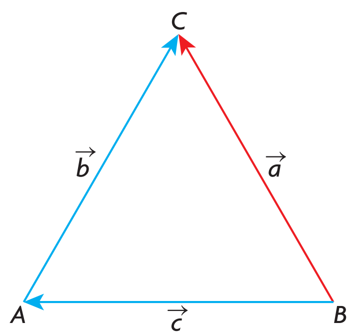
12. If $\vec{a}$ and $\vec{b}$ are any two nonzero vectors, prove each of the following to be true:
    a. $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2$
    b. $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$

13. The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ satisfy the relationship $\vec{a} = \vec{b} + \vec{c}$.
    a. Show that $|\vec{a}|^2 = |\vec{b}|^2 + 2\vec{b} \cdot \vec{c} + |\vec{c}|^2$.
    b. If the vectors $\vec{b}$ and $\vec{c}$ are perpendicular, how does this prove the Pythagorean theorem?

14. Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be three mutually perpendicular vectors of lengths 1, 2, and 3, respectively. Calculate the value of $(\vec{u} + \vec{v} + \vec{w}) \cdot (\vec{u} + \vec{v} + \vec{w})$.

T 15. Prove the identity $|\vec{u} + \vec{v}|^2 + |\vec{u} - \vec{v}|^2 = 2|\vec{u}|^2 + 2|\vec{v}|^2$.

16. The three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are of unit length and form the sides of equilateral triangle *ABC* such that $\vec{a} - \vec{b} - \vec{c} = \vec{0}$ (as shown). Determine the numerical value of $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b} + \vec{c})$.



PART C

17. The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. Determine the value of $\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$ if $|\vec{a}| = 1$, $|\vec{b}| = 2$, and $|\vec{c}| = 3$.

18. The vector $\vec{a}$ is a unit vector, and the vector $\vec{b}$ is any other nonzero vector. If $\vec{c} = (\vec{b} \cdot \vec{a})\vec{a}$ and $\vec{d} = \vec{b} - \vec{c}$, prove that $\vec{d} \cdot \vec{a} = 0$.

Section 7.4—The Dot Product of Algebraic Vectors

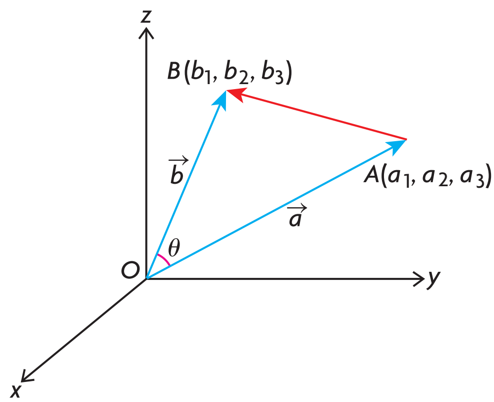
In the previous section, the dot product was discussed in geometric terms. In this section, the dot product will be expressed in terms of algebraic vectors in R^2 and R^3. Recall that a vector expressed as $\vec{a} = (-1, 4, 5)$ is referred to as an algebraic vector. The geometric properties of the dot product developed in the previous section will prove useful in understanding the dot product in algebraic form. The emphasis in this section will be on developing concepts in R^3, but these ideas apply equally well to R^2 or to higher dimensions.

Defining the Dot Product of Algebraic Vectors

Theorem: In R^3, if $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, then

$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} = a_1b_1 + a_2b_2 + a_3b_3$.

Proof: Draw $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, as shown in the diagram.



In ΔOAB, $|\overrightarrow{AB}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2 - 2|\overrightarrow{OA}||\overrightarrow{OB}|\cos\theta$ (Cosine law)

So, $\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$ and

$|\overrightarrow{AB}|^2 = (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2$

We know that $|\overrightarrow{OA}|^2 = a_1^2 + a_2^2 + a_3^2$ and $|\overrightarrow{OB}|^2 = b_1^2 + b_2^2 + b_3^2$.

It should also be noted that $\vec{a} \cdot \vec{b} = |\overrightarrow{OA}||\overrightarrow{OB}|\cos\theta$. (Definition of dot product)

We substitute each of these quantities in the expression for the cosine law.

This gives $(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2 =$

$$a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2\vec{a} \cdot \vec{b}$$

Expanding, we get

$$b_1^2 - 2a_1b_1 + a_1^2 + b_2^2 - 2a_2b_2 + a_2^2 + b_3^2 - 2a_3b_3 + a_3^2$$
$$= a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2\vec{a}\cdot\vec{b} \qquad \text{(Simplify)}$$
$$-2a_1b_1 - 2a_2b_2 - 2a_3b_3 = -2\vec{a}\cdot\vec{b} \qquad \text{(Divide by } -2\text{)}$$
$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Observations about the Algebraic Form of the Dot Product

There are some important observations to be made about this expression for the dot product. First and foremost, the quantity on the right-hand side of the expression, $a_1b_1 + a_2b_2 + a_3b_3$, is evaluated by multiplying corresponding components and then adding them. Each of these quantities, a_1b_1, a_2b_2, and a_3b_3, is just a real number, so their sum is a real number. This implies that $\vec{a}\cdot\vec{b}$ is itself just a real number, or a scalar product. Also, since the right side is an expression made up of real numbers, it can be seen that $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3 = b_1a_1 + b_2a_2 + b_3a_3 = \vec{b}\cdot\vec{a}$. This is a restatement of the commutative law for the dot product of two vectors. All the other rules for computation involving dot products can now be proven using the properties of real numbers and the basic definition of a dot product.

In this proof, we have used vectors in R^3 to calculate a formula for $\vec{a}\cdot\vec{b}$. It is important to understand, however, that this procedure could be used in the same way for two vectors, $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$, in R^2, to obtain the formula $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2$.

EXAMPLE 1 **Proving the distributive property of the dot product in R^3**

Prove that the distributive property holds for dot products in R^3—that is, $\vec{a}\cdot(\vec{b} + \vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$.

Solution

Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$, and $\vec{c} = (c_1, c_2, c_3)$.

In showing this statement to be true, the right side will be expressed in component form and then rearranged to be the same as the left side.

$$\vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} = (a_1, a_2, a_3)\cdot(b_1, b_2, b_3) + (a_1, a_2, a_3)\cdot(c_1, c_2, c_3)$$
$$= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \qquad \text{(Definition of dot product)}$$
$$= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 \qquad \text{(Rearranging terms)}$$
$$= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \qquad \text{(Factoring)}$$
$$= \vec{a}\cdot(\vec{b} + \vec{c})$$

This example shows how to prove the distributive property for the dot product in R^3. The value of writing the dot product in component form is that it allows us to combine the geometric form with the algebraic form, and create the ability to do calculations that would otherwise not be possible.

Computation of the Dot Product of Algebraic Vectors

In R^2, $\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}|\cos\theta = x_1y_1 + x_2x_2$, where $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$.

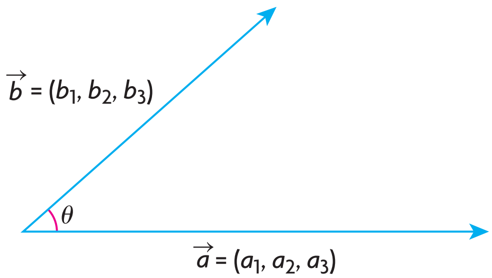
In R^3, $\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}|\cos\theta = x_1y_1 + x_2y_2 + x_3y_3$, where $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$.

In both cases q is the angle between $\vec{x}$ and $\vec{y}$.

The dot product expressed in component form has significant advantages over the geometric form from both a computational and theoretical point of view. At the outset, the calculation appears to be somewhat artificial or contrived, but as we move ahead, we will see its applicability to many situations.

A useful application of the dot product is to calculate the angle between two vectors. Solving for $\cos\theta$ in the formula $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$ gives the following result.

Formula for the Angle between Two Vectors



When two vectors are placed tail to tail, as shown, $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$.

EXAMPLE 2 Selecting a strategy to determine the angle between two algebraic vectors

a. Given the vectors $\vec{a} = (-1, 2, 4)$ and $\vec{b} = (3, 4, 3)$, calculate $\vec{a} \cdot \vec{b}$.

b. Calculate, to the nearest degree, the angle between $\vec{a}$ and $\vec{b}$.

Solution

a. $\vec{a} \cdot \vec{b} = (-1)(3) + (2)(4) + (4)(3) = 17$

b. $|\vec{a}|^2 = (-1)^2 + (2)^2 + (4)^2 = 21,\ |\vec{a}| = \sqrt{21}$

$|\vec{b}|^2 = (3)^2 + (4)^2 + (3)^2 = 34,\ |\vec{b}| = \sqrt{34}$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

$$\cos\theta = \frac{17}{\sqrt{21}\sqrt{34}} \qquad \text{(Substitution)}$$

$$\cos\theta \doteq 0.6362$$

$$\theta \doteq \cos^{-1}(0.6362)$$

$$\theta \doteq 50.5^\circ$$

Therefore, the angle between the two vectors is approximately 50.5°.

In the previous section, we showed that when two nonzero vectors are perpendicular, their dot product equals zero—that is, $\vec{a}\cdot\vec{b} = 0$.

EXAMPLE 3 Using the dot product to solve a problem involving perpendicular vectors

a. For what values of k are the vectors $\vec{a} = (-1, 3, -4)$ and $\vec{b} = (3, k, -2)$ perpendicular?

b. For what values of m are the vectors $\vec{x} = (m, m, -3)$ and $\vec{y} = (m, -3, 6)$ perpendicular?

Solution

a. Since $\vec{a}\cdot\vec{b} = 0$ for perpendicular vectors,

$$-1(3) + 3(k) - 4(-2) = 0$$

$$3k = -5$$

$$k = \frac{-5}{3}$$

In calculations of this type involving the dot product, the calculation should be verified as follows:

$$\begin{aligned}(-1, 3, -4)\cdot\left(3, \tfrac{-5}{3}, -2\right) &= -1(3) + 3\left(\tfrac{-5}{3}\right) - 4(-2)\\ &= -3 - 5 + 8\\ &= 0\end{aligned}$$

This check verifies that the calculation is correct.

b. Using the conditions for perpendicularity of vectors,

$$\begin{aligned}(m, m, -3)\cdot(m, -3, 6) &= 0\\ m^2 - 3m - 18 &= 0\\ (m-6)(m+3) &= 0\\ m = 6 \text{ or } m &= -3\end{aligned}$$

Check:

For $m = 6$, $(6, 6, -3) \cdot (6, -3, 6) = 36 - 18 - 18 = 0$

For $m = -3$, $(-3, -3, -3) \cdot (-3, -3, 6) = 9 + 9 - 18 = 0$

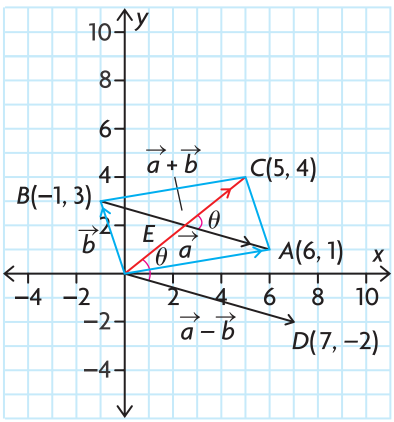
We can combine various operations that we have learned for calculation purposes in R^2 and R^3.

EXAMPLE 4 Using the dot product to solve a problem involving a parallelogram

A parallelogram has its sides determined by $\vec{a} = (6, 1)$ and $\vec{b} = (-1, 3)$. Determine the angle between the diagonals of the parallelogram formed by these vectors.

Solution

The diagonals of the parallelogram are determined by the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, as shown in the diagram. The components of these vectors are
$\vec{a} + \vec{b} = (6 + (-1), 1 + 3) = (5, 4)$ and
$\vec{a} - \vec{b} = (6 - (-1), 1 - 3) = (7, -2)$, as shown in the diagram



At this point, the dot product is applied directly to find θ, the angle between the vectors $\overrightarrow{OD}$ and $\overrightarrow{OC}$.

$$\text{Therefore, } \cos\theta = \frac{(5, 4) \cdot (7, -2)}{|(5, 4)||(7, -2)|}$$

$$\cos\theta = \frac{27}{\sqrt{41}\sqrt{53}}$$

$$\cos\theta \doteq 0.5792$$

$$\text{Therefore, } \theta \doteq 54.61°$$

The angle between the diagonals is approximately 54.6°. The answer given is 54.6°, but its supplement, 125.4°, is also correct.

One of the most important properties of the dot product is its application to determining a perpendicular vector to two given vectors, which will be demonstrated in the following example.

EXAMPLE 5

Selecting a strategy to determine a vector perpendicular to two given vectors

Find a vector (or vectors) perpendicular to each of the vectors $\vec{a} = (1, 5, -1)$ and $\vec{b} = (-3, 1, 2)$.

Solution

Let the required vector be $\vec{x} = (x, y, z)$. Since $\vec{x}$ is perpendicular to each of the two given vectors, $(x, y, z) \cdot (1, 5, -1) = 0$ and $(x, y, z) \cdot (-3, 1, 2) = 0$.

Multiplying gives $x + 5y - z = 0$ and $-3x + y + 2z = 0$, which is a system of two equations in three unknowns.

① $x + 5y - z = 0$

② $-3x + y + 2z = 0$

③ $3x + 15y - 3z = 0$ (Multiplying equation ① by 3)

④ $16y - z = 0$ (Adding equations ② and ③)

$z = 16y$

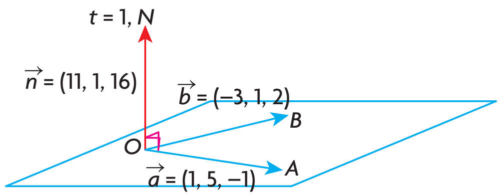
Now, we substitute $z = 16y$ into equation ① to solve for x in terms of y. We obtain $x + 5y - 16y = 0$, or $x = 11y$.

We have solved for x and z by expressing each variable in terms of y. The solution to the system of equations is $(11y, y, 16y)$ or $(11t, t, 16t)$ if we let $y = t$. The substitution of t (called a parameter) for y is not necessarily required for a correct solution and is done more for convenience of notation. This kind of substitution will be used later to great advantage and will be discussed in Chapter 9 at length.

We can find vectors to satisfy the required conditions by replacing t with any real number, $t \neq 0$. Since we can use any real number for t to produce the required vector, this implies that an infinite number of vectors are perpendicular to both $\vec{a}$ and $\vec{b}$. If we use $t = 1$, we obtain $(11, 1, 16)$.

As before, we verify the solution:
$(11, 1, 16) \cdot (1, 5, -1) = 11 + 5 - 16 = 0$ and
$(11, 1, 16) \cdot (-3, 1, 2) = -33 + 1 + 32 = 0$

It is interesting to note that the vector $(11t, t, 16t)$, $t \neq 0$, represents a general vector perpendicular to the plane in which the vectors $\vec{a} = (1, 5, -1)$ and $\vec{b} = (-3, 1, 2)$ lie. This is represented in the diagram shown, where $t = 1$.



Determining the components of a vector perpendicular to two nonzero vectors will prove to be important in later applications.

IN SUMMARY

Key Idea

- The dot product is defined as follows for algebraic vectors in R^2 and R^3, respectively:
  - If $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2,)$, then $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2$
  - If $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, then $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

Need to Know

- The properties of the dot product hold for both geometric and algebraic vectors.
- Two nonzero vectors, $\vec{a}$ and $\vec{b}$, are perpendicular if $\vec{a} \cdot \vec{b} = 0$.
- For two nonzero vectors $\vec{a}$ and $\vec{b}$, where θ is the angle between the vectors, $\cos \theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$.

Exercise 7.4

PART A

1. How many vectors are perpendicular to $\vec{a} = (-1, 1)$? State the components of three such vectors.

2. For each of the following pairs of vectors, calculate the dot product and, on the basis of your result, say whether the angle between the two vectors is acute, obtuse, or 90°.
   a. $\vec{a} = (-2, 1), \vec{b} = (1, 2)$
   b. $\vec{a} = (2, 3, -1), \vec{b} = (4, 3, -17)$
   c. $\vec{a} = (1, -2, 5), \vec{b} = (3, -2, -2)$

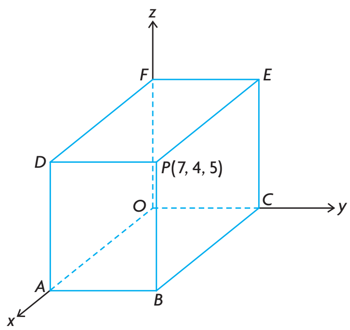
3. Give the components of a vector that is perpendicular to each of the following planes:
   a. xy-plane
   b. xz-plane
   c. yz-plane

4. a. From the set of vectors $\left\{(1, 2, -1), (-4, -5, -6), (4, 3, 10), \left(5, -3, \frac{-5}{6}\right)\right\}$, select two pairs of vectors that are perpendicular to each other.
   b. Are any of these vectors collinear? Explain.
5. In Example 5, a vector was found that was perpendicular to two nonzero vectors.
   a. Explain why it would not be possible to do this in R^2 if we selected the two vectors $\vec{a} = (1, -2)$ and $\vec{b} = (1, 1)$.
   b. Explain, in general, why it is not possible to do this if we select any two vectors in R^2.

PART B

K 6. Determine the angle, to the nearest degree, between each of the following pairs of vectors:
   a. $\vec{a} = (5, 3)$ and $\vec{b} = (-1, -2)$
   b. $\vec{a} = (-1, 4)$ and $\vec{b} = (6, -2)$
   c. $\vec{a} = (2, 2, 1)$ and $\vec{b} = (2, 1, -2)$
   d. $\vec{a} = (2, 3, -6)$ and $\vec{b} = (-5, 0, 12)$
7. Determine k, given two vectors and the angle between them.
   a. $\vec{a} = (-1, 2, -3), \vec{b} = (-6k, -1, k), \theta = 90°$
   b. $\vec{a} = (1, 1), \vec{b} = (0, k), \theta = 45°$
8. In R^2, a square is determined by the vectors $\vec{i}$ and $\vec{j}$.
   a. Sketch the square.
   b. Determine vector components for the two diagonals.
   c. Verify that the angle between the diagonals is 90°.
9. Determine the angle, to the nearest degree, between each pair of vectors.
   a. $\vec{a} = (1 - \sqrt{2}, \sqrt{2}, -1)$ and $\vec{b} = (1, 1)$
   b. $\vec{a} = (\sqrt{2} - 1, \sqrt{2} + 1, \sqrt{2})$ and $\vec{b} = (1, 1, 1)$

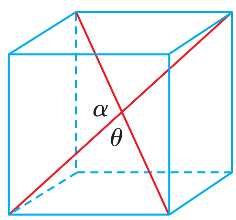
C 10. a. For the vectors $\vec{a} = (2, p, 8)$ and $\vec{b} = (q, 4, 12)$, determine values of p and q so that the vectors are
      i. collinear
      ii. perpendicular
   b. Are the values of p and q unique? Explain why or why not.
11. $\triangle ABC$ has vertices at $A(2, 5)$, $B(4, 11)$, and $C(-1, 6)$. Determine the angles in this triangle.



12. A rectangular box measuring 4 by 5 by 7 is shown in the diagram at the left.
    a. Determine the coordinates of each of the missing vertices.
    b. Determine the angle, to the nearest degree, between $\overrightarrow{AE}$ and $\overrightarrow{BF}$.
13. a. Given the vectors $\vec{p} = (-1, 3, 0)$ and $\vec{q} = (1, -5, 2)$, determine the components of a vector perpendicular to each of these vectors.
    b. Given the vectors $\vec{m} = (1, 3, -4)$ and $\vec{n} = (-1, -2, 3)$, determine the components of a vector perpendicular to each of these vectors.
14. Find the value of p if the vectors $\vec{r} = (p, p, 1)$ and $\vec{s} = (p, -2, -3)$ are perpendicular to each other.
15. a. Determine the algebraic condition such that the vectors $\vec{c} = (-3, p, -1)$ and $\vec{d} = (1, -4, q)$ are perpendicular to each other.
    b. If $q = -3$, what is the corresponding value of p?
16. **A** Given the vectors $\vec{r} = (1, 2, -1)$ and $\vec{s} = (-2, -4, 2)$, determine the components of two vectors perpendicular to each of these vectors. Explain your answer.
17. The vectors $\vec{x} = (-4, p, -2)$ and $\vec{y} = (-2, 3, 6)$ are such that $\cos^{-1}\left(\frac{4}{21}\right) = \theta$, where θ is the angle between $\vec{x}$ and $\vec{y}$. Determine the value(s) of p.

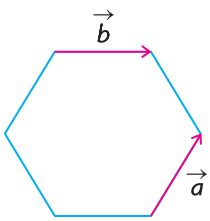
PART C

18. The diagonals of a parallelogram are determined by the vectors $\vec{a} = (3, 3, 0)$ and $\vec{b} = (-1, 1, -2)$.
    a. Show that this parallelogram is a rhombus.
    b. Determine vectors representing its sides and then determine the length of these sides.
    c. Determine the angles in this rhombus.
19. **T** The rectangle $ABCD$ has vertices at $A(-1, 2, 3)$, $B(2, 6, -9)$, and $D(3, q, 8)$.
    a. Determine the coordinates of the vertex C.
    b. Determine the angle between the two diagonals of this rectangle.
20. A cube measures 1 by 1 by 1. A line is drawn from one vertex to a diagonally opposite vertex through the centre of the cube. This is called a body diagonal for the cube. Determine the angles between the body diagonals of the cube.



Mid-Chapter Review

1. a. If $|\vec{a}| = 3$ and $|\vec{b}| = 2$, and the angle between these two vectors is 60°, determine $\vec{a} \cdot \vec{b}$.
   b. Determine the numerical value of $(3\vec{a} + 2\vec{b}) \cdot (4\vec{a} - 3\vec{b})$.
2. A mass of 15 kg is suspended by two cords from a ceiling. The cords have lengths of 15 cm and 20 cm, and the distance between the points where they are attached on the ceiling is 25 cm. Determine the tension in each of the two cords.
3. In a square that has side lengths of 10 cm, what is the dot product of the vectors representing the diagonals?
4. An airplane is travelling at 500 km/h due south when it encounters a wind from $W45°N$ at 100 km/h.
   a. What is the resultant velocity of the airplane?
   b. How long will it take for the airplane to travel 1000 km?
5. A 15 kg block lies on a smooth ramp that is inclined at 40° to the ground.
   a. Determine the force that this block exerts in a direction perpendicular to the ramp.
   b. What is the force, parallel to the inclined plane, needed to prevent the block from slipping?
6. A regular hexagon, with sides of 3 cm, is shown below. Determine $\vec{a} \cdot \vec{b}$.



7. Given the vectors $\vec{a} = (4, -5, 20)$ and $\vec{b} = (1, 2, 2)$, determine the following:
   a. $\vec{a} \cdot \vec{b}$
   b. the cosine of the angle between the two vectors
8. Given the vectors $\vec{a} = \vec{i} + 2\vec{j} + \vec{k}, \vec{b} = 2\vec{i} - 3\vec{j} + 4\vec{k}$, and $\vec{c} = 3\vec{i} - \vec{j} - \vec{k}$, determine the following:
   a. $\vec{a} \cdot \vec{b}$
   b. $\vec{b} \cdot \vec{c}$
   c. $\vec{b} + \vec{c}$
   d. $\vec{a} \cdot (\vec{b} + \vec{c})$
   e. $(\vec{a} + \vec{b}) \cdot (\vec{b} + \vec{c})$
   f. $(2\vec{a} - 3\vec{b}) \cdot (2\vec{a} + \vec{c})$

9. Given the vectors $\vec{p} = x\vec{i} + \vec{j} + 3\vec{k}$ and $\vec{q} = 3x\vec{i} + 10x\vec{j} + \vec{k}$, determine the following:
   a. the value(s) of x that make these vectors perpendicular
   b. the values(s) of x that make these vectors parallel
10. If $\vec{x} = \vec{i} - 2\vec{j} - \vec{k}$ and $\vec{y} = \vec{i} - \vec{j} - \vec{k}$, determine the value of each of the following:
    a. $3\vec{x} - 2\vec{y}$
    b. $3\vec{x} \cdot 2\vec{y}$
    c. $|\vec{x} - 2\vec{y}|$
    d. $(2\vec{x} - 3\vec{y}) \cdot (\vec{x} + 4\vec{y})$
    e. $2\vec{x} \cdot \vec{y} - 5\vec{y} \cdot \vec{x}$
11. Three forces of 3 N, 4 N, and 5 N act on an object so that the object is in equilibrium. Determine the angle between the largest and smallest forces.
12. A force of 3 N and a force of 4 N act on an object. If these two forces make an angle of 60° to each other, find the resultant and equilibrant of these two forces.
13. The sides of a parallelogram are determined by the vectors $\vec{m} = (2, -3, 5)$ and $\vec{n} = (-1, 7, 5)$. Determine
    a. the larger angle between the diagonals of this parallelogram
    b. the smaller angle between the sides
14. Martina is planning to fly to a town 1000 km due north of her present location. There is a 45 km/h wind blowing from $N30°E$.
    a. If her plane travels at 500 km/h, what direction should the pilot head to reach the destination?
    b. How long will the trip take?
15. Determine the coordinates of a unit vector that is perpendicular to $\vec{a} = (-1, 2, 5)$ and $\vec{b} = (1, 3, 5)$.
16. Clarence leaves a dock, paddling a canoe at 3 m/s. He heads downstream at an angle of 45° to the current, which is flowing at 4 m/s.
    a. How far downstream does he travel in 10 s?
    b. What is the length of time required to cross the river if it is 180 m wide?
17. a. Under what conditions does $(\vec{x} + \vec{y}) \cdot (\vec{x} - \vec{y}) = 0$?
    b. Give a geometrical interpretation of the vectors $\vec{a}$, $\vec{b}$, $\vec{a} + \vec{b}$, and $\vec{a} - \vec{b}$.
18. A lawn roller with a mass of 60 kg is being pulled with a force of 350 N. If the handle of the roller makes an angle of 40° with the ground, what horizontal component of the force is causing the roller to move forward?

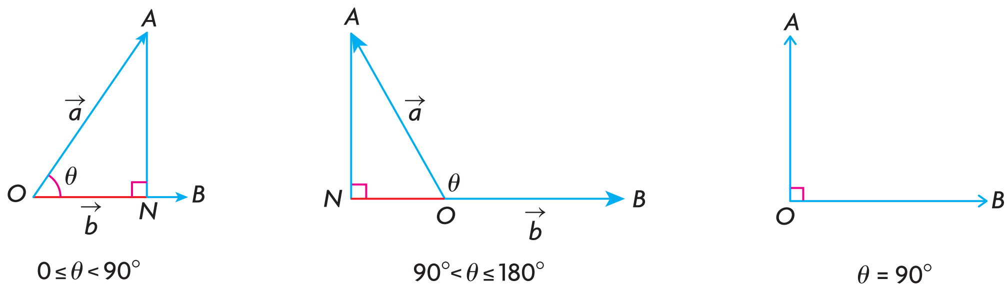
Section 7.5—Scalar and Vector Projections

In the last two sections, the concept of the dot product was discussed, first in geometric form and then in algebraic form. In this section, the dot product will be used along with the concept of **projections**. These concepts are closely related, and each has real significance from both a practical and theoretical point of view.

When two vectors, $\vec{a} = \overrightarrow{OA}$ and $\vec{b} = \overrightarrow{OB}$, are placed tail to tail, and θ is the angle between the vectors, $0° \le \theta \le 180°$, the **scalar projection** of $\vec{a}$ on $\vec{b}$ is ON, as shown in the following diagram. The scalar projection can be determined using right triangle trigonometry and can be applied to either geometric or algebraic vectors equally well.

Scalar Projection of $\vec{a}$ on $\vec{b}$

The scalar projection of vector $\vec{a}$ onto $\vec{b}$ is ON, where $ON = |\vec{a}|\cos\theta$.



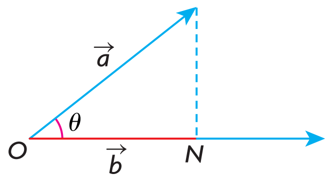
Observations about the Scalar Projection

A number of observations should be made about scalar projections. The scalar projection of $\vec{a}$ on $\vec{b}$ is obtained by drawing a line from the head of vector $\vec{a}$ perpendicular to $\overrightarrow{OB}$, or an extension of $\overrightarrow{OB}$. If the point where this line meets the vector is labelled N, then the scalar projection $\vec{a}$ on $\vec{b}$ is ON. Since ON is a real number, or scalar, and also a projection, it is called a scalar projection. If the angle between two given vectors is such that $0° \le \theta < 90°$, then the scalar projection is positive; otherwise, it is negative for $90° < \theta \le 180°$ and 0 if $\theta = 90°$.

The sign of scalar projections should not be surprising, since it corresponds exactly to the sign convention for dot products that we saw in the previous two sections. An important point is that the scalar projection between perpendicular vectors is always 0 because the angle between the vectors is 90° and $\cos 90° = 0$. Another important point is that it is not possible to take the scalar projection of the vector $\vec{a}$ on $\vec{0}$. This would result in a statement involving division by 0, which is meaningless.

Another observation should be made about scalar projections that is not immediately obvious from the given definition. The scalar projection of vector $\vec{a}$ on vector $\vec{b}$ is in general not equal to the scalar projection of vector $\vec{b}$ on vector $\vec{a}$, which can be seen from the following.

Calculating the scalar projection of $\vec{a}$ on $\vec{b}$:



Projection of $\vec{a}$ onto $\vec{b}$

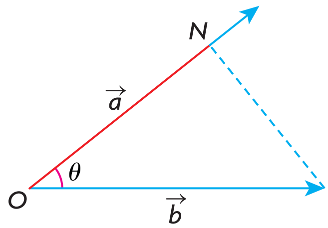
When calculating this projection, what is needed is to solve for $|\vec{a}|\cos\theta$ in the dot product formula.

We know that $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

Rewrite this formula as $\vec{a}\cdot\vec{b} = \left(|\vec{a}|\cos\theta\right)|\vec{b}|$.

Solving for $|\vec{a}|\cos\theta$ gives $|\vec{a}|\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

Calculating the scalar projection of $\vec{b}$ on $\vec{a}$:

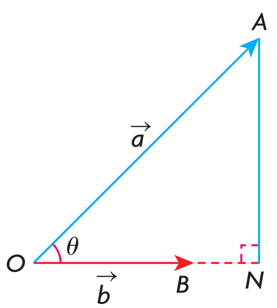


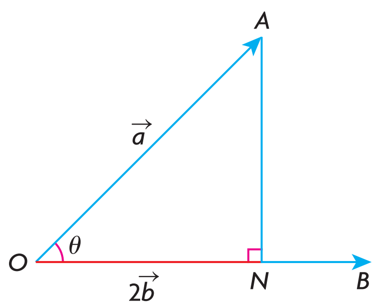
Projection of $\vec{b}$ onto $\vec{a}$

To find the scalar projection of $\vec{b}$ on $\vec{a}$, it is necessary to solve for $|\vec{b}|\cos\theta$ in the dot product formula. This is done in exactly the same way as above, and we find that $|\vec{b}|\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$.

From this, we can see that, in general, $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} \neq \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$. It is correct to say, however, that these scalar projections are equal if $|\vec{a}| = |\vec{b}|$.

Another observation to make about scalar projections is that the scalar projection of $\vec{a}$ on $\vec{b}$ is independent of the length of $\vec{b}$. This is demonstrated in the following diagram:





From the diagram, we can see that the scalar projection of vector $\vec{a}$ on vector $\vec{b}$ equals ON. If we take the scalar projection of $\vec{a}$ on $2\vec{b}$, this results in the exact same line segment ON.

EXAMPLE 1 Reasoning about the characteristics of the scalar projection

a. Show algebraically that the scalar projection of $\vec{a}$ on $\vec{b}$ is identical to the scalar projection of $\vec{a}$ on $2\vec{b}$.

b. Show algebraically that the scalar projection of $\vec{a}$ on $\vec{b}$ is not the same as $\vec{a}$ on $-2\vec{b}$.

Solution

a. The scalar projection of $\vec{a}$ on $\vec{b}$ is given by the formula $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$.

The scalar projection of $\vec{a}$ on $2\vec{b}$ is $\frac{\vec{a} \cdot 2\vec{b}}{|2\vec{b}|}$. If we use the properties of the dot product and the fact that $|2\vec{b}| = 2|\vec{b}|$, this quantity can be written as

$\frac{2(\vec{a} \cdot \vec{b})}{2|\vec{b}|}$, and then simplified to $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$.

From this, we see that what was shown geometrically is verified algebraically.

b. As before, the scalar projection of $\vec{a}$ on $\vec{b}$ is given by the formula $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$.

The scalar projection of $\vec{a}$ on $-2\vec{b}$ is $\frac{\vec{a} \cdot (-2\vec{b})}{|-2\vec{b}|}$. Using the same approach as above and recognizing that $|-2\vec{b}| = 2|\vec{b}|$, this can be rewritten as

$$\frac{\vec{a} \cdot (-2\vec{b})}{|-2\vec{b}|} = \frac{-2\vec{a} \cdot \vec{b}}{2|\vec{b}|} = \frac{-(\vec{a} \cdot \vec{b})}{|\vec{b}|}.$$

In this case, the direction of the vector $-2\vec{b}$ changes the scalar projection to the opposite sign from the projection of $\vec{a}$ on $\vec{b}$.

The following example shows how to calculate scalar projections involving algebraic vectors. All the properties applying to geometric vectors also apply to algebraic vectors.

EXAMPLE 2 Selecting a strategy to calculate the scalar projection involving algebraic vectors

For the vectors $\vec{a} = (-3, 4, 5\sqrt{3})$ and $\vec{b} = (-2, 2, -1)$, calculate each of the following scalar projections:

a. $\vec{a}$ on $\vec{b}$ b. $\vec{b}$ on $\vec{a}$

Solution

a. The required scalar projection is $|\vec{a}|\cos\theta$ and, since $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, as before, $|\vec{a}|\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

We start by calculating $\vec{a}\cdot\vec{b}$.

$$\begin{aligned}\vec{a}\cdot\vec{b} &= -3(-2) + 4(2) + 5\sqrt{3}(-1)\\ &= 14 - 5\sqrt{3}\\ &\doteq 5.34\end{aligned}$$

Since $|\vec{b}| = \sqrt{(-2)^2 + (2)^2 + (-1)^2} = 3$,

$$|\vec{a}|\cos\theta \doteq \frac{5.34}{3} \doteq 1.78$$

The scalar projection of $\vec{a}$ on $\vec{b}$ is approximately 1.78.

b. In this case, the required scalar projection is $|\vec{b}|\cos\theta$. Solving as in the solution to part a. $|\vec{b}|\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

Since $|\vec{a}| = (-3)^2 + (4)^2 + (5\sqrt{3})^2 = 10$,

$$|\vec{b}|\cos\theta \doteq \frac{5.34}{10} \doteq 0.53$$

The scalar projection of $\vec{b}$ on $\vec{a}$ is approximately 0.53.

Calculating Scalar Projections

The scalar projection of $\vec{a}$ on $\vec{b}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

The scalar projection of $\vec{b}$ on $\vec{a}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$.

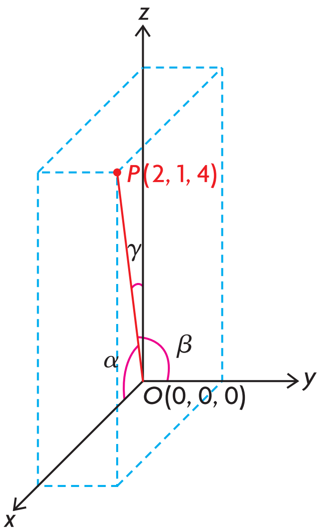
In general, $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} \neq \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$.

Scalar projections are sometimes used to calculate the angle that a position vector $\overrightarrow{OP}$ makes with each of the positive coordinate axes. This concept is illustrated in the next example.

EXAMPLE 3 Selecting a strategy to determine the direction angles of a vector in R^3

Determine the angle that the vector $\overrightarrow{OP} = (2, 1, 4)$ makes with each of the coordinate axes.

Solution



To calculate the required **direction angles**, it is necessary to project $\overrightarrow{OP}$ on each of the coordinate axes. To carry out the calculation, we use the standard basis vectors $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$, and $\vec{k} = (0, 0, 1)$ so that $\overrightarrow{OP}$ can be projected along the x-axis, y-axis, and z-axis, respectively. We define α as the angle between $\overrightarrow{OP}$ and the positive x-axis, β as the angle between $\overrightarrow{OP}$ and the positive y-axis, and γ as the angle between $\overrightarrow{OP}$ and the positive z-axis.

Calculating α:

To calculate the angle that $\overrightarrow{OP}$ makes with the x-axis, we start by writing $\overrightarrow{OP} \cdot \vec{i} = |\overrightarrow{OP}||\vec{i}| \cos \alpha$, which implies $\cos\alpha = \dfrac{\overrightarrow{OP} \cdot \vec{i}}{|\overrightarrow{OP}||\vec{i}|}$.

Since $|\overrightarrow{OP}| = \sqrt{21}$ and $|\vec{i}| = 1$, we substitute to find

$$\cos\alpha = \frac{\overrightarrow{OP} \cdot \vec{i}}{|\overrightarrow{OP}||\vec{i}|}$$

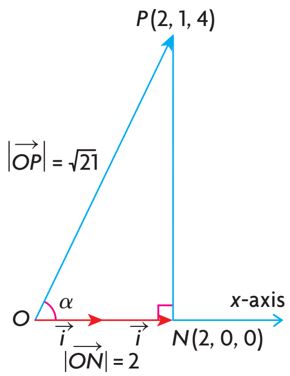
$$\cos\alpha = \frac{(2, 1, 4) \cdot (1, 0, 0)}{\sqrt{21}(1)}$$

$$\cos\alpha = \frac{2}{\sqrt{21}}$$

Thus, $\alpha = \cos^{-1}\left(\dfrac{2}{\sqrt{21}}\right)$ and $\alpha \doteq 64.1°$.

Therefore, the angle that $\overrightarrow{OP}$ makes with the x-axis is approximately $64.1°$. In its simplest terms, the cosine of the required angle α is the scalar projection of $\overrightarrow{OP}$ on $\vec{i}$, divided by $|\overrightarrow{OP}|$—that is, $\cos\alpha = \dfrac{\overrightarrow{OP} \cdot \vec{i}}{|\overrightarrow{OP}|} = \dfrac{2}{\sqrt{21}}$.

This angle is illustrated in the following diagram:



Calculating β and γ:
If we use the same procedure, we can also calculate β and γ, the angles that $\overrightarrow{OP}$ makes with the y-axis and z-axis, respectively.

Thus, $\cos\beta = \dfrac{(2, 1, 4)\cdot(0, 1, 0)}{\sqrt{21}} = \dfrac{1}{\sqrt{21}}$

$$\beta = \cos^{-1}\left(\frac{1}{\sqrt{21}}\right), \beta \doteq 77.4^\circ$$

Similarly, $\cos\gamma = \dfrac{4}{\sqrt{21}}, \gamma \doteq 29.2^\circ$

Therefore, $\overrightarrow{OP}$ makes angles of 64.1°, 77.4°, and 29.2° with the positive x-axis, y-axis and z-axis, respectively.

In our example, specific numbers were used, but the calculation is identical if we consider $\overrightarrow{OP} = (a, b, c)$ and develop a formula for the required direction angles. The cosines of the angles are referred to as the **direction cosines** of α, β, and γ.

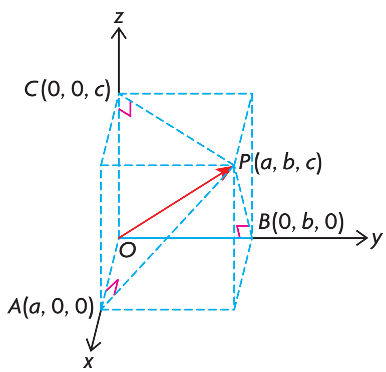
Direction Cosines for $\overrightarrow{OP} = (a, b, c)$

If α, β, and γ are the angles that $\overrightarrow{OP}$ makes with the positive x-axis, y-axis, and z-axis, respectively, then

$$\cos\alpha = \frac{(a, b, c)\cdot(1, 0, 0)}{|\overrightarrow{OP}|} = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$\cos\beta = \frac{b}{\sqrt{a^2 + b^2 + c^2}} \text{ and } \cos\gamma = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

These angles can be visualized by constructing a rectangular box and drawing in the appropriate projections. If we are calculating α, the angle that $\overrightarrow{OP}$ makes with the positive x-axis, the projection of $\overrightarrow{OP}$ on the x-axis is just a, and $|\overrightarrow{OP}| = \sqrt{a^2 + b^2 + c^2}$, so $\cos\alpha = \dfrac{a}{\sqrt{a^2 + b^2 + c^2}}$.
We calculate $\cos\beta$ and $\cos\gamma$ in the same way.



EXAMPLE 4 Calculating a specific direction angle

For the vector $\overrightarrow{OP} = (-2\sqrt{2}, 4, -5)$, determine the direction cosine and the corresponding angle that this vector makes with the positive z-axis.

Solution

We can use the formula to calculate γ.

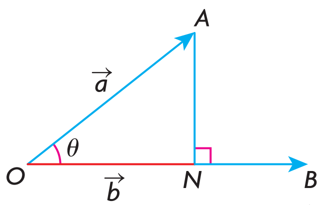
$$\cos\gamma = \frac{-5}{\sqrt{(-2\sqrt{2})^2 + (4)^2 + (-5)^2}} = \frac{-5}{\sqrt{49}} = \frac{-5}{7} \doteq -0.7143$$

and $\gamma \doteq 135.6^\circ$

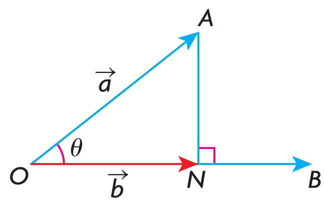
Examining Vector Projections

Thus far, we have calculated scalar projections of a vector onto a vector. This computation can be modified slightly to find the corresponding vector projection of a vector on a vector.

The calculation of the vector projection of $\vec{a}$ on $\vec{b}$ is just the corresponding scalar projection of $\vec{a}$ on $\vec{b}$ multiplied by $\dfrac{\vec{b}}{|\vec{b}|}$. The expression $\dfrac{\vec{b}}{|\vec{b}|}$ is a unit vector pointing in the direction of $\vec{b}$.



Scalar projection of $\vec{a}$ on $\vec{b}$



Vector projection of $\vec{a}$ on $\vec{b}$

Vector Projection of $\vec{a}$ on $\vec{b}$

vector projection of $\vec{a}$ on $\vec{b}$

$= $ (scalar projection of $\vec{a}$ on $\vec{b}$) (unit vector in the direction of $\vec{b}$)

$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\left(\frac{\vec{b}}{|\vec{b}|}\right)$$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}$$

$$= \left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right)\vec{b},\ \vec{b} \neq \vec{0}$$

EXAMPLE 5 Connecting a scalar projection to its corresponding vector projection

Find the vector projection of $\overrightarrow{OA} = (4, 3)$ on $\overrightarrow{OB} = (4, -1)$.

Solution

The formula for the scalar projection of $\overrightarrow{OA}$ on $\overrightarrow{OB}$ is $\dfrac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{|\overrightarrow{OB}|}$.

$$ON = \frac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{|\overrightarrow{OB}|} = \frac{(4, 3) \cdot (4, -1)}{\sqrt{(4)^2 + (-1)^2}}$$
$$= \frac{13}{\sqrt{17}}$$

The vector projection, $\overrightarrow{ON}$, is found by multiplying ON by the unit vector $\dfrac{\overrightarrow{OB}}{|\overrightarrow{OB}|}$.

Since $|\overrightarrow{OB}| = \sqrt{(4)^2 + (-1)^2} = \sqrt{17}$,

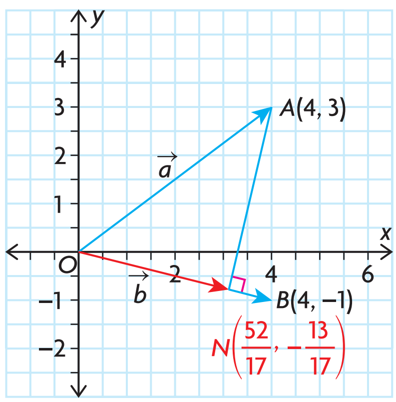
$$\frac{\overrightarrow{OB}}{|\overrightarrow{OB}|} = \frac{1}{\sqrt{17}}(4, -1)$$

The required vector projection is

$\overrightarrow{ON} = (ON)$(a unit vector in the same direction as $\overrightarrow{OB}$)

$$\overrightarrow{ON} = \frac{13}{\sqrt{17}}\left(\frac{1}{\sqrt{17}}(4, -1)\right)$$
$$= \frac{13}{17}(4, -1)$$
$$= \left(\frac{52}{17}, -\frac{13}{17}\right)$$

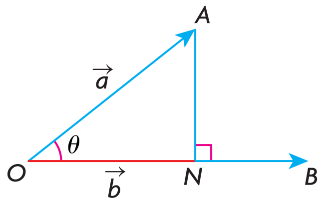
The vector projection $\overrightarrow{ON}$ is shown in red in the following diagram:



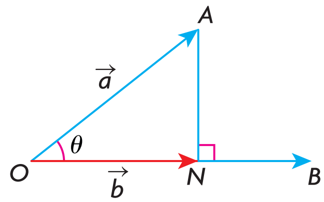
IN SUMMARY

Key Idea

- A projection of one vector onto another can be either a scalar or a vector. The difference is the vector projection has a direction.



Scalar projection of $\vec{a}$ on $\vec{b}$



Vector projection of $\vec{a}$ on $\vec{b}$

Need to Know

- The scalar projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = |\vec{a}| \cos \theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.
- The vector projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\vec{b} = \left(\dfrac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}}\right)\vec{b}$
- The direction cosines for $\overrightarrow{OP} = (a, b, c)$ are

  $\cos \alpha = \dfrac{a}{\sqrt{a^2 + b^2 + c^2}}$, $\cos \beta = \dfrac{b}{\sqrt{a^2 + b^2 + c^2}}$,

  $\cos \gamma = \dfrac{c}{\sqrt{a^2 + b^2 + c^2}}$, where α, β, and γ are the direction angles between the position vector $\overrightarrow{OP}$ and the positive x-axis, y-axis and z-axis, respectively.

Exercise 7.5

PART A

1. a. The vector $\vec{a} = (2, 3)$ is projected onto the x-axis. What is the scalar projection? What is the vector projection?

   b. What are the scalar and vector projections when $\vec{a}$ is projected onto the y-axis?

2. Explain why it is not possible to obtain either a scalar projection or a vector projection when a nonzero vector $\vec{x}$ is projected on $\vec{0}$.

3. Consider two nonzero vectors, $\vec{a}$ and $\vec{b}$, that are perpendicular to each other. Explain why the scalar and vector projections of $\vec{a}$ on $\vec{b}$ must be 0 and $\vec{0}$, respectively. What are the scalar and vector projections of $\vec{b}$ on $\vec{a}$?

4. Draw two vectors, $\vec{p}$ and $\vec{q}$. Draw the scalar and vector projections of $\vec{p}$ on $\vec{q}$. Show, using your diagram, that these projections are not necessarily the same as the scalar and vector projections of $\vec{q}$ on $\vec{p}$.

5. Using the formulas in this section, determine the scalar and vector projections of $\overrightarrow{OP} = (-1, 2, -5)$ on $\vec{i}$, $\vec{j}$, and $\vec{k}$. Explain how you could have arrived at the same answer without having to use the formulas.

PART B

6. a. For the vectors $\vec{p} = (3, 6, -22)$ and $\vec{q} = (-4, 5, -20)$, determine the scalar and vector projections of $\vec{p}$ on $\vec{q}$.

   b. Determine the direction angles for $\vec{p}$.

K 7. For each of the following, determine the scalar and vector projections of $\vec{x}$ on $\vec{y}$.

   a. $\vec{x} = (1, 1), \vec{y} = (1, -1)$

   b. $\vec{x} = (2, 2\sqrt{3}), \vec{y} = (1, 0)$

   c. $\vec{x} = (2, 5), \vec{y} = (-5, 12)$

8. a. Determine the scalar and vector projections of $\vec{a} = (-1, 2, 4)$ on each of the three axes.

   b. What are the scalar and vector projections of $m(-1, 2, 4)$ on each of the three axes?

C 9. a. Given the vector $\vec{a}$, show with a diagram that the vector projection of $\vec{a}$ on $\vec{a}$ is $\vec{a}$ and that the scalar projection of $\vec{a}$ on $\vec{a}$ is $|\vec{a}|$.

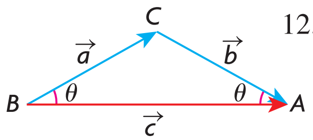
   b. Using the formulas for scalar and vector projections, explain why the results in part a. are correct if we use $\theta = 0°$ for the angle between the two vectors.

10. a. Using a diagram, show that the vector projection of $-\vec{a}$ on $\vec{a}$ is $-\vec{a}$.

   b. Using the formula for determining scalar projections, show that the result in part a. is true.

A 11. a. Find the scalar and vector projections of $\overrightarrow{AB}$ along each of the axes if A has coordinates $(1, 2, 2)$ and B has coordinates $(-1, 3, 4)$.

   b. What angle does $\overrightarrow{AB}$ make with the y-axis?



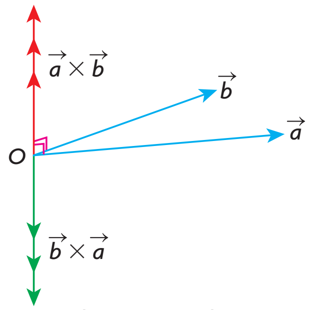
12. In the diagram shown, $\triangle ABC$ is an isosceles triangle where $|\vec{a}| = |\vec{b}|$.
    a. Draw the scalar projection of $\vec{a}$ on $\vec{c}$.
    b. Relocate $\vec{b}$, and draw the scalar projection of $\vec{b}$ on $\vec{c}$.
    c. Explain why the scalar projection of $\vec{a}$ on $\vec{c}$ is the same as the scalar projection of $\vec{b}$ on $\vec{c}$.
    d. Does the vector projection of $\vec{a}$ on $\vec{c}$ equal the vector projection of $\vec{b}$ on $\vec{c}$?
13. Vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 10$ and $|\vec{b}| = 12$, and the angle between them is $135°$.
    a. Show that the scalar projection of $\vec{a}$ on $\vec{b}$ does not equal the scalar projection of $\vec{b}$ on $\vec{a}$.
    b. Draw diagrams to illustrate the corresponding vector projections associated with part a.
14. You are given the vector $\overrightarrow{OD} = (-1, 2, 2)$ and the three points, $A(-2, 1, 4)$, $B(1, 3, 3)$, and $C(-6, 7, 5)$.
    a. Calculate the scalar projection of $\overrightarrow{AB}$ on $\overrightarrow{OD}$.
    b. Verify computationally that the scalar projection of $\overrightarrow{AB}$ on $\overrightarrow{OD}$ added to the scalar projection of $\overrightarrow{BC}$ on $\overrightarrow{OD}$ equals the scalar projection of $\overrightarrow{AC}$ on $\overrightarrow{OD}$.
    c. Explain why this same result is also true for the corresponding vector projections.

T 15. a. If α, β, and γ represent the direction angles for vector $\overrightarrow{OP}$, prove that $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.
    b. Determine the coordinates of a vector $\overrightarrow{OP}$ that makes an angle of $30°$ with the y-axis, $60°$ with the z-axis, and $90°$ with the x-axis.
    c. In Example 3, it was shown that, in general, the direction angles do not always add to 180°—that is, $\alpha + \beta + \gamma \neq 180°$. Under what conditions, however, must the direction angles always add to 180°?

PART C

16. A vector in R^3 makes equal angles with the coordinate axes. Determine the size of each of these angles if the angles are
    a. acute
    b. obtuse
17. If α, β, and γ represent the direction angles for vector $\overrightarrow{OP}$, prove that $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$.
18. Vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are not collinear. The sum of the direction angles of each vector is 180°. Draw diagrams to illustrate possible positions of points A and B.

Section 7.6—The Cross Product of Two Vectors



The Cross Product (Vector Product)

In the previous three sections, the dot product along with some of its applications was discussed. In this section, a second product called the **cross product**, denoted as $\vec{a} \times \vec{b}$, is introduced. The cross product is sometimes referred to as a **vector product** because, when it is calculated, the result is a vector and not a scalar. As we shall see, the cross product can be used in physical applications but also in the understanding of the geometry of R^3.

If we are given two vectors, $\vec{a}$ and $\vec{b}$, and wish to calculate their cross product, what we are trying to find is a particular vector that is perpendicular to each of the two given vectors. As will be observed, if we consider two nonzero, noncollinear vectors, there is an infinite number of vectors perpendicular to the two vectors. If we want to determine the cross product of these two vectors, we choose just one of these perpendicular vectors as our answer. Finding the cross product of two vectors is shown in the following example.

EXAMPLE 1

Calculating the cross product of two vectors

Given the vectors $\vec{a} = (1, 1, 0)$ and $\vec{b} = (1, 3, -1)$, determine $\vec{a} \times \vec{b}$.

Solution

When calculating $\vec{a} \times \vec{b}$, we are determining a vector that is perpendicular to both $\vec{a}$ and $\vec{b}$. We start by letting this vector be $\vec{v} = (x, y, z)$.

Since $\vec{a} \cdot \vec{v} = 0$, $(1, 1, 0) \cdot (x, y, z) = 0$ and $x + y = 0$.

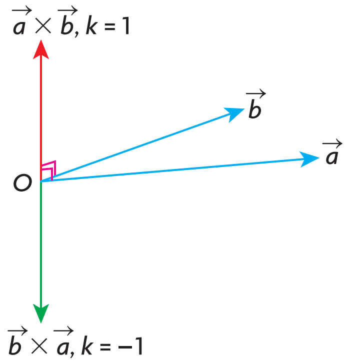
In the same way, since $\vec{b} \cdot \vec{v} = 0$, $(1, 3, -1) \cdot (x, y, z) = 0$ and $x + 3y - z = 0$.

Finding of the cross product of $\vec{a}$ and $\vec{b}$ requires solving a system of two equations in three variables which, under normal circumstances, has an infinite number of solutions. We will eliminate a variable and use substitution to find a solution to this system.

① $x + y = 0$

② $x + 3y - z = 0$

Subtracting eliminates x, $-2y + z = 0$, or $z = 2y$. If we substitute $z = 2y$ in equation ②, it will be possible to express x in terms of y. Doing so gives $x + 3y - (2y) = 0$ or $x = -y$. Since x and z can both be expressed in terms of y, we write the solution as $(-y, y, 2y) = y(-1, 1, 2)$. The solution to this system



is any vector of the form $y(-1, 1, 2)$. It can be left in this form, but is usually written as $k(-1, 1, 2)$, $k \in \mathbf{R}$, where k is a parameter representing any real value. The parameter k indicates that there is an infinite number of solutions and that each of them is a scalar multiple of $(-1, 1, 2)$. In this case, the cross product is defined to be the vector where $k = 1$—that is, $(-1, 1, 2)$. The choosing of $k = 1$ simplifies computation and makes sense mathematically, as we will see in the next section. It should also be noted that the cross product is a vector and, as stated previously, is sometimes called a vector product.

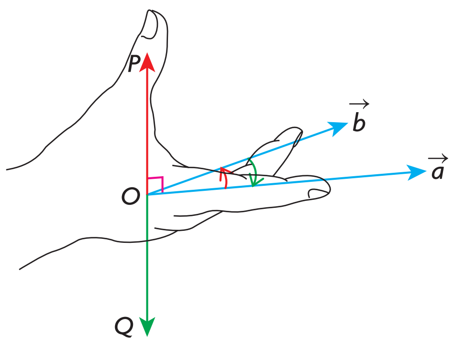
Deriving a Formula for the Cross Product

What is necessary, to be more efficient in calculating $\vec{a} \times \vec{b}$, is a formula.

The vector $\vec{a} \times \vec{b}$ is a vector that is perpendicular to each of the vectors $\vec{a}$ and $\vec{b}$. An infinite number of vectors satisfy this condition, all of which are scalar multiples of each other, but the cross product is one that is chosen in the simplest possible way, as will be seen when the formula is derived below. Another important point to understand about the cross product is that it exists only in R^3. It is not possible to take two noncollinear vectors in R^2 and construct a third vector perpendicular to the two vectors, because this vector would be outside the given plane.

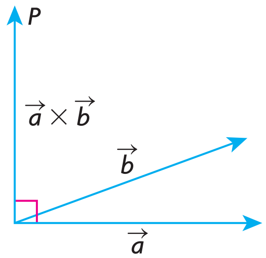
It is also difficult, at times, to tell whether we are calculating $\vec{a} \times \vec{b}$ or $\vec{b} \times \vec{a}$. The formula, properly applied, will do the job without difficulty. There are times when it is helpful to be able to identify the cross product without using a formula. From the diagram below, $\vec{a} \times \vec{b}$ is pictured as a vector perpendicular to the plane formed by $\vec{a}$ and $\vec{b}$, and, when looking down the axis from P on $\vec{a} \times \vec{b}$, $\vec{a}$ would have to be rotated *counterclockwise* in order to be collinear with $\vec{b}$. In other words, $\vec{a}$, $\vec{b}$, and $\vec{a} \times \vec{b}$ form a right-handed system.

The vector $\vec{b} \times \vec{a}$, the opposite to $\vec{a} \times \vec{b}$, is again perpendicular to the plane formed by $\vec{a}$ and $\vec{b}$, but, when looking from Q down the axis formed by $\vec{b} \times \vec{a}$, $\vec{a}$ would have to be rotated *clockwise* in order to be collinear with $\vec{b}$.



Definition of a Cross Product

The cross product of two vectors $\vec{a}$ and $\vec{b}$ in R^3 (3-space) is the vector that is perpendicular to these vectors such that the vectors $\vec{a}$, $\vec{b}$, and $\vec{a} \times \vec{b}$ form a right-handed system.
The vector $\vec{b} \times \vec{a}$ is the opposite of $\vec{a} \times \vec{b}$ and points in the opposite direction.



To develop a formula for $\vec{a} \times \vec{b}$, we follow a procedure similar to that followed in Example 1. Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, and let $\vec{v} = (x, y, z)$ be the vector that is perpendicular to $\vec{a}$ and $\vec{b}$.

So, ① $\vec{a} \cdot \vec{v} = (a_1, a_2, a_3) \cdot (x, y, z) = a_1x + a_2y + a_3z = 0$
and ② $\vec{b} \cdot \vec{v} = (b_1, b_2, b_3) \cdot (x, y, z) = b_1x + b_2y + b_3z = 0$

As before, we have a system of two equations in three unknowns, which we know from before has an infinite number of solutions. To solve this system of equations, we will multiply the first equation by b_1 and the second equation by a_1 and then subtract.

①$\times\, b_1 \rightarrow$ ③ $b_1a_1x + b_1a_2y + b_1a_3z = 0$
②$\times\, a_1 \rightarrow$ ④ $a_1b_1x + a_1b_2y + a_1b_3z = 0$

Subtracting ③ and ④ eliminates x. Move the z-terms to the right.

$(b_1a_2 - a_1b_2)y = (a_1b_3 - b_1a_3)z$

Multiplying each side by -1 and rearranging gives the desired result:

$(a_1b_2 - b_1a_2)y = (b_1a_3 - a_1b_3)z$

Now

$$\frac{y}{a_3b_1 - a_1b_3} = \frac{z}{a_1b_2 - a_2b_1}$$

If we carry out an identical procedure and eliminate z from the system of equations, we have the following:

$$\frac{x}{a_2b_3 - a_3b_2} = \frac{y}{a_3b_1 - a_1b_3}$$

If we combine the two statements and set them equal to a constant k, we have

$$\frac{x}{a_2b_3 - a_3b_2} = \frac{y}{a_3b_1 - a_1b_3} = \frac{z}{a_1b_2 - a_2b_1} = k$$

Note that we can make these fractions equal to k because every proportion can be made equal to a constant k. (For example, if $\frac{3}{6} = \frac{4}{8} = \frac{5}{10} = k$, then k could be either $\frac{1}{2}$ or $\frac{10}{20}$ or any nonzero multiple of the form $\frac{1n}{2n}$.) This expression gives us a general form for a vector that is perpendicular to $\vec{a}$ and $\vec{b}$. The cross product, $\vec{a} \times \vec{b}$, is defined to occur when $k = 1$, and $\vec{b} \times \vec{a}$ occurs when $k = -1$.

Formula for Calculating the Cross Product of Algebraic Vectors

$k(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$, $k \in \mathbf{R}$.
If $k = 1$, then $\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$
If $k = -1$, then $\vec{b} \times \vec{a} = (a_3b_2 - a_2b_3, a_1b_3 - a_3b_1, a_2b_1 - a_1b_2)$

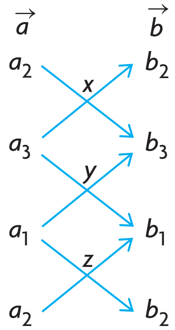
It is not easy to remember this formula for calculating the cross product of two vectors, so we develop a procedure, or a way of writing them, so that the memory work is removed from the calculation.

Method of Calculating $\vec{a} \times \vec{b}$, where $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$

1. List the components of vector $\vec{a}$ in column form on the left side, starting with a_2 and then writing a_3, a_1, and a_2 below each other as shown.
2. Write the components of vector $\vec{b}$ in a column to the right of $\vec{a}$, starting with b_2 and then writing b_3, b_1, and b_2 in exactly the same way as the components of $\vec{a}$.
3. The required formula is now a matter of following the arrows and doing the calculation. To find the x component, for example, we take the down product a_2b_3 and subtract the up product a_3b_2 from it to get $a_2b_3 - a_3b_2$.

(continued)

The other components are calculated in exactly the same way, and the formula for each component is listed below.



$x = a_2b_3 - a_3b_2$

$y = a_3b_1 - a_1b_3$

$z = a_1b_2 - a_2b_1$

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

INVESTIGATION

A. Given two vectors $\vec{a} = (2, 4, 6)$ and $\vec{b} = (-1, 2, -5)$ calculate $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$. What property does this demonstrate does hold not for the cross product? Explain why the property does not hold.

B. How are the vectors $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$ related? Write an expression that relates $\vec{a} \times \vec{b}$ with $\vec{b} \times \vec{a}$.

C. Will the expression you wrote in part B be true for any pair of vectors in R^3? Explain.

D. Using the two vectors given in part A and a third vector $\vec{c} = (4, 3, -1)$ calculate:

i. $\vec{a} \times (\vec{b} + \vec{c})$ ii. $\vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

E. Compare your results from i and ii in part D. What property does this demonstrate? Write an equivalent expression for $\vec{b} \times (\vec{a} + \vec{c})$ and confirm it using the appropriate calculations.

F. Choose any 3 vectors in R^3 and demonstrate that the property you identified in part E holds for your vectors.

G. Using the three vectors given calculate:

i. $(\vec{a} \times \vec{b}) \times \vec{c}$ ii. $\vec{a} \times (\vec{b} \times \vec{c})$

H. Compare your results from i and ii in part G. What property does this demonstrate does hold not for the cross product? Explain why the property does not hold.

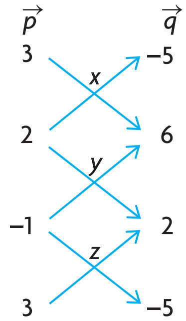
I. Choose any vector that is collinear with $\vec{a}$ (that is, any vector of the form $k\vec{a}$ $k \in \mathbf{R}$). Calculate $\vec{a} \times k\vec{a}$. Repeat using a different value for k. What can you conclude?

EXAMPLE 2 Calculating cross products

If $\vec{p} = (-1, 3, 2)$ and $\vec{q} = (2, -5, 6)$, calculate $\vec{p} \times \vec{q}$ and $\vec{q} \times \vec{p}$.

Solution

Let $\vec{p} \times \vec{q} = (x, y, z)$. The vectors $\vec{p}$ and $\vec{q}$ are listed in column form, with $\vec{p}$ on the left and $\vec{q}$ on the right, starting from the second component and working down.



$x = 3(6) - (2)(-5) = 28$

$y = 2(2) - (-1)(6) = 10$

$z = -1(-5) - 3(2) = -1$

$\vec{p} \times \vec{q} = (28, 10, -1)$

As already mentioned, $\vec{q} \times \vec{p}$ is the opposite of $\vec{p} \times \vec{q}$, so $\vec{q} \times \vec{p} = -1(28, 10, -1) = (-28, -10, 1)$. It is not actually necessary to calculate $\vec{q} \times \vec{p}$. All that is required is to calculate $\vec{p} \times \vec{q}$ and take the opposite vector to get $\vec{q} \times \vec{p}$.

After completing the calculation of the cross product, the answer should be verified to see if it is perpendicular to the given vectors using the dot product.

Check:

$(28, 10, -1) \cdot (-1, 3, 2) = -28 + 30 - 2 = 0$ and

$(28, 10, -1) \cdot (2, -5, 6) = 56 - 50 - 6 = 0$

There are a number of important properties of cross products that are worth noting. Some of these properties will be verified in the exercises.

Properties of the Cross Product

Let $\vec{p}$, $\vec{q}$, and $\vec{r}$ be three vectors in R^3, and let $k \in \mathbf{R}$.

Vector multiplication is not commutative: $\vec{p} \times \vec{q} = -(\vec{q} \times \vec{p})$,

Distributive law for vector multiplication: $\vec{p} \times (\vec{q} + \vec{r}) = \vec{p} \times \vec{q} + \vec{p} \times \vec{r}$,

Scalar law for vector multiplication: $k(\vec{p} \times \vec{q}) = (k\vec{p}) \times \vec{q} = \vec{p} \times (k\vec{q})$, The first property is one that we have seen in this section and is the first instance we have seen where the commutative property for multiplication has failed. Normally, we expect that the order of multiplication does not affect the product. In this case, changing the order of multiplication does change the result. The other two listed results are results that produce exactly what would be expected, and they will be used in this set of exercises and beyond.

IN SUMMARY

Key Idea

- The cross product $\vec{a} \times \vec{b}$, between two vectors $\vec{a}$ and $\vec{b}$, results in a third vector that is perpendicular to the plane in which the given vectors lie.

Need to Know

- $\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$
- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- $(k\vec{a}) \times \vec{b} = \vec{a} \times (k\vec{b}) = k(\vec{a} \times \vec{b})$

Exercise 7.6

PART A

1. The two vectors $\vec{a}$ and $\vec{b}$ are vectors in R^3, and $\vec{a} \times \vec{b}$ is calculated.
   a. Using a diagram, explain why $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$ and $\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$.
   b. Draw the parallelogram determined by $\vec{a}$ and $\vec{b}$, and then draw the vector $\vec{a} + \vec{b}$. Give a simple explanation of why $(\vec{a} + \vec{b}) \cdot (\vec{a} \times \vec{b}) = 0$.
   c. Why is it true that $(\vec{a} - \vec{b}) \cdot (\vec{a} \times \vec{b}) = 0$? Explain.
2. For vectors in R^3, explain why the calculation $(\vec{a} \cdot \vec{b})(\vec{a} \times \vec{b}) = 0$ is meaningless. (Consider whether or not it is possible for the left side to be a scalar.)

PART B

3. For each of the following calculations, say which are possible for vectors in R^3 and which are meaningless. Give a brief explanation for each.
   a. $\vec{a} \cdot (\vec{b} \times \vec{c})$
   b. $(\vec{a} \cdot \vec{b}) \times \vec{c}$
   c. $(\vec{a} \times \vec{b}) \cdot (\vec{c} + \vec{d})$
   d. $(\vec{a} \cdot \vec{b})(\vec{c} \times \vec{d})$
   e. $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$
   f. $\vec{a} \times \vec{b} + \vec{c}$

K 4. Calculate the cross product for each of the following pairs of vectors, and verify your answer by using the dot product.

a. $(2, -3, 5)$ and $(0, -1, 4)$
b. $(2, -1, 3)$ and $(3, -1, 2)$
c. $(5, -1, 1)$ and $(2, 4, 7)$
d. $(1, 2, 9)$ and $(-2, 3, 4)$
e. $(-2, 3, 3)$ and $(1, -1, 0)$
f. $(5, 1, 6)$ and $(-1, 2, 4)$

5. If $(-1, 3, 5) \times (0, a, 1) = (-2, 1, -1)$, determine a.

6. a. Calculate the vector product for $\vec{a} = (0, 1, 1)$ and $\vec{b} = (0, 5, 1)$.
b. Explain geometrically why it makes sense for vectors of the form $(0, b, c)$ and $(0, d, e)$ to have a cross product of the form $(a, 0, 0)$.

7. a. For the vectors $(1, 2, 1)$ and $(2, 4, 2)$, show that their vector product is $\vec{0}$.
b. In general, show that the vector product of two collinear vectors, (a, b, c) and (ka, kb, kc), is always $\vec{0}$.

8. In the discussion, it was stated that $\vec{p} \times (\vec{q} + \vec{r}) = \vec{p} \times \vec{q} + \vec{p} \times \vec{r}$ for vectors in R^3. Verify that this rule is true for the following vectors.
a. $\vec{p} = (1, -2, 4)$, $\vec{q} = (1, 2, 7)$, and $\vec{r} = (-1, 1, 0)$
b. $\vec{p} = (4, 1, 2)$, $\vec{q} = (3, 1, -1)$, and $\vec{r} = (0, 1, 2)$

A 9. Verify each of the following:
a. $\vec{i} \times \vec{j} = \vec{k} = -\vec{j} \times \vec{i}$
b. $\vec{j} \times \vec{k} = \vec{i} = -\vec{k} \times \vec{j}$
c. $\vec{k} \times \vec{i} = \vec{j} = -\vec{i} \times \vec{k}$

C 10. Show algebraically that $k(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \cdot \vec{a} = 0$. What is the meaning of this result?

11. You are given the vectors $\vec{a} = (2, 0, 0)$, $\vec{b} = (0, 3, 0)$, $\vec{c} = (2, 3, 0)$, and $\vec{d} = (4, 3, 0)$.
a. Calculate $\vec{a} \times \vec{b}$ and $\vec{c} \times \vec{d}$.
b. Calculate $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$.
c. Without doing any calculations (that is, by visualizing the four vectors and using properties of cross products), say why $(\vec{a} \times \vec{c}) \times (\vec{b} \times \vec{d}) = \vec{0}$.

PART C

12. Show that the cross product is not associative by finding vectors $\vec{x}$, $\vec{y}$, and $\vec{z}$ such that $(\vec{x} \times \vec{y}) \times \vec{z} \neq \vec{x} \times (\vec{y} \times \vec{z})$.

T 13. Prove that $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2\vec{a} \times \vec{b}$ is true.

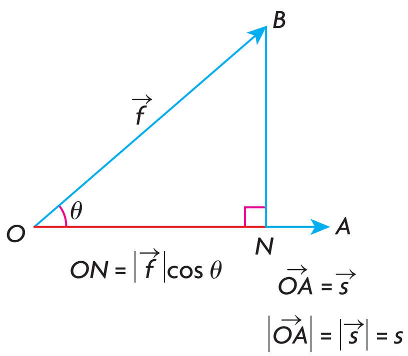
Section 7.7—Applications of the Dot Product and Cross Product

In the previous four sections, the dot product and cross product were discussed in some detail. In this section, some physical and mathematical applications of these concepts will be introduced to give a sense of their usefulness in both physical and mathematical situations.

Physical Application of the Dot Product

When a force is acting on an object so that the object is moved from one point to another, we say that the force has done work. Work is defined as the product of the distance an object has been displaced and the component of the force along the line of displacement.

In the following diagram, $\overrightarrow{OB}$ represents a constant force, $\vec{f}$, acting on an object at O so that this force moves the object from O to A. We will call the distance that the object is displaced s, which is a scalar, where we are assuming that $\vec{s} = \overrightarrow{OA}$ and $s = |\overrightarrow{OA}|$. The scalar projection of $\vec{f}$ on $\overrightarrow{OA}$ equals ON, or $|\vec{f}|\cos\theta$, which is the same calculation for the scalar projection that was done earlier. (This is called the scalar component of $\overrightarrow{OB}$ on $\overrightarrow{OA}$.) The work, W, done by $\vec{f}$ in moving the object is calculated as $W = (|\vec{f}|\cos\theta)(|\overrightarrow{OA}|) = (ON)(s) = \vec{f}\cdot\vec{s}$. As explained before, the force $\vec{f}$ is measured in newtons (N), the displacement is measured in metres (m), and the unit for work is newton-metres, or joules (J). When a 1 N force moves an object 1 m, the amount of work done is 1 J.



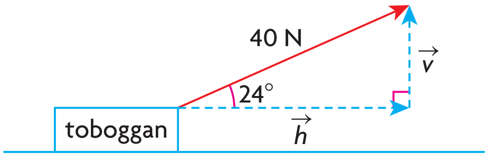
Formula for the Calculation of Work

$W = \vec{f}\cdot\vec{s}$, where $\vec{f}$ is the force acting on an object, measured in newtons (N); $\vec{s}$ is the displacement of the object, measured in metres (m); and W is the work done, measured in joules (J).

EXAMPLE 1 Using the dot product to calculate work

Marianna is pulling her daughter in a toboggan and is exerting a force of 40 N, acting at 24° to the ground. If Marianna pulls the child a distance of 100 m, how much work was done?

Solution



To solve this problem, the 40 N force has been resolved into its vertical and horizontal components. The horizontal component $\vec{h}$ tends to move the toboggan forward, while the vertical component $\vec{v}$ is the force that tends to lift the toboggan.

From the diagram, $|\vec{h}| = 40 \cos 24°$

$$\doteq 40(0.9135)$$

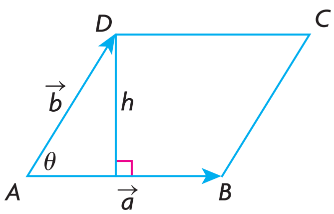
$$\doteq 36.54 \text{ N}$$

The amount of work done is $W \doteq (36.54)(100) \doteq 3654$ J. Therefore, the work done by Marianna is approximately 3654 J.

Geometric Application of the Cross Product

The cross product of two vectors is interesting because calculations involving the cross product can be applied in a number of different ways, giving us results that are important from both a mathematical and physical perspective.

The cross product of two vectors, $\vec{a}$ and $\vec{b}$, can be used to calculate the area of a parallelogram. For any parallelogram, $ABCD$, it is possible to develop a formula for its area, where $\vec{a}$ and $\vec{b}$ are vectors determining its sides and h is its height.



$$\begin{aligned} \text{Area} &= |\vec{a}|h \\ &= |\vec{a}|(|\vec{b}|\sin\theta) \\ &= |\vec{a}||\vec{b}|\sin\theta \end{aligned}$$

where $\sin\theta = \frac{h}{|\vec{b}|}$ or $h = |\vec{b}|\sin\theta$

It can be proven that this formula for the area is equal to $|\vec{a} \times \vec{b}|$. That is, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$.

Theorem: For two vectors, $\vec{a}$ and $\vec{b}$, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$, where θ is the angle between the two vectors.

Proof: The formula for the cross product is

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

Therefore, $|\vec{a} \times \vec{b}|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$

The right-hand side is expanded and then factored to give

$$|\vec{a} \times \vec{b}|^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$

This formula can be simplified by making the following substitutions:

$|\vec{a}|^2 = a_1^2 + a_2^2 + a_3^2$, $|\vec{b}|^2 = b_1^2 + b_2^2 + b_3^2$, and
$|\vec{a}||\vec{b}|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$

Thus, $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta$ (Factor)

$$|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2(1 - \cos^2\theta) = |\vec{a}|^2|\vec{b}|^2\sin^2\theta \quad \text{(Substitution)}$$

$$|\vec{a} \times \vec{b}| = \pm|\vec{a}||\vec{b}|\sin\theta$$

But since $\sin\theta \geq 0$ for $0^\circ \leq \theta \leq 180^\circ$,

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$

This gives us the required formula for the area of a parallelogram, which is equivalent to the magnitude of the cross product between the vectors that define the parallelogram.

EXAMPLE 2 **Solving area problems using the cross product**

a. Determine the area of the parallelogram determined by the vectors $\vec{p} = (-1, 5, 6)$ and $\vec{q} = (2, 3, -1)$.

b. Determine the area of the triangle formed by the points $A(-1, 2, 1)$, $B(-1, 0, 0)$, and $C(3, -1, 4)$.

Solution

a. The cross product is

$$\begin{aligned}\vec{p} \times \vec{q} &= (5(-1) - 3(6), 6(2) - (-1)(-1), -1(3) - 2(5)) \\ &= (-23, 11, -13)\end{aligned}$$

The required area is determined by $|\vec{p} \times \vec{q}|$.

$$\sqrt{(-23)^2 + 11^2 + (-13)^2} = \sqrt{529 + 121 + 169} = \sqrt{819}$$
$$\doteq 28.62 \text{ square units}$$

b. We start by constructing position vectors equal to $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Thus,

$$\overrightarrow{AB} = (-1 - (-1), 0 - 2, 0 - 1) = (0, -2, -1) \text{ and } \overrightarrow{AC} = (4, -3, 3)$$

Calculating,

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= (-2(3) - (-1)(-3), -1(4) - 0(3), 0(-3) - (-2)(4)) \\ &= (-9, -4, 8)\end{aligned}$$

And

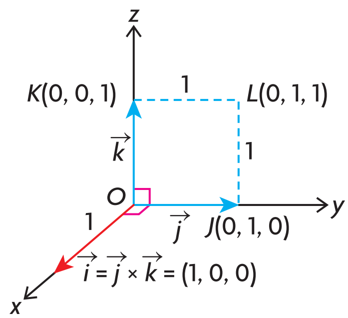
$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(-9)^2 + (-4)^2 + (8)^2} = \sqrt{161}$$

Therefore, the area of $\triangle ABC$ is one half of the area of the parallelogram formed by vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, which is $\frac{1}{2}\sqrt{161} \doteq 6.34$ square units.

This connection between the magnitude of the cross product and area allows us further insight into relationships in R^3. This calculation makes a direct and precise connection between the length of the cross product and the area of the parallelogram formed by two vectors. These two vectors can be anywhere in 3-space, not necessarily in the plane. As well, it also allows us to determine, in particular cases, the cross product of two vectors without having to carry out any computation.

EXAMPLE 3 Reasoning about a cross product involving the standard unit vectors

Without calculating, explain why the cross product of $\vec{j}$ and $\vec{k}$ is $\vec{i}$—that is, $\vec{j} \times \vec{k} = \vec{i}$.



Solution

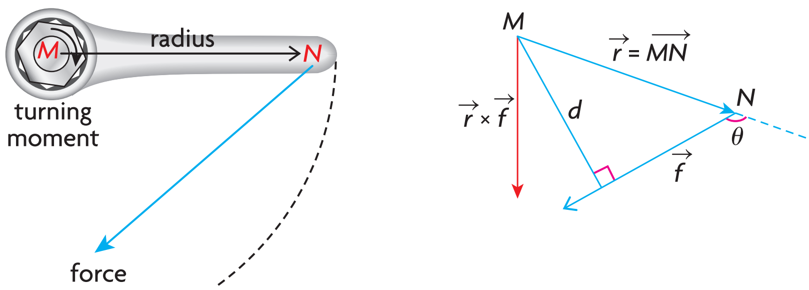
As shown in the diagram, the area of square *OJLK* is 1. The cross product, $\vec{j} \times \vec{k}$, is a vector perpendicular to the plane determined by $\vec{j}$ and $\vec{k}$, and must therefore lie along either the positive or negative x-axis. Using the definition of the cross product, and knowing that these vectors form a right-handed system, the only possibility is that the cross product must then lie along the *positive* x-axis. The length of the cross product must equal the area of the square *OJLK*, which is 1. So, the required cross product is $\vec{i}$ since $|\vec{i}| = 1$.

Using the same kind of reasoning, it is interesting to note that $\vec{k} \times \vec{j}$ is $-\vec{i}$, which could be determined by using the definition of a right-handed system and verified by calculation.

Physical Application of the Cross Product

The cross product can also be used in the consideration of forces that involve rotation, or turning about a point or an axis. The rotational or turning effect of a force is something that is commonly experienced in everyday life. A typical example might be the tightening or loosening of a nut using a wrench. A second example is the application of force to a bicycle pedal to make the crank arm rotate. The simple act of opening a door by pushing or pulling on it is a third example of how force can be used to create a turning effect. In each of these cases, there is rotation about either a point or an axis.

In the following situation, a bolt with a right-hand thread is being screwed into a piece of wood by a wrench, as shown. A force $\vec{f}$ is applied to the wrench at point N and is rotating about point M. The vector $\vec{r} = \overrightarrow{MN}$ is the position vector of N with respect to M—that is, it defines the position of N relative to M.



The torque, or the turning effect, of the force $\vec{f}$ about the point M is defined to be the vector $\vec{r} \times \vec{f}$. This vector is perpendicular to the plane formed by the vectors $\vec{r}$ and $\vec{f}$, and gives the direction of the axis through M about which the force tends to twist. In this situation, the vector representing the cross product is directed down as the bolt tightens into the wood and would normally be directed along the axis of the bolt. The magnitude of the torque depends upon two factors: the exerted force, and the distance between the line of the exerted force and the point of rotation, M. The exerted force is $\vec{f}$, and the distance between M and the line of the exerted force is d. The magnitude of the torque is the product of the magnitude of the force (that is, $|\vec{f}|$) and the distance d. Since $d = |\vec{r}|\sin\theta$, the magnitude of the torque $\vec{f}$ about M is $(|\vec{r}|\sin\theta)(|\vec{f}|) = |\vec{r} \times \vec{f}|$. The magnitude of the torque measures the twisting effect of the applied force.

The force $\vec{f}$ is measured in newtons, and the distance d is measured in metres, so the unit of magnitude for torque is (newton)(metres), or joules (J), which is the same unit that work is measured in.

EXAMPLE 4 Using the cross product to calculate torque

A 20 N force is applied at the end of a wrench that is 40 cm in length. The force is applied at an angle of 60° to the wrench. Calculate the magnitude of the torque about the point of rotation M.

Solution

$$|\vec{r} \times \vec{f}| = (|\vec{r}|\sin\theta)|\vec{f}| = (0.40)(20)\frac{\sqrt{3}}{2} \doteq 6.93 \text{ J}$$

One of the implications of calculating the magnitude of torque, $|\vec{r} \times \vec{f}| = |\vec{r}||\vec{f}|\sin\theta$, is that it is maximized when $\sin\theta = 1$ and when the force is applied as far as possible from the turning point—that is, $|\vec{r}|$ is as large as possible. To get the best effect when tightening a bolt, this implies that force should be applied at right angles to the wrench and as far down the handle of the wrench as possible from the turning point.

IN SUMMARY

Key Idea

- Both the dot and cross products have useful applications in geometry and physics.

Need to Know

- $W = \vec{F} \cdot \vec{s}$, where $\vec{F}$ is the force applied to an object, measured in newtons (N); $\vec{s}$ is the displacement of the object, measured in metres (m); and W is work, measured in joules (J).
- $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$
- Area of a parallelogram, with sides $\vec{a}$ and $\vec{b}$, equals $|\vec{a} \times \vec{b}|$
- Area of a triangle, with sides $\vec{a}$ and $\vec{b}$, equals $\frac{1}{2}|\vec{a} \times \vec{b}|$.
- Torque equals $\vec{r} \times \vec{f} = |\vec{r}||\vec{f}|\sin\theta$.
- $|\vec{r} \times \vec{f}|$, the magnitude of the torque, measures the overall twisting effect of applied force.

Exercise 7.7

PART A

C 1. A door is opened by pushing inward. Explain, in terms of torque, why this is most easily accomplished when pushing at right angles to the door as far as possible from the hinge side of the door.

2. a. Calculate $|\vec{a} \times \vec{b}|$, where $\vec{a} = (1, 2, 1)$ and $\vec{b} = (2, 4, 2)$.
   b. If $\vec{a}$ and $\vec{b}$ represent the sides of a parallelogram, explain why your answer for part a. makes sense, in terms of the formula for the area of a parallelogram.

PART B

3. Calculate the amount of work done in each situation.
   a. A stove is slid 3 m across the floor against a frictional force of 150 N.
   b. A 40 kg rock falls 40 m down a slope at an angle of 50° to the vertical.
   c. A wagon is pulled a distance of 250 m by a force of 140 N applied at an angle of 20° to the road.
   d. A lawnmower is pushed 500 m by a force of 100 N applied at an angle of 45° to the horizontal.
4. Determine each of the following by using the method shown in Example 3:
   a. $\vec{i} \times \vec{j}$ b. $-\vec{i} \times \vec{j}$ c. $\vec{i} \times \vec{k}$ d. $-\vec{i} \times \vec{k}$

K 5. Calculate the area of the parallelogram formed by the following pairs of vectors:
   a. $\vec{a} = (1, 1, 0)$ and $\vec{b} = (1, 0, 1)$ b. $\vec{a} = (1, -2, 3)$ and $\vec{b} = (1, 2, 4)$
6. The area of the parallelogram formed by the vectors $\vec{p} = (a, 1, -1)$ and $\vec{q} = (1, 1, 2)$ is $\sqrt{35}$. Determine the value(s) of a for which this is true.
7. In R^3, points $A(-2, 1, 3)$, $B(1, 0, 1)$, and $C(2, 3, 2)$ form the vertices of $\triangle ABC$.
   a. By constructing position vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, determine the area of the triangle.
   b. By constructing position vectors $\overrightarrow{BC}$ and $\overrightarrow{CA}$, determine the area of the triangle.
   c. What conclusion can be drawn?

A 8. A 10 N force is applied at the end of a wrench that is 14 cm long. The force makes an angle of 45° with the wrench. Determine the magnitude of the torque of this force about the other end of the wrench.

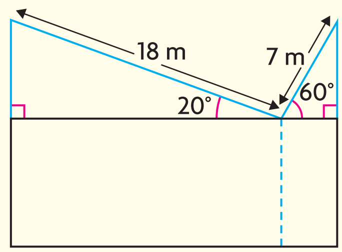
T 9. Parallelogram $OBCA$ has its sides determined by $\overrightarrow{OA} = \vec{a} = (4, 2, 4)$ and $\overrightarrow{OB} = \vec{b} = (3, 1, 4)$. Its fourth vertex is point C. A line is drawn from B perpendicular to side AC of the parallelogram to intersect AC at N. Determine the length of BN.

PART C

10. For the vectors $\vec{p} = (1, -2, 3)$, $\vec{q} = (2, 1, 3)$, and $\vec{r} = (1, 1, 0)$, show the following to be true.
    a. The vector $(\vec{p} \times \vec{q}) \times \vec{r}$ can be written as a linear combination of $\vec{p}$ and $\vec{q}$.
    b. $(\vec{p} \times \vec{q}) \times \vec{r} = (\vec{p} \cdot \vec{r})\vec{q} - (\vec{q} \cdot \vec{r})\vec{p}$

CHAPTER 7: STRUCTURAL ENGINEERING

A structural engineer is designing a special roof for a building. The roof is designed to catch rainwater and hold solar panels to collect sunlight for electricity. Each angled part of the roof exerts a downward force of 50 kg/m^2, including the loads of the panels and rainwater. The building will need a load-bearing wall at the point where each angled roof meets.



a. Calculate the force of the longer angled roof at the point where the roofs meet.

b. Calculate the force of the shorter angled roof at the point where the roofs meet.

c. Calculate the resultant force that the load-bearing wall must counteract to support the roof.

d. Use the given lengths and angles to calculate the width of the building.

e. If the point where the two roofs meet is moved 2 m to the left, calculate the angles that the sloped roofs will make with the horizontal and the length of each roof. Assume that only the point where the roofs meet can be adjusted and that the height of each roof will not change.

f. Repeat parts a. to c., using the new angles you calculated in part e.

g. Make a conjecture about the angles that the two roofs must make with the horizontal (assuming again that the heights are the same but the point where the roofs meet can be adjusted) to minimize the downward force that the load-bearing wall will have to counteract.

h. Calculate the downward force for the angles you conjectured in part g. Then perform the calculations for other angles to test your conjecture.

Key Concepts Review

In Chapter 7, you were introduced to applications of geometric vectors involving force and velocity. You were also introduced to the dot product and cross product between two vectors and should be familiar with the differences in their formulas and applications. Consider the following summary of key concepts:

- When two or more forces are applied to an object, the net effect of the forces can be represented by the resultant vector determined by adding the vectors that represent the forces.

- A system is in a state of equilibrium when the net effect of all the forces acting on an object causes no movement of the object. If there are three forces, this implies that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$.

- The velocity of a moving object can be influenced by external forces, such as wind and the current of a river. The resultant velocity is determined by adding the vectors that represent the object in motion and the effect of the external force: $\vec{v}_r = \vec{v}_{object} + \vec{v}_{external\ force}$

- The dot product between two geometric vectors $\vec{a}$ and $\vec{b}$ is a scalar quantity defined as $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between the two vectors.

- The dot product between two algebraic vectors $\vec{a}$ and $\vec{b}$ is:

  $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2$ in R^2 $\qquad$ $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ in R^3

- If $\vec{a} \cdot \vec{b} = 0$, then $\vec{a} \perp \vec{b}$.

- The cross product $\vec{a} \times \vec{b}$ between two vectors $\vec{a}$ and $\vec{b}$ results in a third vector that is perpendicular to the plane in which the given vectors lie:

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \text{ and } |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta.$$

Geometrically, $|\vec{a} \times \vec{b}|$ is equivalent to the area of the parallelogram formed by vectors $\vec{a}$ and $\vec{b}$.

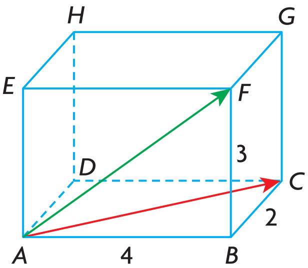
- Work is an application of the dot product, while torque is an application of the cross product.
  - $W = \vec{F} \cdot \vec{s}$, where $\vec{F}$ is the force applied to an object measured in newtons (N), $\vec{s}$ is the objects displacement measured in meters(m), and W is work measured in Joules (J).
  - Torque $= \vec{r} \times \vec{f} = |\vec{r}||\vec{f}|\sin\theta$, where $\vec{r}$ is the vector determined by the lever arm acting from the axis of rotation, $\vec{f}$ is the applied force and θ is the angle between the force and the lever arm.

Review Exercise

1. Given that $\vec{a} = (-1, 2, 1)$, $\vec{b} = (-1, 0, 1)$, and $\vec{c} = (-5, 4, 5)$, determine each of the following:
   a. $\vec{a} \times \vec{b}$
   b. $\vec{b} \times \vec{c}$
   c. $|\vec{a} \times \vec{b}| \times |\vec{b} \times \vec{c}|$
   d. Why is it possible to conclude that the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are coplanar?
2. Given that $\vec{i}$, $\vec{j}$, and $\vec{k}$ represent the standard basis vectors, $\vec{a} = 2\vec{i} - \vec{j} + 2\vec{k}$ and $\vec{b} = 6\vec{i} + 3\vec{j} - 2\vec{k}$, determine each of the following:
   a. $|\vec{a}|$
   b. $|\vec{b}|$
   c. $|\vec{a} - \vec{b}|$
   d. $|\vec{a} + \vec{b}|$
   e. $\vec{a} \cdot \vec{b}$
   f. $\vec{a} \cdot (\vec{a} - 2\vec{b})$
3. a. For what value(s) of a are the vectors $\vec{x} = (3, a, 9)$ and $\vec{y} = (a, 12, 18)$ collinear?
   b. For what value(s) of a are these vectors perpendicular?
4. Determine the angle between the vectors $\vec{x} = (4, 5, 20)$ and $\vec{y} = (-3, 6, 22)$.
5. A parallelogram has its sides determined by $\overrightarrow{OA} = (5, 1)$ and $\overrightarrow{OB} = (-1, 4)$.
   a. Draw a sketch of the parallelogram.
   b. Determine the angle between the two diagonals of this parallelogram.
6. An object of mass 10 kg is suspended by two pieces of rope that make an angle of 30° and 45° with the horizontal. Determine the tension in each of the two pieces of rope.
7. An airplane has a speed of 300 km/h and is headed due west. A wind is blowing from the south at 50 km/h. Determine the resultant velocity of the airplane.
8. The diagonals of a parallelogram are determined by the vectors $\vec{x} = (3, -3, 5)$ and $\vec{y} = (-1, 7, 5)$.
   a. Construct x, y, and z coordinate axes and draw the two given vectors. In addition, draw the parallelogram formed by these vectors.
   b. Determine the area of the parallelogram.
9. Determine the components of a unit vector perpendicular to $(0, 3, -5)$ and to $(2, 3, 1)$.
10. A triangle has vertices $A(2, 3, 7)$, $B(0, -3, 4)$, and $C(5, 2, -4)$.
    a. Determine the largest angle in the triangle.
    b. Determine the area of $\triangle ABC$.

11. A mass of 10 kg is suspended by two pieces of string, 30 cm and 40 cm long, from two points that are 50 cm apart and at the same level. Find the tension in each piece of string.

12. A particle is acted upon by the following four forces: 25 N pulling east, 30 N pulling west, 54 N pulling north, and 42 N pulling south.
    a. Draw a diagram showing these four forces.
    b. Calculate the resultant and equilibrant of these forces.



13. A rectangular box is drawn as shown in the diagram at the left. The lengths of the edges of the box are $AB = 4$, $BC = 2$, and $BF = 3$.
    a. Select an appropriate origin, and then determine coordinates for the other vertices.
    b. Determine the angle between $\overrightarrow{AF}$ and $\overrightarrow{AC}$.
    c. Determine the scalar projection of $\overrightarrow{AF}$ on $\overrightarrow{AC}$.

14. If $\vec{a}$ and $\vec{b}$ are unit vectors, and $|\vec{a} + \vec{b}| = \sqrt{3}$, determine $(2\vec{a} - 5\vec{b}) \cdot (\vec{b} + 3\vec{a})$.

15. Kayla wishes to swim from one side of a river, which has a current speed of 2 km/h, to a point on the other side directly opposite from her starting point. She can swim at a speed of 3 km/h in still water.
    a. At what angle to the bank should Kayla swim if she wishes to swim directly across?
    b. If the river has a width of 300 m, how long will it take for her to cross the river?
    c. If Kayla's speed and the river's speed had been reversed, explain why it would not have been possible for her to swim across the river.

16. A parallelogram has its sides determined by the vectors $\overrightarrow{OA} = (3, 2, -6)$ and $\overrightarrow{OB} = (-6, 6, -2)$.
    a. Determine the coordinates of vectors representing the diagonals.
    b. Determine the angle between the sides of the parallelogram.

17. You are given the vectors $\vec{p} = (2, -2, -3)$ and $\vec{q} = (a, b, 6)$.
    a. Determine values of a and b if $\vec{q}$ is collinear with $\vec{p}$.
    b. Determine an algebraic condition for $\vec{p}$ and $\vec{q}$ to be perpendicular.
    c. Using the answer from part b., determine the components of a unit vector that is perpendicular to $\vec{p}$.

18. For the vectors $\vec{m} = (\sqrt{3}, -2, -3)$ and $\vec{n} = (2, \sqrt{3}, -1)$, determine the following:

a. the angle between these two vectors, to the nearest degree

b. the scalar projection of $\vec{n}$ on $\vec{m}$

c. the vector projection of $\vec{n}$ on $\vec{m}$

d. the angle that $\vec{m}$ makes with the z-axis

19. A number of unit vectors, each of which is perpendicular to the other vectors in the set, is said to form a *special* set. Determine which of the following sets are special.

a. $(1, 0, 0), (0, 0, -1), (0, 1, 0)$

b. $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), (0, 0, -1)$

20. If $\vec{p} = \vec{i} - 2\vec{j} + \vec{k}$, $\vec{q} = 2\vec{i} - \vec{j} + \vec{k}$, and $\vec{r} = \vec{j} - 2\vec{k}$, determine each of the following:

a. $\vec{p} \times \vec{q}$

b. $(\vec{p} - \vec{q}) \times (\vec{p} + \vec{q})$

c. $(\vec{p} \times \vec{r}) \cdot \vec{r}$

d. $(\vec{p} \times \vec{q}) \times \vec{r}$

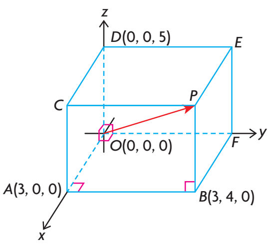
21. Two forces of equal magnitude act on an object so that the angle between their directions is 60°. If their resultant has a magnitude of 20 N, find the magnitude of the equal forces.

22. Determine the components of a vector that is perpendicular to the vectors $\vec{a} = (3, 2, -1)$ and $\vec{b} = (5, 0, 1)$.

23. If $|\vec{x}| = 2$ and $|\vec{y}| = 5$, determine the dot product between $\vec{x} - 2\vec{y}$ and $\vec{x} + 3\vec{y}$ if the angle between $\vec{x}$ and $\vec{y}$ is 60°.

24. The magnitude of the scalar projection of $(1, m, 0)$ on $(2, 2, 1)$ is 4. Determine the value of m.

25. Determine the angle that the vector $\vec{a} = (12, -3, 4)$ makes with the y-axis.



26. A rectangular solid measuring 3 by 4 by 5 is placed on a coordinate axis as shown in the diagram at the left.

a. Determine the coordinates of points C and F.

b. Determine $\overrightarrow{CF}$.

c. Determine the angle between the vectors $\overrightarrow{CF}$ and $\overrightarrow{OP}$.

27. The vectors $\vec{d}$ and $\vec{e}$ are such that $|\vec{d}| = 3$ and $|\vec{e}| = 5$, where the angle between the two given vectors is 50°. Determine each of the following:

a. $|\vec{d} + \vec{e}|$ b. $|\vec{d} - \vec{e}|$ c. $|\vec{e} - \vec{d}|$

28. Find the scalar and vector projections of $\vec{i} + \vec{j}$ on each of the following vectors:

a. $\vec{i}$ b. $\vec{j}$ c. $\vec{k} + \vec{j}$

29. a. Determine which of the following are unit vectors:

$\vec{a} = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$, $\vec{b} = \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$, $\vec{c} = \left(\frac{1}{2}, \frac{-1}{\sqrt{2}}, \frac{1}{2}\right)$, and

$\vec{d} = (-1, 1, 1)$

b. Which one of vectors $\vec{a}$, $\vec{b}$, or $\vec{c}$ is perpendicular to vector $\vec{d}$? Explain.

30. A 25 N force is applied at the end of a 60 cm wrench. If the force makes a 30° angle with the wrench, calculate the magnitude of the torque.

31. a. Verify that the vectors $\vec{a} = (2, 5, -1)$ and $\vec{b} = (3, -1, 1)$ are perpendicular.

b. Find the direction cosines for each vector.

c. If $\overrightarrow{m_1} = (\cos \alpha_a, \cos \beta_a, \cos \gamma_a)$, the direction cosines for $\vec{a}$, and if $\overrightarrow{m_2} = (\cos \alpha_b, \cos \beta_b, \cos \gamma_b)$, the direction cosines for $\overrightarrow{b_2}$, verify that $\overrightarrow{m_1} \cdot \overrightarrow{m_2} = 0$.

32. The diagonals of quadrilateral *ABCD* are $3\vec{i} + 3\vec{j} + 10\vec{k}$ and $-\vec{i} + 9\vec{j} - 6\vec{k}$. Show that quadrilateral *ABCD* is a rectangle.

33. The vector $\vec{v}$ makes an angle of 30° with the *x*-axis and equal angles with both the *y*-axis and *z*-axis.

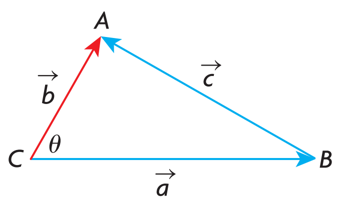
a. Determine the direction cosines for $\vec{v}$.

b. Determine the angle that $\vec{v}$ makes with the *z*-axis.

34. The vectors $\vec{a}$ and $\vec{b}$ are unit vectors that make an angle of 60° with each other. If $\vec{a} - 3\vec{b}$ and $m\vec{a} + \vec{b}$ are perpendicular, determine the value of m.

35. If $\vec{a} = (0, 4, -6)$ and $\vec{b} = (-1, -5, -2)$, verify that

$\vec{a} \cdot \vec{b} = \frac{1}{4}|\vec{a} + \vec{b}|^2 - \frac{1}{4}|\vec{a} - \vec{b}|^2$.



36. Use the fact that $|\vec{c}|^2 = \vec{c} \cdot \vec{c}$ to prove the cosine law for the triangle shown in the diagram with sides $\vec{a}$, $\vec{b}$, and $\vec{c}$.

37. Find the lengths of the sides, the cosines of the angles, and the area of the triangle whose vertices are $A(1, -2, 1)$, $B(3, -2, 5)$, and $C(2, -2, 3)$.

Chapter 7 Test

1. Given the vectors $\vec{a} = (-1, 1, 1)$, $\vec{b} = (2, 1, -3)$, and $\vec{c} = (5, 1, -7)$, calculate the value of each of the following:

   a. $\vec{a} \times \vec{b}$ b. $\vec{b} \times \vec{c}$ c. $\vec{a} \cdot (\vec{b} \times \vec{c})$ d. $(\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{c})$

2. Given the vectors $\vec{a} = \vec{i} - \vec{j} + \vec{k}$ and $\vec{b} = 2\vec{i} - \vec{j} - 2\vec{k}$, determine the following:

   a. the scalar projection and vector projection of $\vec{a}$ on $\vec{b}$

   b. the angle that $\vec{b}$ makes with each of the coordinate axes

   c. the area of the parallelogram formed by the vectors $\vec{a}$ and $\vec{b}$

3. Two forces of 40 N and 50 N act at an angle of 60° to each other. Determine the resultant and equilibrant of these forces.

4. An airplane is heading due north at 1000 km/h when it encounters a wind from the east at 100 km/h. Determine the resultant velocity of the airplane.

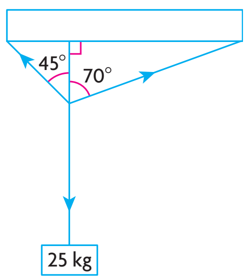
5. A canoeist wishes to cross a 200 m river to get to a campsite directly across from the starting point. The canoeist can paddle at 2.5 m/s in still water, and the current has a speed of 1.2 m/s.

   a. How far downstream would the canoeist land if headed directly across the river?

   b. In what direction should the canoeist head in order to arrive directly across from the starting point?

6. Calculate the area of a triangle with vertices $A(-1, 3, 5)$, $B(2, 1, 3)$, and $C(-1, 1, 4)$.

7. A 25 kg mass is suspended from a ceiling by two cords. The cords make angles of 45° and 70° with a perpendicular drawn to the ceiling, as shown. Determine the tension in each cord.



8. a. Using the vectors $\vec{x} = (3, 3, 1)$ and $\vec{y} = (-1, 2, -3)$, verify that the following formula is true: $\vec{x} \cdot \vec{y} = \frac{1}{4}|\vec{x} + \vec{y}|^2 - \frac{1}{4}|\vec{x} - \vec{y}|^2$

   b. Prove that this formula is true for any two vectors.

Chapter 8

EQUATIONS OF LINES AND PLANES

In this chapter, you will work with vector concepts you learned in the preceding chapters and use them to develop equations for lines and planes. We begin with lines in R^2 and then move to R^3, where lines are once again considered along with planes. The determination of equations for lines and planes helps to provide the basis for an understanding of geometry in R^3. All of these concepts provide the foundation for the solution of systems of linear equations that result from intersections of lines and planes, which are considered in Chapter 9.

CHAPTER EXPECTATIONS

In this chapter, you will

- determine the vector and parametric equations of a line in two-space, **Section 8.1**
- make connections between Cartesian, vector, and parametric equations of a line in two-space, **Section 8.2**
- determine the vector, parametric, and symmetric equations of a line in 3-space, **Section 8.3**
- determine the vector, parametric, and Cartesian equations of a plane, **Sections 8.4, 8.5**
- determine some geometric properties of a plane, **Section 8.5**
- determine the equation of a plane in Cartesian, vector, or parametric form, given another form, **Section 8.5**
- sketch a plane in 3-space, **Section 8.6**

Review of Prerequisite Skills

In this chapter, we will develop the equation of a line in two- and three-dimensional space and the equation of a plane in three-dimensional space. You will find it helpful to review the following concepts:

- geometric and algebraic vectors
- the dot product
- the cross product
- plotting points and vectors in three-space

We will begin this chapter by examining equations of lines. Lines are not vectors, but vectors are used to describe lines. The table below shows their similarities and differences.

Lines	Vectors
Lines are bi-directional. A line defines a direction, but there is nothing to distinguish forward from backwards.	Vectors are unidirectional. A vector defines a direction with a clear distinction between forward and backwards.
A line is infinite in extent in both directions. A line segment has a finite length.	Vectors have a finite magnitude.
Lines and line segments have a definite location. The opposite sides of a parallelogram are two different line segments.	A vector has no fixed location. The opposite sides of a parallelogram are described by the same vector.
Two lines are the same when they have the same direction and same location. These lines are said to be coincident.	Two vectors are the same when they have the same direction and the same magnitude. These vectors are said to be equal.

Exercise

1. Determine a single vector that is equivalent to each of the following expressions:

a. $(3, -2, 1) - (1, 7, -5)$

b. $5(2, -3, -4) + 3(1, 1, -7)$

2. Determine if the following sets of points are collinear:

a. $A(1, -3), B(4, 2), C(-8, -18)$

b. $J(-4, 3), K(4, 5), L(0, 4)$

c. $A(1, 2, 1), B(4, 7, 0), C(7, 12, -1)$

d. $R(1, 2, -3), S(4, 1, 3), T(2, 4, 0)$

3. Determine if $\triangle ABC$ is a right-angled triangle, given $A(1, 6, -2)$, $B(2, 5, 3)$, and $C(5, 3, 2)$.

4. Given $\vec{u} = (t, -1, 3)$ and $\vec{v} = (2, t, -6)$, for what values of t are the vectors perpendicular?

5. State a vector perpendicular to each of the following:
   a. $\vec{a} = (1, -3)$
   b. $\vec{b} = (6, -5)$
   c. $\vec{c} = (-7, -4, 0)$

6. Calculate the area of the parallelogram formed by the vectors $(4, 10, 9)$ and $(3, 1, -2)$.

7. Use the cross product to determine a vector perpendicular to each of the following pairs of vectors. Check your answer using the dot product.
   a. $\vec{a} = (2, 1, -4)$ and $\vec{b} = (3, -5, -2)$
   b. $\vec{a} = (-1, -2, 0)$ and $\vec{b} = (-2, -1, 0)$

8. For each of the following, draw the x-axis, y-axis, and z-axis, and accurately draw the position vectors:
   a. $A(1, 2, 3)$
   b. $B(1, 2, -3)$
   c. $C(1, -2, 3)$
   d. $D(-1, 2, 3)$

9. Determine the position vector that passes from the first point to the second.
   a. $(4, 8)$ and $(-3, 5)$
   b. $(-7, -6)$ and $(3, 8)$
   c. $(1, 2, 4)$ and $(3, -6, 9)$
   d. $(4, 0, -4)$ and $(0, 5, 0)$

10. State the vector that is opposite to each of the vectors you found in question 9.

11. Determine the slope and y-intercept of each of the following linear equations. Then sketch its graph.
    a. $y = -2x - 5$
    b. $4x - 8y = 8$
    c. $3x - 5y + 1 = 0$
    d. $5x = 5y - 15$

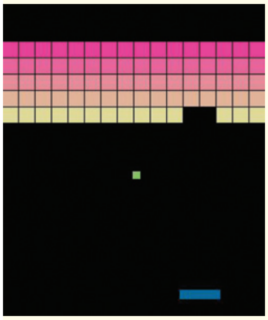
12. State a vector that is collinear to each of the following and has the same direction:
    a. $(4, 7)$
    b. $(-5, 4, 3)$
    c. $2\vec{i} + 6\vec{j} - 4\vec{k}$
    d. $-5\vec{i} + 8\vec{j} + 2\vec{k}$

13. If $\vec{u} = (4, -9, -1)$ and $\vec{v} = 4\vec{i} - 2\vec{j} + \vec{k}$, determine each of the following:
    a. $\vec{u} \cdot \vec{v}$
    b. $-\vec{v} \cdot \vec{u}$
    c. $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v})$
    d. $\vec{u} \times \vec{v}$
    e. $\vec{v} \times \vec{u}$
    f. $(2\vec{u} + \vec{v}) \times (\vec{u} - 2\vec{v})$

14. Both the dot product and the cross product are ways to multiply two vectors. Explain how these products differ.

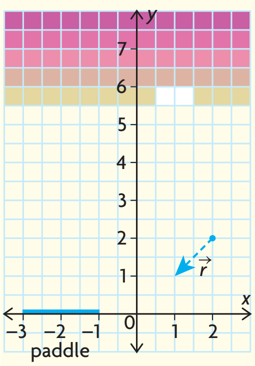
CHAPTER 8: COMPUTER PROGRAMMING WITH VECTORS

Computer programmers use vectors for a variety of graphics applications. Any time that two- and three-dimensional images are designed, they are represented in the form of vectors. Vectors allow the programmer to move the figure easily to any new location on the screen. If the figure were expressed point by point using coordinate geometry, each and every point would have to be recalculated each time the figure needed to be moved. By using vectors drawn from an anchor point to draw the figure, only the coordinates of the anchor point need to be recalculated on the screen to move the entire figure. This method is used in many different types of software, including games, flight simulators, drafting and architecture tools, and visual design tools.



Case Study—Breakout

Breakout is a classic video game, and a prime example of using vectors in computer graphics. A paddle is used to bounce a ball into a section of bricks to slowly break down a wall. Each time the ball hits the paddle, it bounces off at an angle to the paddle. The path the ball takes from the paddle can be described by a vector, which is dependent upon the angle and speed of the ball. If the wall were not in the path of the ball, the ball would continue along its path at that speed until it "fell" off of the screen.



DISCUSSION QUESTIONS

1. The coordinate plane represents the screen in the game "Breakout." The ball is travelling toward the paddle along the vector $\vec{r}$. Find the equation of the line determined by vector $\vec{r}$ in its current position. Draw a direction vector for the line.
2. Find where the line crosses the x-axis to show where the paddle must move in order to bounce the ball back.
3. Since the angle of entry for the ball is 45°, the ball will bounce off the paddle along a path perpendicular to $\vec{r}$. Draw a vector $\vec{s}$ perpendicular to $\vec{r}$ that emanates from the origin in the direction the ball will travel when it bounces off the paddle. Then draw a line parallel to vector $\vec{s}$ that passes through the point where $\vec{r}$ crosses the x-axis.

Section 8.1—Vector and Parametric Equations of a Line in R^2

In this section, we begin with a discussion about how to find the **vector** and **parametric equations** of a line in R^2. To find the vector and parametric equations of a line, we must be given either two distinct points or one point and a vector that defines the direction of the line. In either situation, a **direction vector** for the line is necessary. A direction vector is defined to be a nonzero vector $\vec{m} = (a, b)$ parallel (collinear) to the given line. The direction vector $\vec{m} = (a, b)$ is represented by a vector with its tail at the origin and its head at the point (a, b). The x and y components of this direction vector are called its **direction numbers**. For the vector (a, b), the direction numbers are a and b.

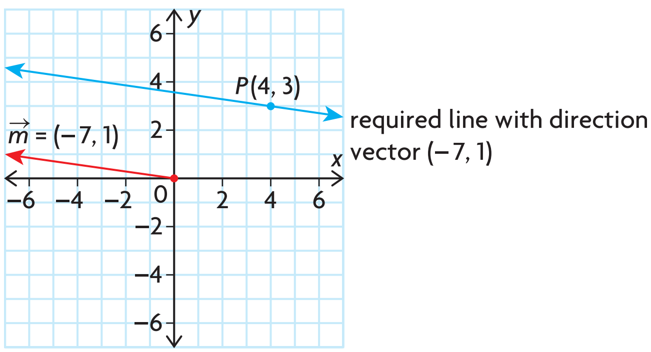
EXAMPLE 1 Representing lines using vectors

a. A line passing through $P(4, 3)$ has $\vec{m} = (-7, 1)$ as its direction vector. Sketch this line.

b. A line passes through the points $A\left(\frac{1}{2}, -3\right)$ and $B\left(\frac{3}{4}, \frac{1}{2}\right)$. Determine a direction vector for this line, and write it using integer components.

Solution

a. The vector $\vec{m} = (-7, 1)$ is a direction vector for the line and is shown on the graph. The required line is parallel to $\vec{m}$ and passes through $P(4, 3)$. This line is drawn through $P(4, 3)$, parallel to $\vec{m}$.



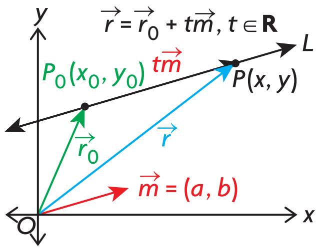
b. When determining a direction vector for the line through $A\left(\frac{1}{2}, -3\right)$ and $B\left(\frac{3}{4}, \frac{1}{2}\right)$, we determine a vector equivalent to either $\overrightarrow{AB}$ or $\overrightarrow{BA}$.

$$\overrightarrow{AB} = \left(\frac{3}{4} - \frac{1}{2}, \frac{1}{2} - (-3)\right) = \left(\frac{1}{4}, \frac{7}{2}\right) \text{ or } \overrightarrow{BA} = \left(-\frac{1}{4}, -\frac{7}{2}\right)$$

Both of these vectors can be multiplied by 4 to ensure that both direction numbers are integers. As a result, either $\vec{m} = (1, 14)$ or $\vec{m} = (-1, -14)$ are the best choices for a direction vector. When we determine the direction vector, any scalar multiple of this vector of the form $t(1, 14)$ is correct, provided that $t \neq 0$. If $t = 0$, $(0, 0)$ would be the direction vector, meaning that the line would not have a defined direction.

Expressing the Equations of Lines Using Vectors

In general, we would like to determine the equation of a line if we have a direction for the line and a point on it. In the following diagram, the given point $P_0(x_0, y_0)$ is on the line L and is associated with vector $\overrightarrow{OP_0}$, designated as $\vec{r_0}$. The direction of the line is given by $\vec{m} = (a, b)$, where $t\vec{m},\ t \in \mathbf{R}$ is any vector collinear with $\vec{m}$. $P(x, y)$ represents a general point on the line, where $\overrightarrow{OP}$ is the vector associated with this point.



To find the vector equation of line L, the triangle law of addition is used.

In $\triangle OP_0P$, $\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P}$.

Since $\vec{r} = \overrightarrow{OP}$, $\vec{r_0} = \overrightarrow{OP_0}$, and $t\vec{m} = \overrightarrow{P_0P}$, the vector equation of the line is written as $\vec{r} = \vec{r_0} + t\vec{m}, t \in \mathbf{R}$.

When writing an equation of a line using vectors, the vector form of the line is sometimes modified and put in parametric form. The parametric equations of a line come directly from its vector equation. How to change the equation of a line from vector to parametric form is shown below.

The general vector equation of a line is $\vec{r} = \vec{r_0} + t\vec{m}, t \in \mathbf{R}$.

In component form, this is written as $(x, y) = (x_0, y_0) + t(a, b), t \in \mathbf{R}$. Expanding the right side, $(x, y) = (x_0, y_0) + (ta, tb) = (x_0 + ta, y_0 + tb), t \in \mathbf{R}$. If we equate the respective x and y components, the required parametric form is $x = x_0 + ta$ and $y = y_0 + tb$, $t \in \mathbf{R}$.

Vector and Parametric Equations of a Line in R^2

Vector Equation: $\vec{r} = \vec{r_0} + t\vec{m}, t \in \mathbf{R}$
Parametric Equations: $x = x_0 + ta,\ y = y_0 + tb,\ t \in \mathbf{R}$
where $\vec{r_0}$ is the vector from $(0, 0)$ to the point (x_0, y_0) and $\vec{m}$ is a direction vector with components (a, b).

In either vector or parametric form, t is called a **parameter**. This means that t can be replaced by any real number to obtain the coordinates of points on the line.

EXAMPLE 2 Reasoning about the vector and parametric equations of a line

a. Determine the vector and parametric equations of a line passing through point $A(1, 4)$ with direction vector $\vec{m} = (-3, 3)$.
b. Sketch the line, and determine the coordinates of four points on the line.
c. Is either point $Q(-21, 23)$ or point $R(-29, 34)$ on this line?

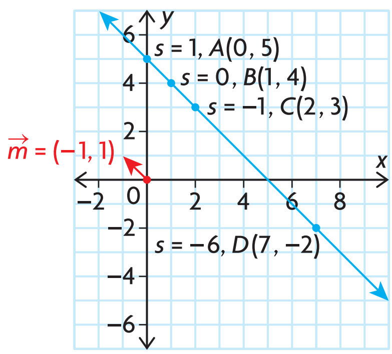
Solution

a. Since $A(1, 4)$ is on the line, $\overrightarrow{OP_0} = \vec{r}_0 = (1, 4)$ and $\vec{m} = (-3, 3)$. The vector equation is $\vec{r} = (1, 4) + t(-3, 3),\ t \in \mathbf{R}$. The parametric equations are $x = 1 - 3t, y = 4 + 3t,\ t \in \mathbf{R}$.

It is also possible to use other scalar multiples of $\vec{m} = (-3, 3)$ as a direction vector, such as $(-1, 1)$, which gives the respective vector and parametric equations $\vec{r} = (1, 4) + s(-1, 1),\ s \in \mathbf{R}$, and $x = 1 - s,\ y = 4 + s, s \in \mathbf{R}$. The vector $(-1, 1)$ has been chosen as our direction vector for the sake of simplicity. Note that we have written the second equation with parameter s to avoid confusion between the two lines. Although the two equations, $\vec{r} = (1, 4) + t(-3, 3),\ t \in \mathbf{R}$, and $\vec{r} = (1, 4) + s(-1, 1),\ s \in \mathbf{R}$, appear with different parameters, the lines they represent are identical.

b. To determine the coordinates of points on the line, the parametric equations $x = 1 - s, y = 4 + s, s \in \mathbf{R}$, were used, with s chosen to be 0, 1, -1, and -6. To find the coordinates of a particular point, such as D, $s = -6$ was substituted into the parametric equations and $x = 1 - (-6) = 7, y = 4 + (-6) = -2$.

The required point is $D(7, -2)$. The coordinates of the other points are determined in the same way, using the other values of s.



c. If the point $Q(-21, 23)$ lies on the line, then there must be consistency with the parameter s. We substitute this point into the parametric equations to check for the required consistency. Substituting gives $-21 = 1 - s$ and $23 = 4 + s$.

In the first equation, $s = 22$, and in the second equation, $s = 19$. Since these values are inconsistent, the point Q is not on the line.

If the point $R(-29, 34)$ is on the line, then $-29 = 1 - s$ and $34 = 4 + s$, $s = 30$, for both equations.

Since each of these equations has the same solution, $s = 30$, we conclude that $R(-29, 34)$ is on the line.

Sometimes, the equation of the line must be found when two points are given. This is shown in the following example.

EXAMPLE 3 Connecting vector and parametric equations with two points on a line

a. Determine vector and parametric equations for the line containing points $E(-1, 5)$ and $F(6, 11)$.
b. What are the coordinates of the point where this line crosses the x-axis?
c. Can the equation $\vec{r} = (-15, -7) + t\left(\frac{14}{3}, 4\right), t \in \mathbf{R}$, also represent the line containing points E and F?

Solution

a. A direction vector for the line containing points E and F is $\vec{m} = \overrightarrow{EF} = (6 - (-1), 11 - 5) = (7, 6)$. A vector equation for the line is $\vec{r} = (-1, 5) + s(7, 6), s \in \mathbf{R}$, and its parametric equations are $x = -1 + 7s, y = 5 + 6s, s \in \mathbf{R}$.

The equation given for this line is not unique. This is because there are an infinite number of choices for the direction vector, and any point on the line could have been used. In writing a second equation for the line, the parametric equations $x = 6 + 7s, y = 11 + 6s,\ s \in \mathbf{R}$, would also have been correct because $(6, 11)$ is on the line and the direction vector is $(7, 6)$.

b. The line intersects the x-axis at a point with coordinates of the form $(a, 0)$. At the point of intersection, $y = 0$ and, so, $5 + 6s = 0$, $s = \frac{-5}{6}$. Therefore,

$$\begin{aligned} a &= -1 + 7s \\ &= -1 + 7\left(\frac{-5}{6}\right) \\ &= -\frac{41}{6}, \end{aligned}$$

and the line intersects the x-axis at the point $\left(-\frac{41}{6}, 0\right)$.

c. If this equation represents the same line as the equation in part a., it is necessary for the two lines to have the same direction and contain the same set of points.

The line $\vec{r} = (-15, -7) + t\left(\frac{14}{3}, 4\right), t \in \mathbf{R}$, has $\left(\frac{14}{3}, 4\right)$ as its direction vector.

The two lines will have the same direction vectors because $\frac{3}{2}\left(\frac{14}{3}, 4\right) = (7, 6)$.

The two lines have the same direction, and if these lines have a point in common, then the equations represent the same line. The easiest approach is to substitute $(-15, -7)$ into the first equation to see if this point is on the line. Substituting gives $(-15, -7) = (-1, 5) + s(7, 6)$ or $-15 = -1 + 7s$ and $-7 = 5 + 6s$. Since the solution to both of these equations is $s = -2$, the point $(-15, -7)$ is on the line, and the two equations represent the same line.

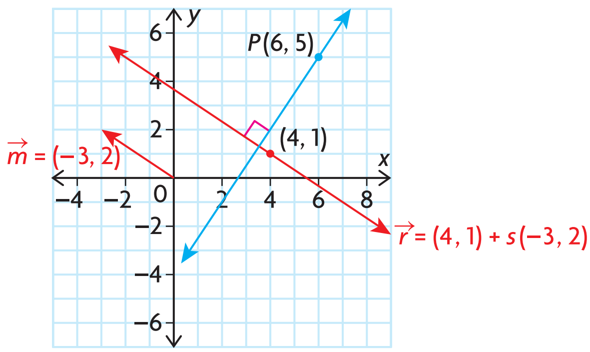
In the next example, vector properties will be used to determine equations for lines that involve perpendicularity.

EXAMPLE 4 Selecting a strategy to determine the vector equation of a perpendicular line

Determine a vector equation for the line that is perpendicular to $\vec{r} = (4, 1) + s(-3, 2), s \in \mathbf{R}$, and passes through point $P(6, 5)$.

Solution

The direction vector for the given line is $\vec{m} = (-3, 2)$, and this line is drawn through (4, 1), as shown in red in the diagram. A sketch of the required line, passing through (6, 5) and perpendicular to the given line, is drawn in blue.



Let the direction vector for the required blue line be $\vec{v} = (a, b)$. Since the direction vector of the given line is perpendicular to that of the required line, $\vec{v} \cdot \vec{m} = 0$.

Therefore, $(a, b) \cdot (-3, 2) = 0$ or $-3a + 2b = 0$.

The simplest integer values for a and b, which satisfy this equation, are $a = 2$ and $b = 3$. This gives the direction vector $(2, 3)$ and the required vector equation for the perpendicular line is $\vec{r} = (6, 5) + t(2, 3),\ t \in \mathbf{R}$.

In this section, the vector and parametric equations of a line in R^2 were discussed. In Section 8.3, the discussion will be extended to R^3, where many of the ideas seen in this section apply to lines in three-space.

The following investigation is designed to aid in understanding the concept of parameter, when dealing with either the vector or parametric equations of a line.

INVESTIGATION

A. i. On graph paper, draw the lines $L_1: \vec{r} = t(0, 1),\ t \in \mathbf{R}$, and $L_2: \vec{r} = p(1, 0)$, $p \in \mathbf{R}$. Make sure that you clearly show a direction vector for each line.

ii. Describe geometrically what each of the two equations represent.

iii. Give a vector equation and corresponding parametric equations for each of the following:

- the line parallel to the x-axis, passing through $P(2, 4)$
- the line parallel to the y-axis, passing through $Q(-2, -1)$

iv. Sketch $L_3: x = -3,\ y = 1 + s,\ s \in \mathbf{R}$, and $L_4: x = 4 + t,\ y = 1,\ t \in \mathbf{R}$, using your own axes.

v. By examining parametric equations of a line, how is it possible to determine by inspection whether the line is parallel to either the x-axis or y-axis?

vi. Write an equation of a line in both vector and parametric form that is parallel to the x-axis.

vii. Write an equation of a line in both vector and parametric form that is parallel to the y-axis.

B. i. Sketch the line $L: \vec{r} = (-3, 0) + s(2, -1), s \in \mathbf{R}$, on graph paper.

ii. On the set of axes used for part i., sketch each of the following:

- $L_1: \vec{r} = (-2, 1) + s(2, -1), s \in \mathbf{R}$
- $L_2: \vec{r} = (-3, 1) + s(2, -1), s \in \mathbf{R}$
- $L_3: \vec{r} = (2, -1) + s(2, -1), s \in \mathbf{R}$
- $L_4: \vec{r} = (4, 2) + s(2, -1), s \in \mathbf{R}$

If you are given the equation $\vec{r} = \vec{r_0} + s(2, -1), s \in \mathbf{R}$, what is the mathematical effect of changing the value of $\vec{r_0}$?

iii. For the line $L_1: \vec{r} = (-2, 1) + s(2, -1), s \in \mathbf{R}$, show that each of the following points are on this line by finding corresponding values of s: $(4, -2), (-4, 2), (198, -99)$, and $(-202, 101)$.

iv. Which part of the equation $\vec{r} = \vec{r_0} + t\vec{m}, t \in \mathbf{R}$, indicates that there are an infinite number of points on this line? Explain your answer.

IN SUMMARY

Key Ideas

- The vector equation of a line in R^2 is $\vec{r} = \vec{r_0} + t\vec{m}$, $t \in \mathbf{R}$, where $\vec{m} = (a, b)$ is the direction vector and $\vec{r_0}$ is the vector from the origin to any point on the line whose general coordinates are (x_0, y_0). This is equivalent to the equation $(x, y) = (x_0, y_0) + t(a, b)$.
- The parametric form of the equation of a line is $x = x_0 + ta$ and $y = y_0 + tb$, $t \in \mathbf{R}$.

Need to Know

- In both the vector and parametric equations, t is a parameter. Every real number for t generates a different point that lies on the line.

Exercise 8.1

PART A

1. A vector equation is given as $\vec{r} = \left(\frac{1}{2}, -\frac{3}{4}\right) + s\left(\frac{1}{3}, \frac{1}{6}\right)$, $s \in \mathbf{R}$. Explain why $\vec{m} = (-2, -1)$, $\vec{m} = (2, 1)$, and $\vec{m} = \left(\frac{2}{7}, \frac{1}{7}\right)$ are acceptable direction vectors for this line.
2. Parametric equations of a line are $x = 1 + 3t$ and $y = 5 - 2t$, $t \in \mathbf{R}$.
   a. Write the coordinates of three points on this line.
   b. Show that the point $P(-14, 15)$ lies on the given line by determining the parameter value of t corresponding to this point.
3. Identify the direction vector and a point on each of the following lines:
   a. $\vec{r} = (3, 4) + t(2, 1)$, $t \in \mathbf{R}$
   b. $x = 1 + 2t$, $y = 3 - 7t$, $t \in \mathbf{R}$
   c. $\vec{r} = (4, 1 + 2t)$, $t \in \mathbf{R}$
   d. $x = -5t$, $y = 6$, $t \in \mathbf{R}$

PART B

4. A line passes through the points $A(2, 1)$ and $B(-3, 5)$. Write two different vector equations for this line.
5. A line is defined by the parametric equations $x = -2 - t$ and $y = 4 + 2t$, $t \in \mathbf{R}$.
   a. Does $R(-9, 18)$ lie on this line? Explain.
   b. Write a vector equation for this line using the given parametric equations.
   c. Write a second vector equation for this line, different from the one you wrote in part b.

6. a. If the equation of a line is $\vec{r} = s(3, 4), s \in \mathbf{R}$, name the coordinates of three points on this line.

   b. Write a vector equation, different from the one given, in part a., that also passes through the origin.

   c. Describe how the line with equation $\vec{r} = (9, 12) + t(3, 4), t \in \mathbf{R}$ relates to the line given in part a.

C 7. A line has $\vec{r} = \left(\frac{1}{3}, \frac{1}{7}\right) + p(-2, 3), p \in \mathbf{R}$, as its vector equation. A student decides to "simplify" this equation by clearing the fractions and multiplies the vector $\left(\frac{1}{3}, \frac{1}{7}\right)$ by 21. The student obtains $\vec{r} = (7, 3) + p(-2, 3), p \in \mathbf{R}$, as a "correct" form of the line. Explain why multiplying a point in this way is incorrect.

8. A line passes through the points $Q(0, 7)$ and $R(0, 9)$.

   a. Sketch this line.

   b. Determine vector and parametric equations for this line.

K 9. A line passes through the points $M(4, 5)$ and $N(9, 5)$.

   a. Sketch this line.

   b. Determine vector and parametric equations for this line.

10. For the line $L: \vec{r} = (1, -5) + s(3, 5), s \in \mathbf{R}$, determine the following:

    a. an equation for the line perpendicular to L, passing through $P(2, 0)$

    b. the point at which the line in part a. intersects the y-axis

A 11. The parametric equations of a line are given as $x = -10 - 2s$, $y = 8 + s$, $s \in \mathbf{R}$. This line crosses the x-axis at the point with coordinates $A(a, 0)$ and crosses the y-axis at the point with coordinates $B(0, b)$. If O represents the origin, determine the area of the triangle AOB.

T 12. A line has $\vec{r} = (1, 2) + s(-2, 3)$, $s \in \mathbf{R}$, as its vector equation. On this line, the points A, B, C, and D correspond to parametric values $s = 0, 1, 2,$ and 3, respectively. Show that each of the following is true:

    a. $\overrightarrow{AC} = 2\overrightarrow{AB}$ b. $\overrightarrow{AD} = 3\overrightarrow{AB}$ c. $\overrightarrow{AC} = \frac{2}{3}\overrightarrow{AD}$

PART C

13. The line L has $x = 2 + t$, $y = 9 + t$, $t \in \mathbf{R}$, as its parametric equations. If L intersects the circle with equation $x^2 + y^2 = 169$ at points A and B, determine the following:

    a. the coordinates of points A and B

    b. the length of the chord AB

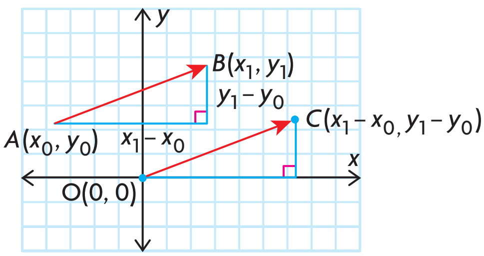
14. Are the lines $2x - 3y + 15 = 0$ and $(x, y) = (1, 6) + t(6, 4)$ parallel? Explain.

Section 8.2—Cartesian Equation of a Line

In the previous section, we discussed the vector and parametric equations of lines in R^2. In this section, we will show how lines of the form $y = mx + b$ (slope–y-intercept form) and $Ax + By + C = 0$ (Cartesian equation of a line, also called a scalar equation of a line) are related to the vector and parametric equations of the line.

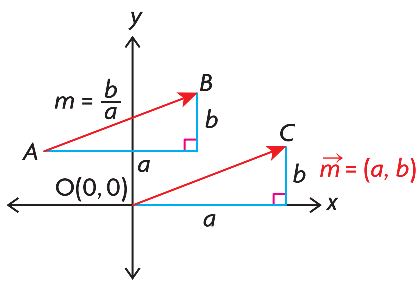
The Relationship between Vector and Scalar Equations of Lines in R^2

The direction, or inclination, of a line can be described in two ways: by its slope and by a direction vector. The slope of the line joining two points $A(x_0, y_0)$ and $B(x_1, y_1)$ is given by the formula $m = \frac{\text{rise}}{\text{run}} = \frac{y_1 - y_0}{x_1 - x_0}$. It is also possible to describe the direction of a line using the vector defined by the two points A and B, $\overrightarrow{AB} = \vec{m} = (x_1 - x_0, y_1 - y_0)$. This direction vector is equivalent to a vector with its tail at the origin and its head at $C(x_1 - x_0, y_1 - y_0)$ and is shown in the diagram below.



Direction Vectors and Slope

In the diagram, a line segment AB with slope $m = \frac{b}{a}$ is shown with a run of a and a rise of b. The vector $\vec{m} = (a, b)$ is used to describe the direction of this line or any line parallel to it, with no restriction on the direction numbers a and b. In practice, a and b can be any two real numbers when describing a direction vector. If the direction vector of a line is $\vec{m} = (a, b)$, this corresponds to a slope of $m = \frac{b}{a}$ except when $a = 0$ (which corresponds to a vertical line).



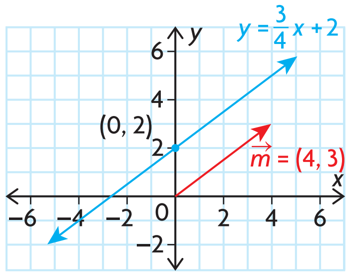
In the following example, we will show how to take a line in slope–y-intercept form and convert it to vector and parametric form.

EXAMPLE 1 Representing the Cartesian equation of a line in vector and parametric form

Determine the equivalent vector and parametric equations of the line $y = \frac{3}{4}x + 2$.

Solution

In the diagram below, the line $y = \frac{3}{4}x + 2$ is drawn. This line passes through $(0, 2)$, has a slope of $m = \frac{3}{4}$, and, as a result, has a direction vector $\vec{m} = (4, 3)$. A vector equation for this line is $\vec{r} = (0, 2) + t(4, 3), t \in \mathbf{R}$, with parametric equations $x = 4t$, $y = 2 + 3t$, $t \in \mathbf{R}$.



In the next example, we will show the conversion of a line in vector form to one in slope–y-intercept form.

EXAMPLE 2 Representing a vector equation of a line in Cartesian form

For the line with equation $\vec{r} = (3, -6) + s(-1, -4), s \in \mathbf{R}$, determine the equivalent slope–y-intercept form.

Solution

Method 1:

The direction vector for this line is $\vec{m} = (-1, -4)$, with slope $m = \frac{-4}{-1} = 4$. This line contains the point $(3, -6)$. If $P(x, y)$ represents a general point on this line, then we can use slope–point form to determine the required equation.

Thus, $\frac{y - (-6)}{x - 3} = 4$

$$4(x - 3) = y + 6$$
$$4x - 12 = y + 6$$
$$4x - 18 = y$$

The required equation for this line is $y = 4x - 18$ in slope–y-intercept form.

Method 2:
We start by writing the given line in parametric form, which is $(x, y) = (3, -6) + s(-1, -4)$ or $(x, y) = (3 - s, -6 - 4s)$. This gives the parametric equations $x = 3 - s$ and $y = -6 - 4s$. To find the required equation, we solve for s in each component. Thus, $s = \frac{x-3}{-1}$ and $s = \frac{y+6}{-4}$. Since these equations for s are equal,

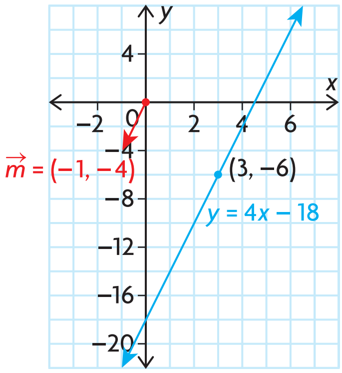
$$\frac{x-3}{-1} = \frac{y+6}{-4}$$

$$\frac{-4(x-3)}{-1} = y + 6$$

$$y + 6 = 4(x - 3)$$

$$y = 4x - 18$$

Therefore, the required equation is $y = 4x - 18$, which is the same answer we obtained using Method 1. The graph of this line is shown below.



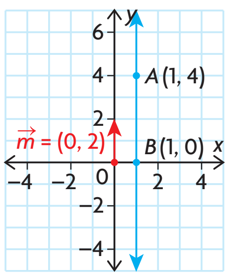
In the example that follows, we examine the situation in which the direction vector of the line is of the form $\vec{m} = (0, b)$.

EXAMPLE 3 Reasoning about equations of vertical lines

Determine the Cartesian form of the line with the equation $\vec{r} = (1, 4) + s(0, 2), s \in \mathbf{R}$.

Solution

The given line passes through the point $(1, 4)$, with direction vector $(0, 2)$, as shown in the diagram below.

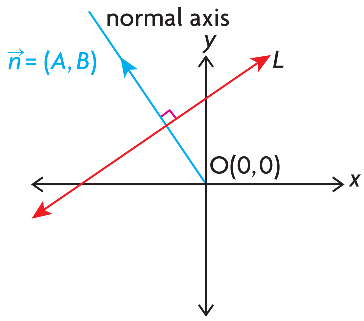


It is not possible, in this case, to calculate the slope because the line has direction vector $(0, 2)$, meaning its slope would be $\frac{2}{0}$, which is undefined. Since the line is parallel to the y-axis, it must have the form $x = a$, where $(a, 0)$ is the point where the line crosses the x-axis. The equation of this line is $x - 1 = 0$ or $x = 1$.

Developing the Cartesian Equation from a Direction Vector

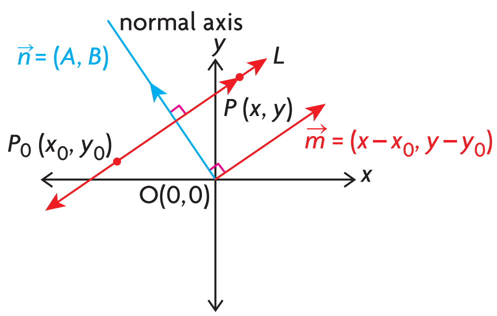
In addition to making the connection between lines in either slope–y-intercept form or Cartesian form with those in vector form, we would like to consider how direction vectors can be used to obtain the equations of lines in Cartesian form.

In the following diagram, the line L represents a general line in R^2. A line has been drawn from the origin, perpendicular to L. This perpendicular line is called the **normal** axis for the line and is the only line that can be drawn from the origin perpendicular to the given line. If the origin is joined to any point on the normal axis, other than itself, the vector formed is described as a normal to the given line. Since there are an infinite number of points on the normal axis, this is a way of saying that any line in R^2 has an infinite number of normals, none of which is the zero vector. A general point on the normal axis is given the coordinates $N(A, B)$, and so a normal vector, denoted by $\vec{n}$, is the vector $\vec{n} = (A, B)$.



The important property of the normal vector is that it is perpendicular to any vector on the given line. This property of normal vectors is what allows us to derive the Cartesian equation of the line.

In the following diagram, the line L is drawn, along with a normal $\vec{n} = (A, B)$, to L. The point $P(x, y)$ represents any point on the line, and the point $P_0(x_0, y_0)$ represents a given point on the line.



To derive the Cartesian equation for this line, we first determine $\overrightarrow{P_0P}$. In coordinate form, this vector is $\overrightarrow{P_0P} = (x - x_0, y - y_0)$, which represents a direction vector for the line. In the diagram, this vector has been shown as $\vec{m} = (x - x_0, y - y_0)$. Since the vectors $\vec{n}$ and $\overrightarrow{P_0P}$ are perpendicular to each other, $\vec{n} \cdot \overrightarrow{P_0P} = 0$.

$(A, B) \cdot (x - x_0, y - y_0) = 0$ (Expand)

$Ax - Ax_0 + By - By_0 = 0$ (Rearrange)

$Ax + By - Ax_0 - By_0 = 0$

Since the point $P_0(x_0, y_0)$ is a point whose coordinates are known, as is $\vec{n} = (A, B)$, we substitute C for the quantity $-Ax_0 - By_0$ to obtain $Ax + By + C = 0$ as the Cartesian equation of the line.

EXAMPLE 4 Connecting the Cartesian equation of a line to its normal

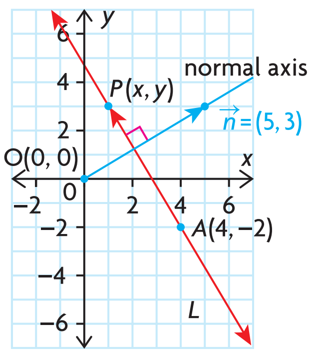
Determine the Cartesian equation of the line passing through $A(4, -2)$, which has $\vec{n} = (5, 3)$ as its normal.

Solution

The required line is sketched by first drawing the normal $\vec{n} = (5, 3)$ and then constructing a line L through $A(4, -2)$ perpendicular to this normal.

Cartesian Equation of a Line in R^2

In R^2, the Cartesian equation of a line (or scalar equation) is given by $Ax + By + C = 0$, where a normal to this line is $\vec{n} = (A, B)$. A normal to this line is a vector drawn from the origin perpendicular to the given line to the point $N(A, B)$.



Method 1:

Let $P(x, y)$ be any point on the required line L, other than A. Let $\overrightarrow{AP}$ be a vector parallel to L.

$\overrightarrow{AP} = (x - 4, y - (-2)) = (x - 4, y + 2)$.

Since $\vec{n}$ and $\overrightarrow{AP}$ are perpendicular, $\vec{n} \cdot \overrightarrow{AP} = 0$.

Therefore, $(5, 3) \cdot (x - 4, y + 2) = 0$ or $5(x - 4) + 3(y + 2) = 0$

Thus, $5x - 20 + 3y + 6 = 0$ or $5x + 3y - 14 = 0$

Method 2:
Since $\vec{n} = (5, 3)$, the Cartesian equation of the line is of the form $5x + 3y + C = 0$, with C to be determined. Since the point $A(4, -2)$ is a point on this line, it must satisfy the following equation:

$5(4) + 3(-2) + C = 0$

So, $C = -14$, and $5x + 3y - 14 = 0$

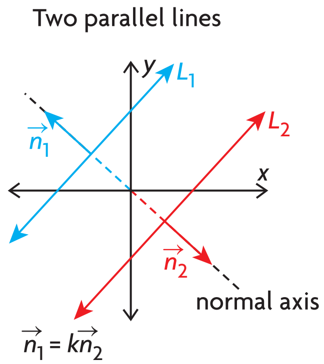
Using either method, the required Cartesian equation is $5x + 3y - 14 = 0$.

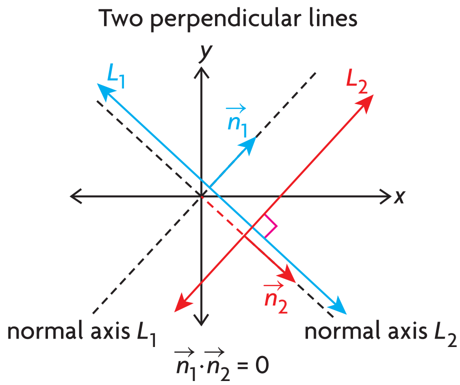
Since it has been established that the line with equation $Ax + By + C = 0$ has a normal vector of $\vec{n} = (A, B)$, this now provides an easy test to determine whether lines are parallel or perpendicular.

Parallel and Perpendicular Lines and their Normals

If the lines L_1 and L_2 have normals $\vec{n_1}$ and $\vec{n_2}$, respectively, we know the following:

1. The two lines are parallel if and only if their normals are scalar multiples. $\vec{n_1} = k\vec{n_2}, k \in \mathbf{R}, k \neq 0$ It follows that the lines direction vectors are also scalar multiples in this case.
2. The two lines are perpendicular if and only if their dot product is zero. $\vec{n_1} \cdot \vec{n_2} = 0$ It follows that dot product of the direction vectors is also zero in this case.





The next examples demonstrate these ideas.

EXAMPLE 5 Reasoning about parallel and perpendicular lines in R^2

a. Show that the lines $L_1: 3x - 4y - 6 = 0$ and $L_2: 6x - 8y + 12 = 0$ are parallel and non-coincident.
b. For what value of k are the lines $L_3: kx + 4y - 4 = 0$ and $L_4: 3x - 2y - 3 = 0$ perpendicular lines?

Solution

a. The lines are parallel because when the two normals, $\vec{n_1} = (3, -4)$ and $\vec{n_2} = (6, -8)$, are compared, the two vectors are scalar multiples $\vec{n_2} = (6, -8) = 2(3, -4) = 2\vec{n_1}$. The lines are non-coincident, since there is no value of t such that $6x - 8y + 12 = t(3x - 4y - 6)$. In simple terms, lines can only be coincident if their equations are scalar multiples of each other.
b. If the lines are perpendicular, then the normal vectors $\vec{n_3} = (k, 4)$ and $\vec{n_4} = (3, -2)$ have a dot product equal to zero—that is $(k, 4) \cdot (3, -2) = 0$ or $3k - 8 = 0$, $k = \frac{8}{3}$. This implies that the lines $3x - 2y - 3 = 0$ and $\frac{8}{3}x + 4y - 4 = 0$ are perpendicular.

The following investigation helps in understanding the relationship between normals and perpendicular lines.

EXAMPLE 6 Selecting a strategy to determine the angle between two lines in R^2

Determine the acute angle formed at the point of intersection created by the following pair of lines:
L_1: $(x, y) = (2, 2) + s(-1, 3), s \in \mathbf{R}$
L_2: $(x, y) = (5, 1) + t(3, 4), t \in \mathbf{R}$

Solution

The direction of each line is determined by their respective direction vectors, so the angle formed at the point of intersection is equivalent to the angle formed by the direction vectors when drawn tail to tail. For L_1 its direction vector is $\vec{a} = (-1, 3)$ and for L_2 its direction vectors is $\vec{b} = (3, 4)$. These lines are clearly not parallel as their direction vectors are not scalar multiples. They are also not perpendicular because the dot product of their direction vectors is a nonzero value. The angle between two vectors is determined by:

$$\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}\right)$$ (Substitute)

$$\theta = \cos^{-1}\left(\frac{(-1, 3) \cdot (3, 4)}{(\sqrt{(-1)^2 + (3)^2})(\sqrt{(3)^2 + (4)^2})}\right)$$ (Simplify)

$$\theta = \cos^{-1}\left(\frac{-3 + 12}{(\sqrt{10})(\sqrt{25})}\right)$$

$$\theta = \cos^{-1}\left(\frac{9}{5\sqrt{10}}\right) \qquad \text{(Evaluate)}$$

$$\theta \doteq 55.3°$$

The acute angle formed at the point of intersection of the given lines is about 55.3°.

INVESTIGATION

A. A family of lines has $kx - 2y - 4 = 0$ as its equation. On graph paper, sketch the three members of this family when $k = 1$, $k = -1$, and $k = 2$.

B. What point do the three lines you sketched in part A have in common?

C. A second family of lines has $4x - ty - 8 = 0$ as its equation. Sketch the three members of this family used in part A for $t = -2$, $t = 2$, and $t = -4$.

D. What points do the lines in part C have in common?

E. Select the three pairs of perpendicular lines from the two families. Verify that you are correct by calculating the dot products of their respective normals.

F. By selecting different values for k and t, determine another pair of lines that are perpendicular.

G. In general, if you are given a line in R^2, how many different lines is it possible to draw through a particular point perpendicular to the given line? Explain your answer.

IN SUMMARY

Key Idea

- The Cartesian (or scalar) equation of a line in R^2 is $Ax + By + C = 0$, where $\vec{n} = (A, B)$ is a normal to the line.

Need to Know

- Two planes whose normals are $\vec{n_1}$ and $\vec{n_2}$:
  - are parallel if and only if $\vec{n_1} = k\vec{n_2}$, where k is any nonzero real number.
  - are perpendicular if and only if $\vec{n_1} \cdot \vec{n_2} = 0$.
- The angle between two lines is defined by the angle between their direction vectors, $\vec{a}$ and $\vec{b}$, where $\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}\right)$.

Exercise 8.2

PART A

1. A line has $y = -\frac{5}{6}x + 9$ as its equation.
   a. Give a direction vector for a line that is parallel to this line.
   b. Give a direction vector for a line that is perpendicular to this line.
   c. Give the coordinates of a point on the given line.
   d. In both vector and parametric form, give the equations of the line parallel to the given line and passing through $A(7, 9)$.
   e. In both vector and parametric form, give the equations of the line perpendicular to the given line and passing through $B(-2, 1)$.
2. a. Sketch the line defined by the equation $\vec{r} = (2, 1) + s(-2, 5), s \in \mathbf{R}$.
   b. On the same axes, sketch the line $\vec{q} = (-2, 5) + t(2, 1),\ t \in \mathbf{R}$.
   c. Discuss the impact of switching the components of the direction vector with the coordinates of the point on the line in the vector equation of a line in R^2.
3. For each of the given lines, determine the vector and parametric equations.

   a. $y = \frac{7}{8}x - 6$ b. $y = \frac{3}{2}x + 5$ c. $y = -1$ d. $x = 4$

C 4. Explain how you can show that the lines with equations $x - 3y + 4 = 0$ and $6x - 18y + 24 = 0$ are coincident.

5. Two lines have equations $2x - 3y + 6 = 0$ and $4x - 6y + k = 0$.
   a. Explain, with the use of normal vectors, why these lines are parallel.
   b. For what value of k will these lines be coincident?

PART B

6. Determine the Cartesian equation for the line with a normal vector of $(4, 5)$, passing through the point $A(-1, 5)$.
7. A line passes through the points $A(-3, 5)$ and $B(-2, 4)$. Determine the Cartesian equation of this line.
8. A line is perpendicular to the line $2x - 4y + 7 = 0$ and that passes through the point $P(7, 2)$. Determine the equation of this line in Cartesian form.

K 9. A line has parametric equations $x = 3 - t$, $y = -2 - 4t$, $t \in \mathbf{R}$.
   a. Sketch this line.
   b. Determine a Cartesian equation for this line.

A

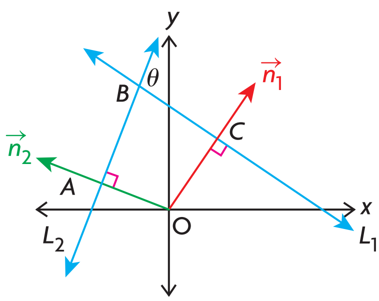
10. For each pair of lines, determine the size of the acute angle, to the nearest degree, that is created by the intersection of the lines.
   a. $(x, y) = (3, 6) + t(2, -5)$ and $(x, y) = (-3, 4) - t(-4, -1)$
   b. $x = 2 - 5t, y = 3 + 4t$ and $x = -1 + t, y = 2 - 6t$
   c. $y = 0.5x + 6$ and $y = -0.75x - 1$
   d. $(x, y) = (-1, -1) + t(2, 4)$ and $2x - 4y = 8$
   e. $x = 2t, y = 1 - 5t$ and $(x, y) = (4, 0) + t(-4, 1)$
   f. $x = 3$ and $5x - 10y + 20 = 0$
11. The angle between any pair of lines in Cartesian form is also the angle between their normal vectors. For the lines $x - 3y + 6 = 0$ and $x + 2y - 7 = 0$, do the following:
   a. Sketch the lines.
   b. Determine the acute and obtuse angles between these two lines.

T

12. The line segment joining $A(-3, 2)$ and $B(8, 4)$ is the hypotenuse of a right triangle. The third vertex, C, lies on the line with the vector equation $(x, y) = (-6, 6) + t(3, -4)$.
   a. Determine the coordinates of C.
   b. Illustrate with a diagram.
   c. Use vectors to show that $\angle ACB = 90°$.

PART C

13. Lines L_1 and L_2 have $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ as their respective normals. Prove that the angle between the two lines is the same as the angle between the two normals.



(*Hint:* Show that $\angle AOC = \theta$ by using the fact that the sum of the angles in a quadrilateral is 360°.)

14. The lines $x - y + 1 = 0$ and $x + ky - 3 = 0$ have an angle of 60° between them. For what values of k is this true?

Section 8.3—Vector, Parametric, and Symmetric Equations of a Line in R^3

In Section 8.1, we discussed vector and parametric equations of a line in R^2. In this section, we will continue our discussion, but, instead of R^2, we will examine lines in R^3.

The derivation and form of the vector equation for a line in R^3 is the same as in R^2. If we wish to find a vector equation for a line in R^3, it is necessary that either two points or a point and a direction vector be given. If we are given two points and wish to determine a direction vector for the corresponding line, the coordinates of this vector must first be calculated.

EXAMPLE 1

Determining a direction vector of a line in R^3

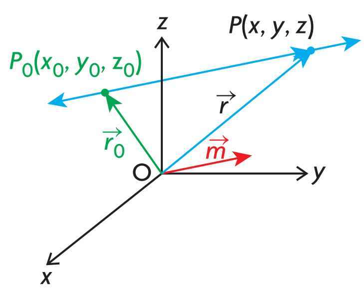
A line passes through the points $A(-1, 3, 5)$ and $B(-3, 3, -4)$. Calculate possible direction vectors for this line.

Solution

A possible direction vector is $\vec{m} = (-1 - (-3), 3 - 3, 5 - (-4)) = (2, 0, 9)$. In general, any vector of the form $t(2, 0, 9)$, $t \in \mathbf{R}$, $t \neq 0$, can be used as a direction vector for this line. As before, the best choice for a direction vector is one in which the direction numbers are integers, with common divisors removed. This implies that either $(2, 0, 9)$ or $(-2, 0, -9)$ are the best choices for a direction vector for this line. Generally speaking, if a line has $\vec{m} = (a, b, c)$ as its direction vector, then any scalar multiple of this vector of the form $t(a, b, c)$, $t \in \mathbf{R}$, $t \neq 0$, can be used as a direction vector.

Vector and Parametric Equations of Lines in R^3

Consider the following diagram.



When determining the vector equation of the line passing through P_0 and P, we know that the point $P_0(x_0, y_0, z_0)$ is a given point on the line, and $\vec{m} = (a, b, c)$

is its direction vector. If $P(x, y, z)$ represents a general point on the line, then $\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$ is a direction vector for this line. This allows us to form the vector equation of the line.

In $\triangle OP_0P$, $\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P}$.

Since $\overrightarrow{OP_0} = \vec{r_0}$ and $\overrightarrow{P_0P} = t\vec{m}$, the vector equation of the line is $\vec{r} = \vec{r_0} + t\vec{m}$, $t \in \mathbf{R}$. In component form, this can be written as $(x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$, $t \in \mathbf{R}$. The parametric equations of the line are found by equating the respective x, y, and z components, giving $x = x_0 + ta$, $y = y_0 + tb$, $z = z_0 + tc$, $t \in \mathbf{R}$.

Vector and Parametric Equations of a Line in R^3

Vector Equation: $\vec{r} = \vec{r_0} + t\vec{m}$, $t \in \mathbf{R}$
Parametric Equations: $x = x_0 + ta$, $y = y_0 + tb$, $z = z_0 + tc$, $t \in \mathbf{R}$
where $\vec{r_0} = (x_0, y_0, z_0)$, the vector from the origin to a point on the line and
$\vec{m} = (a, b, c)$ is a direction vector of the line

EXAMPLE 2 Representing the equation of a line in R^3 in vector and parametric form

Determine the vector and parametric equations of the line passing through $P(-2, 3, 5)$ and $Q(-2, 4, -1)$.

Solution

A direction vector is $\vec{m} = (-2 - (-2), 3 - 4, 5 - (-1)) = (0, -1, 6)$. A vector equation is $\vec{r} = (-2, 3, 5) + s(0, -1, 6)$, $s \in \mathbf{R}$, and its parametric equations are $x = -2$, $y = 3 - s$, $z = 5 + 6s$, $s \in \mathbf{R}$. It would also have been correct to choose any multiple of $(0, -1, 6)$ as a direction vector and any point on the line. For example, the vector equation $\vec{r} = (-2, 4, -1) + t(0, -2, 12)$, $t \in \mathbf{R}$, would also have been correct.

Since a vector equation of a line can be written in many ways, it is useful to be able to tell if different forms are actually equivalent. In the following example, an algebraic approach to this problem is considered.

EXAMPLE 3 Reasoning to establish the equivalence of two lines

a. Show that the following are vector equations for the same line:
$L_1: \vec{r} = (-1, 0, 4) + s(-1, 2, 5)$, $s \in \mathbf{R}$, and
$L_2: \vec{r} = (4, -10, -21) + m(-2, 4, 10)$, $m \in \mathbf{R}$

b. Show that the following are vector equations for different lines:
$L_3: \vec{r} = (1, 6, 1) + l(-1, 1, 2)$, $l \in \mathbf{R}$, and
$L_4: \vec{r} = (-3, 10, 12) + k\left(\frac{1}{2}, -\frac{1}{2}, -1\right)$, $k \in \mathbf{R}$

Solution

a. Since the direction vectors are parallel—that is, $2(-1, 2, 5) = (-2, 4, 10)$—this means that the two lines are parallel. To show that the equations are equivalent, we must show that a point on one of the lines is also on the other line. This is based on the logic that, if the lines are parallel and they share a common point, then the two equations must represent the same line. To check whether $(4, -10, -21)$ is also on L_1 substitute into its vector equation.
$(4, -10, -21) = (-1, 0, 4) + s(-1, 2, 5)$

Using the x component, we find $4 = -1 - s$, or $s = -5$. Substituting $s = -5$ into the above equation, $(-1, 0, 4) + -5(-1, 2, 5) = (4, -10, -21)$. This verifies that the point $(4, -10, -21)$ is also on L_1.

Since the lines have the same direction, and a point on one line is also on the second line, the two given equations represent the same line.

b. The first check is to compare the direction vectors of the two lines. Since $-2\left(\frac{1}{2}, -\frac{1}{2}, -1\right) = (-1, 1, 2)$, the lines must be parallel. As in part a, $(-3, 10, 12)$ must be a point on L_3 for the equations to be equivalent. Therefore, $(-3, 10, 12) = (1, 6, 1) + l(-1, 1, 2)$ must give a consistent value of l for each component. If we solve, this gives an inconsistent result since $l = 4$ for the x and y components, and $12 = 1 + 2l$, or $l = \frac{11}{2}$ for the z component. This verifies that the two equations are not equations for the same line.

Symmetric Equations of Lines in R^3

We introduce a new form for a line in R^3, called its **symmetric equation**. The symmetric equation of a line is derived from using its parametric equations and solving for the parameter in each component, as shown below

$$x = x_0 + ta \leftrightarrow t = \frac{x - x_0}{a}, a \neq 0$$

$$y = y_0 + tb \leftrightarrow t = \frac{y - y_0}{b}, b \neq 0$$

$$z = z_0 + tc \leftrightarrow t = \frac{z - z_0}{c}, c \neq 0$$

Combining these statements gives $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}, a, b, c \neq 0$.

These equations are called the symmetric equations of a line in R^3.

Symmetric Equations of a Line in R^3

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}, \quad a \neq 0, b \neq 0, c \neq 0$$

where (x_0, y_0, z_0) is the vector from the origin to a point on the line, and (a, b, c) is a direction vector of the line.

EXAMPLE 4 Representing the equation of a line in R^3 in symmetric form

a. Write the symmetric equations of the line passing through the points $A(-1, 5, 7)$ and $B(3, -4, 8)$.
b. Write the symmetric equations of the line passing through the points $P(-2, 3, 1)$ and $Q(4, 3, -5)$.
c. Write the symmetric equations of the line passing through the points $X(-1, 2, 5)$ and $Y(-1, 3, 9)$.

Solution

a. A direction vector for this line is $\vec{m} = (-1 - 3, 5 - (-4), 7 - 8) = (-4, 9, -1)$. Using the point $A(-1, 5, 7)$, the parametric equations of the line are $x = -1 - 4t$, $y = 5 + 9t$, and $z = 7 - t$, $t \in \mathbf{R}$. Solving each equation for t gives the required symmetric equations, $\frac{x + 1}{-4} = \frac{y - 5}{9} = \frac{z - 7}{-1}$. It is usually not necessary to find the parametric equations before finding the symmetric equations. The symmetric equations of a line can be written by inspection if the direction vector and a point on the line are known. Using point B and the direction vector found above, the symmetric equations of this line by inspection are $\frac{x - 3}{-4} = \frac{y + 4}{9} = \frac{z - 8}{-1}$.

b. A direction vector for the line is $\vec{m} = (-2 - 4, 3 - 3, 1 - (-5)) = (-6, 0, 6)$. The vector $(1, 0, -1)$ will be used as the direction vector. In a situation like this, where the y direction number is 0, using point P the equation is written as $\frac{x + 2}{1} = \frac{z - 1}{-1}$, $y = 3$.

c. A direction vector for the line is $\vec{m} = (-1 - (-1), 2 - 3, 5 - 9) = (0, -1, -4)$. Using point X, possible symmetric equations are $\frac{y - 2}{-1} = \frac{z - 5}{-4}$, $x = -1$.

IN SUMMARY

Key Idea

- In R^3, if $\vec{r_0} = (x_0, y_0, z_0)$ is determined by a point on a line and $\vec{m} = (a, b, c)$ is a direction vector of the same line, then
  - the vector equation of the line is $\vec{r} = \vec{r_0} + t\vec{m}$, $t \in \mathbf{R}$ or equivalently $(x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$
  - the parametric form of the equation of the line is $x = x_0 + ta$, $y = y_0 + tb$, and $z = z_0 + tc$, $t \in \mathbf{R}$
  - the symmetric form of the equation of the line is $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} (= t)$, $a \neq 0$, $b \neq 0$, $c \neq 0$

Need to Know

- Knowing one of these forms of the equation of a line enables you to find the other two, since all three forms depend on the same information about the line.

Exercise 8.3

PART A

1. State the coordinates of a point on each of the given lines.
   a. $\vec{r} = (-3, 1, 8) + s(-1, 1, 9), s \in \mathbf{R}$
   b. $\dfrac{x-1}{2} = \dfrac{y+1}{1} = \dfrac{z-3}{-1}$
   c. $x = -2 + 3t, y = 1 + (-4t), z = 3 - t, t \in \mathbf{R}$
   d. $\dfrac{x+2}{-1} = \dfrac{z-1}{2}, y = -3$
   e. $x = 3, y = -2, z = -1 + 2k, k \in \mathbf{R}$
   f. $\dfrac{x - \frac{1}{3}}{\frac{1}{2}} = \dfrac{y + \frac{3}{4}}{\frac{-1}{4}} = \dfrac{z - \frac{2}{5}}{\frac{1}{2}}$
2. State a direction vector for each line in question 1, making certain that the components for each are integers.

PART B

3. A line passes through the points $A(-1, 2, 4)$ and $B(3, -3, 5)$.
   a. Write two vector equations for this line.
   b. Write the two sets of parametric equations associated with the vector equations you wrote in part a.
4. A line passes through the points $A(-1, 5, -4)$ and $B(2, 5, -4)$.
   a. Write a vector equation for the line containing these points.
   b. Write parametric equations corresponding to the vector equation you wrote in part a.
   c. Explain why there are no symmetric equations for this line.

K

5. State where possible vector, parametric, and symmetric equations for each of the following lines.
   a. the line passing through the point $P(-1, 2, 1)$ with direction vector $(3, -2, 1)$
   b. the line passing through the points $A(-1, 1, 0)$ and $B(-1, 2, 1)$
   c. the line passing through the point $B(-2, 3, 0)$ and parallel to the line passing through the points $M(-2, -2, 1)$ and $N(-2, 4, 7)$
   d. the line passing through the points $D(-1, 0, 0)$ and $E(-1, 1, 0)$
   e. the line passing through the points $X(-4, 3, 0)$ and $O(0, 0, 0)$
   f. the line passing through the point $Q(1, 2, 4)$ and parallel to the z-axis

6. a. Determine parametric equations for each of the following lines:
$\frac{x+6}{1} = \frac{y-10}{-1} = \frac{z-7}{1}$ and $\frac{x+7}{1} = \frac{y-11}{-1}, z = 5$

b. Determine the angle between the two lines.

C 7. Show that the following two sets of symmetric equations represent the same straight line: $\frac{x+7}{8} = \frac{y+1}{2} = \frac{z-5}{-2}$ and $\frac{x-1}{-4} = \frac{y-1}{-1} = \frac{z-3}{1}$

8. a. Show that the points $A(6, -2, 15)$ and $B(-15, 5, -27)$ lie on the line that passes through (0, 0, 3) and has the direction vector $(-3, 1, -6)$.

b. Use parametric equations with suitable restrictions on the parameter to describe the line segment from A to B.

A 9. Determine the value of k for which the direction vectors of the lines $\frac{x-1}{k} = \frac{y-2}{2} = \frac{z+1}{k-1}$ and $\frac{x+3}{-2} = \frac{z}{1}, y = -1$ are perpendicular.

10. Determine the coordinates of three different points on each line.

a. $(x, y, z) = (4, -2, 5) + t(-4, -6, 8)$

b. $x = -4 + 5s, y = 2 - s, z = 9 - 6s$

c. $\frac{x+1}{3} = \frac{y-2}{-1} = \frac{z}{4}$

d. $x = -4, \frac{y-2}{3} = \frac{z-3}{5}$

11. Express each equation in question 10 in two other equivalent forms. (i.e. vector, parametric or symmetric form)

PART C

12. Determine the parametric equations of the line whose direction vector is perpendicular to the direction vectors of the two lines $\frac{x}{-4} = \frac{y+10}{-7} = \frac{z+2}{3}$ and $\frac{x-5}{3} = \frac{y-5}{2} = \frac{z+5}{4}$ and passes through the point $(2, -5, 0)$.

T 13. A line with parametric equations $x = 10 + 2s, y = 5 + s, z = 2, s \in \mathbf{R}$, intersects a sphere with the equation $x^2 + y^2 + z^2 = 9$ at the points A and B. Determine the coordinates of these points.

14. You are given the two lines $L_1: x = 4 + 2t, y = 4 + t, z = -3 - t, t \in \mathbf{R}$, and $L_2: x = -2 + 3s, y = -7 + 2s, z = 2 - 3s, s \in \mathbf{R}$. If the point P_1 lies on L_1 and the point P_2 lies on L_2, determine the coordinates of these two points if $\overrightarrow{P_1P_2}$ is perpendicular to each of the two lines. (*Hint*: The vector $\overrightarrow{P_1P_2}$ is perpendicular to the direction vector of each of the two lines.)

15. Determine the angle formed by the intersection of the lines defined by $\frac{x-1}{2} = \frac{y+3}{1}, z = -3$ and $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z}{1}$.

Mid-Chapter Review

1. Name three points on each of the following lines:
   a. $x = 2t - 5,\ y = 3t + 1,\ t \in \mathbf{R}$
   b. $\vec{r} = (2, 3) + s(3, -2),\ s \in \mathbf{R}$
   c. $3x + 5y - 8 = 0$
   d. $\dfrac{x-1}{3} = \dfrac{y+2}{2} = \dfrac{z-5}{1}$
2. Find x- and y-intercepts for each of the following lines:
   a. $\vec{r} = (3, 1) + t(-3, 5),\ t \in \mathbf{R}$
   b. $x = -6 + 2s$ and $y = 3 - 2s,\ s \in \mathbf{R}$
3. Two lines $L_1: \vec{r} = (5, 3) + p(-4, 7), p \in \mathbf{R}$, and $L_2: \vec{r} = (5, 3) + q(2, 1), q \in \boldsymbol{R}$, intersect at the point with coordinates $(5, 3)$. What is the angle between L_1 and L_2?
4. Determine the angle that the line with equation $\vec{r} = t(4, -5), t \in \mathbf{R}$, makes with the x-axis and y-axis.
5. Determine a Cartesian equation for the line that passes through the point $(4, -3)$ and is perpendicular to the line $\vec{r} = (2, -3) + t(5, -7),\ t \in \mathbf{R}$.
6. Determine an equation in symmetric form of a line parallel to $\frac{x-3}{3} = \frac{y-5}{-4} = \frac{z+7}{4}$ and passing through $(0, 0, 2)$.
7. Determine parametric equations of the line passing through $(1, 2, 5)$ and parallel to the line passing through $K(2, 4, 5)$ and $L(3, -5, 6)$.
8. Determine direction angles (the angles the direction vector makes with the x-axis, y-axis, and z-axis) for the line with parametric equations $x = 5 + 2t$, $y = 12 - 8t,\ z = 5 + 7t,\ t \in \mathbf{R}$.
9. Determine an equation in symmetric form for the line passing through $P(3, -4, 6)$ and having direction angles 60°, 90°, and 30°.
10. Write an equation in parametric form for each of the three coordinate axes in R^3.
11. The two lines with equations $\vec{r} = (1, 2, -4) + t(k + 1, 3k + 1, k - 3)$, $t \in \mathbf{R}$, and $x = 2 - 3s,\ y = 1 - 10s,\ z = 3 - 5s,\ s \in \mathbf{R}$, are given.
    a. Determine a value for k if these lines are parallel.
    b. Determine a value for k if these lines are perpendicular.
12. Determine the perimeter and area of the triangle whose vertices are the origin and the x- and y-intercepts of the line $\frac{x-6}{3} = \frac{y+8}{-2}$.

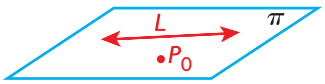
13. The Cartesian equation of a line is given by $3x + 4y - 24 = 0$.
   a. Determine a vector equation for this line.
   b. Determine the parametric equations of this line.
   c. Determine the acute angle that this line makes with the x-axis.
   d. Determine a vector equation of the line that is perpendicular to the given line and passes through the origin.
14. Determine the scalar, vector, and parametric equations of the line that passes through points $A(-4, 6)$ and $B(8, 4)$.
15. Determine a unit vector normal to the line defined by the parametric equations $x = 1 + 2t$ and $y = -5 - 4t$.
16. Determine the parametric equations of each line.
   a. the line that passes through $(-5, 10)$ and has a slope of $-\frac{2}{3}$
   b. the line that passes through $(1, -1)$ and is perpendicular to the line $(x, y) = (4, -6) + t(2, -2)$
   c. the line that passes through $(0, 7)$ and $(0, 10)$
17. Given the line $(x, y, z) = (12, -8, -4) + t(-3, 4, 2)$,
   a. determine the intersections with the coordinate planes, if any
   b. determine the intercepts with the coordinate axes, if any
   c. graph the line in an x-, y-, z-coordinate system.
18. For each of the following, determine vector, parametric, and, if possible, symmetric equations of the line that passes through P_0 and has direction vector $\vec{d}$.
   a. $P_0 = (1, -2, 8), \vec{d} = (-5, -2, 1)$
   b. $P_0 = (3, 6, 9), \vec{d} = (2, 4, 6)$
   c. $P_0 = (0, 0, 6), \vec{d} = (-1, 5, 1)$
   d. $P_0 = (2, 0, 0), \vec{d} = (0, 0, -2)$
19. Determine a vector equation of the line that passes through the origin and is parallel to the line through the points $(-4, 5, 6)$ and $(6, -5, 4)$.
20. Determine the parametric equations of the line through $(0, -8, 1)$ and which passes through the midpoint of the segment joining $(2, 6, 10)$ and $(-4, 4, -8)$.
21. The symmetric equations of two lines are given. Show that these lines are parallel.
   $L_1: \frac{x-2}{1} = \frac{y+3}{3} = \frac{z-4}{-5}$ and $L_2: \frac{x+1}{-3} = \frac{y-2}{-9} = \frac{z+1}{15}$
22. Does the point $D(7, -1, 8)$ lie on the line with symmetric equations $\frac{x-4}{3} = \frac{y+2}{1} = \frac{z-6}{2}$? Explain.

Section 8.4—Vector and Parametric Equations of a Plane

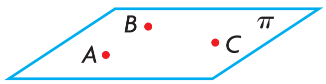
In the previous section, the vector, parametric, and symmetric equations of lines in R^3 were developed. In this section, we will develop vector and parametric equations of planes in R^3. Planes are flat surfaces that extend infinitely far in all directions. To represent planes, parallelograms are used to represent a small part of the plane and are designated with the Greek letter π. This is the usual method for representing planes. In real life, part of a plane might be represented by the top of a desk, by a wall, or by the ice surface of a hockey rink.

Before developing the equation of a plane, we start by showing that planes can be determined in essentially four ways. That is, a plane can be determined if we are given any of the following:

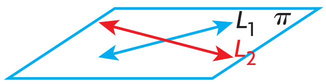
a. a line and a point not on the line
b. three noncollinear points (three points not on a line)
c. two intersecting lines
d. two parallel and non-coincident lines

a. 

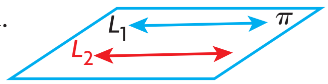
A line and a point not on the line

c. 

Three noncollinear points

b. 

Two intersecting lines

d. 

Two parallel and non-coincident lines

If we are given any one of these conditions, we are guaranteed that we can form a plane, and the plane formed will be unique. For example, in condition a, we are given line L and point P_0 not on this line; there is just one plane that can be formed using this point and this line.

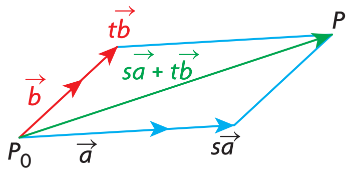
Linear Combinations and their Relationship to Planes

The ideas of linear combination and spanning sets are the two concepts needed to understand how to obtain the vector and parametric equations of planes. For example, suppose that vectors $\vec{a} = (1, 2, -1)$ and $\vec{b} = (0, 2, 1)$ and a linear combination of these vectors—that is, $\overrightarrow{P_0P} = s(1, 2, -1) + t(0, 2, 1)$, $s, t \in \mathbf{R}$—are formed. As different values are chosen for s and t, a new vector is formed. Different values for these parameters have been selected, and the corresponding calculations have been done in the table shown, with vector $\overrightarrow{P_0P}$ also calculated.

s	t	$s(1, 2, -1) + t(0, 2, 1), s, t \in \mathbf{R}$	$\overrightarrow{P_0P}$
-2	1	$-2(1, 2, -1) + 1(0, 2, 1)$	$(-2, -4, 2) + (0, 2, 1) = (-2, -2, 3)$
4	-3	$4(1, 2, -1) - 3(0, 2, 1)$	$(4, 8, -4) + (0, -6, -3) = (4, 2, -7)$
10	-7	$10(1, 2, -1) - 7(0, 2, 1)$	$(10, 20, -10) + (0, -14, -7) = (10, 6, -17)$
-2	-1	$-2(1, 2, -1) - 1(0, 2, 1)$	$(-2, -4, 2) + (0, -2, -1) = (-2, -6, 1)$

$\overrightarrow{P_0P}$ is on the plane determined by the vectors $\vec{a}$ and $\vec{b}$, as is its head. The parameters s and t are chosen from the set of real numbers, meaning that there are an infinite number of vectors formed by selecting all possible combinations of s and t. Each one of these vectors is different, and every point on the plane can be obtained by choosing appropriate parameters. This observation is used in developing the vector and parametric equations of a plane.

In the following diagram, two noncollinear vectors, $\vec{a}$ and $\vec{b}$, are given. The linear combinations of these vectors, $s\vec{a} + t\vec{b}$, form a diagonal of the parallelogram determined by $s\vec{a}$ and $t\vec{b}$.



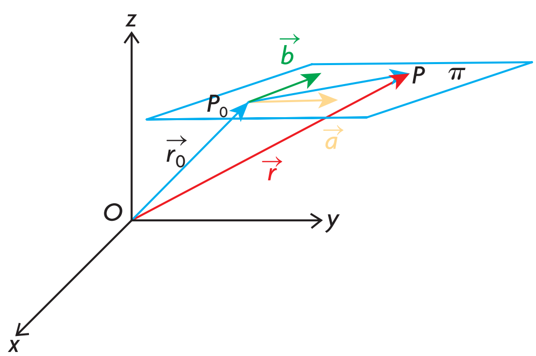
EXAMPLE 1

Developing the vector and parametric equations of a plane

Two noncollinear vectors, $\vec{a}$ and $\vec{b}$, are given, and the point P_0. Determine the vector and parametric equations of the plane π formed by taking all linear combinations of these vectors.

Solution

The vectors $\vec{a}$ and $\vec{b}$ can be translated anywhere in R^3. When drawn tail to tail they form and infinite number of parallel planes, but only one such plane contains the point P_0. We start by drawing a parallelogram to represent part of this plane π. This plane contains P_0, $\vec{a}$, and $\vec{b}$.



From the diagram, it can be seen that vector $\vec{r_0}$ represents the vector for a particular point P_0 on the plane, and $\vec{r}$ represents the vector for any point P on the plane. $\overrightarrow{P_0P}$ is on the plane and is a linear combination of $\vec{a}$ and $\vec{b}$—that is, $\overrightarrow{P_0P} = s\vec{a} + t\vec{b}$, $s, t \in \mathbf{R}$. Using the triangle law of addition in $\triangle OP_0P$, $\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P}$. Thus, $\vec{r} = \vec{r_0} + s\vec{a} + t\vec{b}$, $s, t \in \mathbf{R}$.

The vector equation for the plane is $\vec{r} = \vec{r_0} + s\vec{a} + t\vec{b}$, $s, t \in \mathbf{R}$, and can be used to generate parametric equations for the plane.

If $\vec{r} = (x, y, z)$, $\vec{r_0} = (x_0, y_0, z_0)$, $\vec{a} = (a_1, a_2, a_3)$, and $\vec{b} = (b_1, b_2, b_3)$, these expressions can be substituted into the vector equation to obtain $(x, y, z) = (x_0, y_0, z_0) + s(a_1, a_2, a_3) + t(b_1, b_2, b_3)$, $s, t \in \mathbf{R}$.

Expanding, $(x, y, z) = (x_0, y_0, z_0) + (sa_1, sa_2, sa_3) + (tb_1, tb_2, tb_3)$

Simplifying, $(x, y, z) = (x_0 + sa_1 + tb_1, y_0 + sa_2 + tb_2, z_0 + sa_3 + tb_3)$

Equating the respective components gives the parametric equations $x = x_0 + sa_1 + tb_1$, $y = y_0 + sa_2 + tb_2$, $z = z_0 + sa_3 + tb_3$, $s, t \in \mathbf{R}$.

Vector and Parametric Equations of a Plane in R^3

In R^3, a plane is determined by a vector $\vec{r_0} = (x_0, y_0, z_0)$ where (x_0, y_0, z_0) is a point on the plane, and two noncollinear vectors vector $\vec{a} = (a_1, a_2, a_3)$ and vector $\vec{b} = (b_1, b_2, b_3)$.

Vector Equation of a Plane: $\vec{r} = \vec{r_0} + s\vec{a} + t\vec{b}$, $s, t \in \mathbf{R}$ or equivalently $(x, y, z) = (x_0, y_0, z_0) + s(a_1, a_2, a_3) + t(b_1, b_2, b_3)$.

Parametric Equations of a Plane: $x = x_0 + sa_1 + tb_1$,
$y = y_0 + sa_2 + tb_2$,
$z = z_0 + sa_3 + tb_3$, $s, t \in \mathbf{R}$

The vectors $\vec{a}$ and $\vec{b}$ are the direction vectors for the plane. When determining the equation of a plane, it is necessary to have two direction vectors. As will be seen in the examples, any pair of noncollinear vectors are coplanar, so they can be used as direction vectors for a plane. The vector equation of a plane always requires two parameters, s and t, each of which are real numbers. Because two parameters are required to define a plane, the plane is described as two-dimensional. Earlier, we saw that the vector equation of a line, $\vec{r} = \vec{r_0} + t\vec{m}$, $t \in \mathbf{R}$, required just one parameter. For this reason, a line is described as one-dimensional. A second observation about the vector equation of the plane is that there is a one-to-one correspondence between the chosen parametric values (s, t) and points on the plane. Each time values for s and t are selected, this generates a different point on the plane, and because s and t can be any real number, this will generate all points on the plane.

After deriving vector and parametric equations of lines, a symmetric form was also developed. Although it is possible to derive vector and parametric equations of planes, it is not possible to derive a corresponding symmetric equation of a plane.

The next example shows how to derive an equation of a plane passing through three points.

EXAMPLE 2 **Selecting a strategy to represent the vector and parametric equations of a plane**

a. Determine a vector equation and the corresponding parametric equations for the plane that contains the points $A(-1, 3, 8)$, $B(-1, 1, 0)$, and $C(4, 1, 1)$.
b. Do either of the points $P(14, 1, 3)$ or $Q(14, 1, 5)$ lie on this plane?

Solution

a. In determining the required vector equation, it is necessary to have two direction vectors for the plane. The following shows the calculations for each of the direction vectors.

Direction Vector 1:
When calculating the first direction vector, any two points can be used and a position vector determined. If the points $A(-1, 3, 8)$ and $B(-1, 1, 0)$ are used, then $\overrightarrow{AB} = (-1 - (-1), 1 - 3, 0 - 8) = (0, -2, -8)$.

Since, $(0, -2, -8) = -2(0, 1, 4)$, a possible first direction vector is $\vec{a} = (0, 1, 4)$.

Direction Vector 2:
When finding the second direction vector, any two points (other than A and B) can be chosen. If B and C are used, then
$\overrightarrow{BC} = (4 - (-1), 1 - 1, 1 - 0) = (5, 0, 1)$.

A second direction vector is $\vec{b} = (5, 0, 1)$.

To determine the equation of the plane, any of the points A, B, or C can be used. An equation for the plane is $\vec{r} = (-1, 3, 8) + s(0, 1, 4) + t(5, 0, 1)$, $s, t \in \mathbf{R}$.

Writing the vector equation in component form will give the parametric equations. Thus, $(x, y, z) = (-1, 3, 8) + (0, s, 4s) + (5t, 0, t)$.

The parametric equations are $x = -1 + 5t$, $y = 3 + s$, and $z = 8 + 4s + t$, $s, t \in \mathbf{R}$.

b. If the point $P(14, 1, 3)$ lies on the plane, there must be parameters that correspond to this point. To find these parameters, $x = 14$ and $y = 1$ are substituted into the corresponding parametric equations.

Thus, $14 = -1 + 5t$ and $1 = 3 + s$.

Solving for s and t, we find that $14 + 1 = 5t$, or $t = 3$ and $1 - 3 = s$, or $s = -2$. Using these values, consistency will be checked with the z component. If $s = -2$ and $t = 3$ are substituted into $z = 8 + 4s + t$, then $z = 8 + 4(-2) + 3 = 3$. Since the z value for the point is also 3, this tells us that the point with coordinates $P(14, 1, 3)$ is on the given plane.

From this, it can be seen that the parametric values used for the x and y components, $s = -2$ and $t = 3$, will not produce consistent values for $z = 5$. So, the point $Q(14, 1, 5)$ is not on the plane.

In the following example, we will show how to use vector and parametric equations to find the point of intersection of planes with the coordinate axes.

EXAMPLE 3

Solving a problem involving a plane

A plane π has $\vec{r} = (6, -2, -3) + s(1, 3, 0) + t(2, 2, -1)$, $s, t \in \mathbf{R}$, as its equation. Determine the point of intersection between π and the z-axis.

Solution

We start by writing this equation in parametric form. The parametric equations of the plane are $x = 6 + s + 2t$, $y = -2 + 3s + 2t$, and $z = -3 - t$.

The plane intersects the z-axis at a point with coordinates of the form $P(0, 0, c)$—that is, where $x = y = 0$. Substituting these values into the parametric equations for x and y gives $0 = 6 + s + 2t$ and $0 = -2 + 3s + 2t$. Simplifying gives the following system of equations:

① $s + 2t = -6$

② $3s + 2t = 2$

Subtracting equation ① from equation ② gives $2s = 8$, so $s = 4$.

The value of t is found by substituting into either of the two equations. Using equation ①, $4 + 2t = -6$, or $t = -5$.

Solving for z using the equation of the third component, we find that $z = -3 - (-5) = 2$.

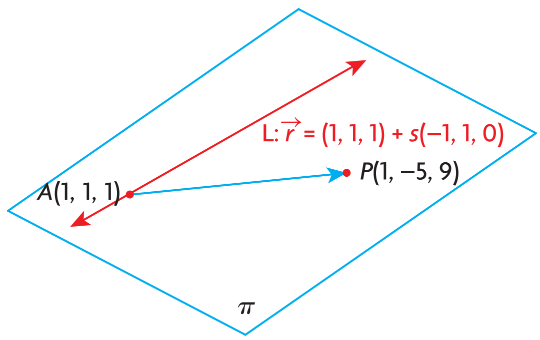
Thus, the plane cuts the z-axis at the point $P(0, 0, 2)$.

EXAMPLE 4 Representing the equations of a plane from a point and a line

Determine the vector and parametric equations of the plane containing the point $P(1, -5, 9)$ and the line $L: \vec{r} = (1, 1, 1) + s(-1, 1, 0), s \in \mathbf{R}$.

Solution

In the following diagram, a representation of the line L and the point P are given.



To find the equation of the plane, it is necessary to find two direction vectors and a point on the plane. The line $L: \vec{r} = (1, 1, 1) + s(-1, 1, 0), s \in \mathbf{R}$, gives a point and one direction vector, so all that is required is a second direction vector. Using $A(1, 1, 1)$ and $P(1, -5, 9)$, $\overrightarrow{AP} = (1 - 1, -5 - 1, 9 - 1) = (0, -6, 8) = -2(0, 3, -4)$. The equation of the plane is $\vec{r} = (1, 1, 1) + s(-1, 1, 0) + t(0, 3, -4),\ s, t \in \mathbf{R}$.

The corresponding parametric equations are $x = 1 - s, y = 1 + s + 3t$, and $z = 1 - 4t, s, t \in \mathbf{R}$.

IN SUMMARY

Key Idea

- In R^3, if $\vec{r_0} = (x_0, y_0, z_0)$ is determined by a point on a plane and $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ are direction vectors, then
  - the vector equation of the plane is $\vec{r} = \vec{r_0} + s\vec{a} + t\vec{b}, s, t \in \mathbf{R}$ or equivalently $(x, y, z) = (x_0, y_0, z_0) + s(a_1, a_2\ a_3) + t(b_1, b_2, b_3)$
  - the parametric form of the equation of the plane is $x = x_0 + sa_1 + tb_1$, $y = y_0 + sa_2 + tb_2, z = z_0 + sa_3 + tb_3, s, t \in \mathbf{R}$
  - there are no symmetric equations of the plane

Need to Know

- Replacing the parameters in the vector and parametric equations of a plane with numbers generates points on the plane. Because there are an infinite number of real numbers that can be used for s and t, there are an infinite number of points that lie on a plane.

Exercise 8.4

PART A

1. State which of the following equations define lines and which define planes. Explain how you made your decision.
   a. $\vec{r} = (1, 2, 3) + s(1, 1, 0) + t(3, 4, -6), s, t \in \mathbf{R}$
   b. $\vec{r} = (-2, 3, 0) + m(3, 4, 7),\ m \in \mathbf{R}$
   c. $x = -3 - t, y = 5, z = 4 + t, t \in \mathbf{R}$
   d. $\vec{r} = m(4, -1, 2) + t(4, -1, 5), m, t \in \mathbf{R}$
2. A plane has vector equation $\vec{r} = (2, 1, 3) + s\left(\frac{1}{3}, -2, \frac{3}{4}\right) + t(6, -12, 30)$, $s, t \in \mathbf{R}$.
   a. Express the first direction vector with only integers.
   b. Reduce the second direction vector.
   c. Write a new equation for the plane using the calculations from parts a. and b.
3. A plane has $x = 2m$, $y = -3m + 5n$, $z = -1 - 3m - 2n$, $m, n \in \mathbf{R}$, as its parametric equations.
   a. By inspection, identify the coordinates of a point that is on this plane.
   b. What are the direction vectors for this plane?
   c. What point corresponds to the parameter values of $m = -1$ and $n = -4$?
   d. What are the parametric values corresponding to the point $A(0, 15, -7)$?
   e. Using your answer for part d., explain why the point $B(0, 15, -8)$ cannot be on this plane.
4. A plane passes through the points $P(-2, 3, 1)$, $Q(-2, 3, 2)$, and $R(1, 0, 1)$.
   a. Using $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ as direction vectors, write a vector equation for this plane.
   b. Using $\overrightarrow{QR}$ and one other direction vector, write a second vector equation for this plane.

C 5. Explain why the equation $\vec{r} = (-1, 0, -1) + s(2, 3, -4) + t(4, 6, -8)$, $s, t \in \mathbf{R}$, does not represent the equation of a plane. What does this equation represent?

PART B

6. Determine vector equations and the corresponding parametric equations of each plane.
   a. the plane with direction vectors $\vec{a} = (4, 1, 0)$ and $\vec{b} = (3, 4, -1)$, passing through the point $A(-1, 2, 7)$
   b. the plane passing through the points $A(1, 0, 0)$, $B(0, 1, 0)$, and $C(0, 0, 1)$
   c. the plane passing through points $A(1, 1, 0)$ and $B(4, 5, -6)$, with direction vector $\vec{a} = (7, 1, 2)$

7. a. Determine parameters corresponding to the point $P(5, 3, 2)$, where P is a point on the plane with equation
$\pi : \vec{r} = (2, 0, 1) + s(4, 2, -1) + t(-1, 1, 2), s, t \in \mathbf{R}$.
   b. Show that the point $A(0, 5, -4)$ does not lie on π.

8. A plane has $\vec{r} = (-3, 5, 6) + s(-1, 1, 2) + v(2, 1, -3), s, v \in \mathbf{R}$ as its equation.
   a. Give the equations of two intersecting lines that lie on this plane.
   b. What point do these two lines have in common?

A

9. Determine the coordinates of the point where the plane with equation $\vec{r} = (4, 1, 6) + s(11, -1, 3) + t(-7, 2, -2), s, t \in \mathbf{R}$, crosses the z-axis.

10. Determine the equation of the plane that contains the point $P(-1, 2, 1)$ and the line $\vec{r} = (2, 1, 3) + s(4, 1, 5), s \in \mathbf{R}$.

11. Determine the equation of the plane that contains the point $A(-2, 2, 3)$ and the line $\vec{r} = m(2, -1, 7), m \in \mathbf{R}$.

12. a. Determine two pairs of direction vectors that can be used to represent the xy-plane in R^3.
    b. Write a vector and parametric equations for the xy-plane in R^3.

K

13. a. Determine a vector equation for the plane containing the points $O(0, 0, 0)$, $A(-1, 2, 5)$, and $C(3, -1, 7)$.
    b. Determine a vector equation for the plane containing the points $P(-2, 2, 3), Q(-3, 4, 8)$, and $R(1, 1, 10)$.
    c. How are the planes found in parts a. and b. related? Explain your answer.

14. Show that the following equations represent the same plane:
$\vec{r} = u(-3, 2, 4) + v(-4, 7, 1), u, v \in \mathbf{R}$, and
$\vec{r} = s(-1, 5, -3) + t(-1, -5, 7), s, t \in \mathbf{R}$
(*Hint*: Express each direction vector in the first equation as a linear combination of the direction vectors in the second equation.)

T

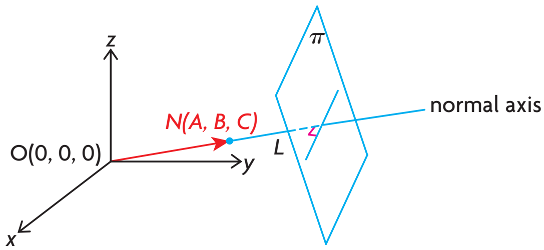
15. The plane with equation $\vec{r} = (1, 2, 3) + m(1, 2, 5) + n(1, -1, 3)$ intersects the y- and z-axes at the points A and B, respectively. Determine the equation of the line that contains these two points.

PART C

16. Suppose that the lines L_1 and L_2 are defined by the equations $\vec{r} = \overrightarrow{OP_0} + s\vec{a}$ and $\vec{r} = \overrightarrow{OP_0} + t\vec{b}$, respectively, where $s, t \in \mathbf{R}$, and $\vec{a}$ and $\vec{b}$ are noncollinear vectors. Prove that the plane defined by the equation $\vec{r} = \overrightarrow{OP_0} + s\vec{a} + t\vec{b}$ contains both of these lines.

Section 8.5—The Cartesian Equation of a Plane

In the previous section, the vector and parametric equations of a plane were found. In this section, we will show how to derive the Cartesian (or scalar) equation of a plane. The process is very similar to the process used to find the Cartesian equation of a line in R^2.

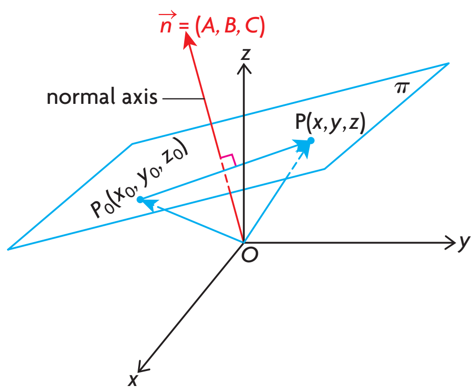


In the diagram above, a plane π is shown, along with a line L drawn from the origin, so that L is perpendicular to the given plane. For any plane in R^3, there is only one possible line that can be drawn through the origin perpendicular to the plane. This line is called the normal axis for the plane. The direction of the normal axis is given by a vector joining the origin to any point on the normal axis. The direction vector is called a normal to the plane. In the diagram, $\overrightarrow{ON}$ is a normal to the plane because it joins the origin to $N(A, B, C)$, a point on the normal axis. Any nonzero vector is a normal to a plane if it lies along the normal axis. This implies that an infinite number of normals exist for all planes.

A plane is completely determined when we know a point $(P_0(x_0, y_0, z_0))$ on the plane and a normal to the plane. This single idea can be used to determine the Cartesian equation of a plane.

Deriving the Cartesian Equation of a Plane

Consider the following diagram:



To derive the equation of this plane, we need two points on the plane, $P_0(x_0, y_0, z_0)$ (its coordinates given) and a general point, $P(x, y, z)$, different from P_0. The vector $\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$ represents any vector on the plane. If $\vec{n} = \overrightarrow{ON} = (A, B, C)$ is a known normal to the plane, then the relationship, $\vec{n} \cdot \overrightarrow{P_0P} = 0$ can be used to derive the equation of the plane, since $\vec{n}$ and any vector on the plane are perpendicular.

$\vec{n} \cdot \overrightarrow{P_0P} = 0$

$(A, B, C) \cdot (x - x_0, y - y_0, z - z_0) = 0$

$Ax - Ax_0 + By - By_0 + Cz - Cz_0 = 0$

Or, $Ax + By + Cz + (-Ax_0 - By_0 - Cz_0) = 0$

Since the quantities in the expression $-Ax_0 - By_0 - Cz_0$ are known, we'll replace this with D to make the equation simpler. The Cartesian equation of the plane is, thus, $Ax + By + Cz + D = 0$.

Cartesian Equation of a Plane

The Cartesian (or scalar) equation of a plane in R^3 is of the form $Ax + By + Cz + D = 0$ with normal $\vec{n} = (A, B, C)$. The normal $\vec{n}$ is a nonzero vector perpendicular to all vectors in the plane.

EXAMPLE 1 Representing a plane by its Cartesian equation

The point $A(1, 2, 2)$ is a point on the plane with normal $\vec{n} = (-1, 2, 6)$. Determine the Cartesian equation of this plane.

Solution

Two different methods can be used to determine the Cartesian equation of this plane. Both methods will give the same answer.

Method 1:
Let $P(x, y, z)$ be any point on the plane.

Therefore, $\overrightarrow{AP} = (x - 1, y - 2, z - 2)$ represents any vector on the plane. Since $\vec{n} = (-1, 2, 6)$ and $\vec{n} \cdot \overrightarrow{AP} = 0$,

$$\begin{aligned}(-1, 2, 6) \cdot (x - 1, y - 2, z - 2) &= 0\\ -1(x - 1) + 2(y - 2) + 6(z - 2) &= 0\\ -x + 1 + 2y - 4 + 6z - 12 &= 0\\ -x + 2y + 6z - 15 &= 0\end{aligned}$$

Multiplying each side by -1, $x - 2y - 6z + 15 = 0$.

Either $-x + 2y + 6z - 15 = 0$ or $x - 2y - 6z + 15 = 0$ is a correct equation for the plane, but usually we write the equation with integer coefficients and with a positive coefficient for the x-term.

Method 2:
Since the required equation has the form $Ax + By + Cz + D = 0$, where $\vec{n} = (A, B, C) = (-1, 2, 6)$, the direction numbers for the normal can be substituted directly into the equation. This gives $-x + 2y + 6z + D = 0$, with D to be determined. Since the point $A(1, 2, 2)$ is on the plane, it satisfies the equation.

Substituting the coordinates of this point into the equation gives $-(1) + 2(2) + 6(2) + D = 0$, and thus $D = -15$.

If $D = -15$ is substituted into $-x + 2y + 6z + D = 0$, the equation will be $-x + 2y + 6z + (-15) = 0$ or $x - 2y - 6z + 15 = 0$.

To find the Cartesian equation of a plane, either Method 1 or Method 2 can be used.

The Cartesian equation of a plane is simpler than either the vector or the parametric form and is used most often.

EXAMPLE 2 Determining the Cartesian equation of a plane from three coplanar points

Determine the Cartesian equation of the plane containing the points $A(-1, 2, 5)$, $B(3, 2, 4)$, and $C(-2, -3, 6)$.

Solution

A normal to this plane is determined by calculating the cross product of the direction vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. This results in a vector perpendicular to the plane in which both these vectors lie.

$\overrightarrow{AB} = (3 - (-1), 2 - 2, 4 - 5) = (4, 0, -1)$ and

$\overrightarrow{AC} = (-2 - (-1), -3 - 2, 6 - 5) = (-1, -5, 1)$

$$\begin{aligned}\text{Thus, } \overrightarrow{AB} \times \overrightarrow{AC} &= (0(1) - (-1)(-5), -1(-1) - (4)(1), 4(-5) - (0)(-1)) \\ &= (-5, -3, -20) \\ &= -1(5, 3, 20)\end{aligned}$$

If we let $P(x, y, z)$ be any point on the plane, then $\overrightarrow{AP} = (x + 1, y - 2, z - 5)$, and since a normal to the plane is $(5, 3, 20)$,

$$(5, 3, 20) \cdot (x + 1, y - 2, z - 5) = 0$$

$$5x + 5 + 3y - 6 + 20z - 100 = 0$$

After simplifying, the required equation of the plane is $5x + 3y + 20z - 101 = 0$.

A number of observations can be made about this calculation. If we had used $\overrightarrow{AB}$ and $\overrightarrow{BC}$ as direction vectors, for example, we would have found that $\overrightarrow{BC} = (-5, -5, 2)$ and $\overrightarrow{AB} \times \overrightarrow{BC} = (4, 0, -1) \times (-5, -5, 2) = -1(5, 3, 20)$. When finding the equation of a plane, it is possible to use any pair of direction vectors on the plane to find a normal to the plane. Also, when finding the value for D, if we had used the method of substitution, it would have been possible to substitute any one of the three given points in the equation.

In the next example, we will show how to convert from vector or parametric form to Cartesian form. We will also show how to obtain the vector form of a plane if given its Cartesian form.

EXAMPLE 3 **Connecting the various forms of the equation of a plane**

a. Determine the Cartesian form of the plane whose equation in vector form is $\vec{r} = (1, 2, -1) + s(1, 0, 2) + t(-1, 3, 4)$, $s, t \in \mathbf{R}$.

b. Determine the vector and parametric equations of the plane with Cartesian equation $x - 2y + 5z - 6 = 0$.

Solution

a. To find the Cartesian equation of the plane, two direction vectors are needed so that a normal to the plane can be determined. The two given direction vectors for the plane are $(1, 0, 2)$ and $(-1, 3, 4)$. Their cross product is

$$\begin{aligned}(1, 0, 2) \times (-1, 3, 4) &= (0(4)-2(3), 2(-1)-1(4), 1(3)-0(-1)) \\ &= -3(2, 2, -1)\end{aligned}$$

A normal to the plane is $(2, 2, -1)$, and the Cartesian equation of the plane is of the form $2x + 2y - z + D = 0$. Substituting the point $(1, 2, -1)$ into this equation gives $2(1) + 2(2)-(-1) + D = 0$, or $D = -7$.

Therefore, the Cartesian equation of the plane is $2x + 2y - z - 7 = 0$.

b. *Method 1:*

To find the corresponding vector and parametric equations of a plane, the equation of the plane is first converted to its parametric form. The simplest way to do this is to choose any two of the variables and replace them with a parameter. For example, if we substitute $y = s$ and $z = t$ and solve for x, we obtain $x - 2s + 5t - 6 = 0$ or $x = 2s - 5t + 6$.

This gives us the required parametric equations $x = 2s - 5t + 6$, $y = s$, and $z = t$. The vector form of the plane can be found by rearranging the parametric form.

Therefore, $(x, y, z) = (2s - 5t + 6, s, t)$

$$(x, y, z) = (6, 0, 0) + (2s, s, 0) + (-5t, 0, t)$$

$$\vec{r} = (6, 0, 0) + s(2, 1, 0) + t(-5, 0, 1),\ s, t \in \mathbf{R}$$

The parametric equations of this plane are $x = 2s - 5t + 6$, $y = s$, and $z = t$, and the corresponding vector form is $\vec{r} = (6, 0, 0) + s(2, 1, 0) + t(-5, 0, 1)$, $s, t \in \mathbf{R}$.

Check:
This vector equation of the plane can be checked by converting to Cartesian form. A normal to the plane is $(2, 1, 0) \times (-5, 0, 1) = (1, -2, 5)$. The plane has the form $x - 2y + 5z + D = 0$. If $(6, 0, 0)$ is substituted into the equation to find D, we find that $6 - 2(0) + 5(0) + D = 0$, so $D = -6$ and the equation is the given equation $x - 2y + 5z - 6 = 0$.

Method 2:
We rewrite the given equation as $x = 2y - 5z + 6$. We are going to find the coordinates of three points on the plane, and writing the equation in this way allows us to choose integer values for y and z that will give an integer value for x. The values in the table are chosen to make the computation as simple as possible.

The following table shows our choices for y and z, along with the calculation for x.

y	z	$x = 2y - 5z + 6$	Resulting Point
0	0	$x = 2(0) - 5(0) + 6 = 6$	$A(6, 0, 0)$
-1	0	$x = 2(-1) - 5(0) + 6 = 4$	$B(4, -1, 0)$
-1	1	$x = 2(-1) - 5(1) + 6 = -1$	$C(-1, -1, 1)$

$\overrightarrow{AB} = (4 - 6, -1 - 0, 0 - 0) = (-2, -1, 0)$ and

$\overrightarrow{AC} = (-1 - 6, -1 - 0, 1 - 0) = (-7, -1, 1)$

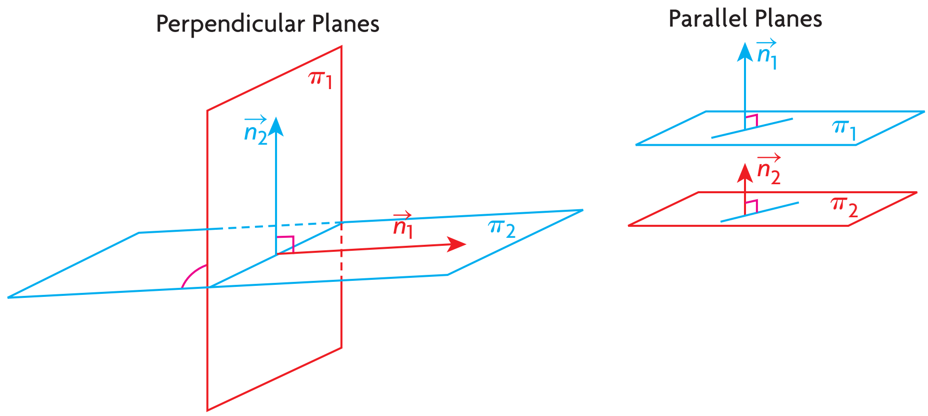
A vector equation is $\vec{r} = (6, 0, 0) + p(-2, -1, 0) + q(-7, -1, 1)$, $p, q \in \mathbf{R}$.

The corresponding parametric form is $x = 6 - 2p - 7q$, $y = -p - q$, and $z = q$.

Check:
To check that these equations are correct, the same procedure shown in Method 1 is used. This gives the identical Cartesian equation, $x - 2y + 5z - 6 = 0$.

When we considered lines in R^2, we showed how to determine whether lines were parallel or perpendicular. It is possible to use the same formula to determine whether planes are parallel or perpendicular.

Parallel and Perpendicular Planes



1. If π_1 and π_2 are two perpendicular planes, with normals $\vec{n_1}$ and $\vec{n_2}$, respectively, their normals are perpendicular (that is, $\vec{n_1} \cdot \vec{n_2} = 0$).
2. If π_1 and π_2 are two parallel planes, with normals $\vec{n_1}$ and $\vec{n_2}$, respectively, their normals are parallel (that is, $\vec{n_1} = k\vec{n_2}$) for all nonzero real numbers k.

EXAMPLE 4 Reasoning about parallel and perpendicular planes

a. Show that the planes $\pi_1: 2x - 3y + z - 1 = 0$ and $\pi_2: 4x - 3y - 17z = 0$ are perpendicular.

b. Show that the planes $\pi_3: 2x - 3y + 2z - 1 = 0$ and $\pi_4: 2x - 3y + 2z - 3 = 0$ are parallel but not coincident.

Solution

a. For π_1: $\vec{n_1} = (2, -3, 1)$ and for $\pi_2: \vec{n_2} = (4, -3, -17)$.

$$\begin{aligned}\vec{n_1} \cdot \vec{n_2} &= (2, -3, 1) \cdot (4, -3, -17) \\ &= 2(4) - 3(-3) + 1(-17) \\ &= 8 + 9 - 17 \\ &= 0\end{aligned}$$

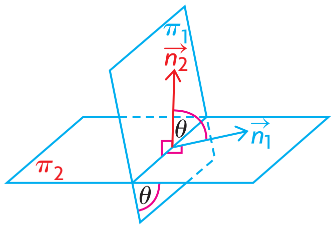
Since $\vec{n_1} \cdot \vec{n_2} = 0$, the two planes are perpendicular to each other.

b. For π_3 and π_4, $\vec{n_3} = \vec{n_4} = (2, -3, 2)$, so the planes are parallel. Because the planes have different constants (that is, -1 and -3), the planes are not coincident.

In general, if planes are coincident, it means that the planes have equations that are scalar multiples of each other. For example, the two planes $2x - y + z - 13 = 0$ and $-6x + 3y - 3z + 39 = 0$ are coincident because $-6x + 3y - 3z + 39 = -3(2x - y + z - 13)$.

It is also possible to find the angle between intersecting planes using their normals and the dot product formula for calculating the angle between two vectors. The angle between two planes is the same as the angle between their normals.

Angle between Intersecting Planes



The angle θ between two planes, π_1 and π_2, with normals of $\vec{n_1}$ and $\vec{n_2}$, respectively, can be calculated using the formula $\cos\theta = \dfrac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$.

EXAMPLE 5 Calculating the angle formed between two intersecting planes

Determine the angle between the two planes $\pi_1: x - y - 2z + 3 = 0$ and $\pi_2: 2x + y - z + 2 = 0$.

Solution

For $\pi_1: \vec{n_1} = (1, -1, -2)$. For $\pi_2: \vec{n_2} = (2, 1, -1)$.

Since $|\vec{n_1}| = \sqrt{(1)^2 + (-1)^2 + (-2)^2} = \sqrt{6}$ and

$|\vec{n_2}| = \sqrt{(2)^2 + (1)^2 + (-1)^2} = \sqrt{6}$,

$$\cos\theta = \frac{(1, -1, -2)\cdot(2, 1, -1)}{\sqrt{6}\sqrt{6}}$$

$$\cos\theta = \frac{2 - 1 + 2}{6}$$

$$\cos\theta = \frac{1}{2}$$

Therefore, the angle between the two planes is 60°. Normally, the angle between planes is given as an acute angle, but it is also correct to express it as 120°.

IN SUMMARY

Key Idea

- The Cartesian equation of a plane in R^3 is $Ax + By + Cz + D = 0$, where $\vec{n} = (A, B, C)$ is a normal to the plane and $\vec{n} = \vec{a} \times \vec{b}$. $\vec{a}$ and $\vec{b}$ are any two noncollinear direction vectors of the plane.

Need to Know

- Two planes whose normals are $\vec{n_1}$ and $\vec{n_2}$
  - are parallel if and only if $\vec{n_1} = k\vec{n_2}$ for any nonzero real number k.
  - are perpendicular if and only if $\vec{n_1} . \vec{n_2} = 0$.
  - have an angle θ between the planes determined by $\theta = \cos^{-1}\left(\frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}\right)$.

Exercise 8.5

PART A

1. A plane is defined by the equation $x - 7y - 18z = 0$.
   a. What is a normal vector to this plane?
   b. Explain how you know that this plane passes through the origin.
   c. Write the coordinates of three points on this plane.
2. A plane is defined by the equation $2x - 5y = 0$.
   a. What is a normal vector to this plane?
   b. Explain how you know that this plane passes through the origin.
   c. Write the coordinates of three points on this plane.
3. A plane is defined by the equation $x = 0$.
   a. What is a normal vector to this plane?
   b. Explain how you know that this plane passes through the origin.
   c. Write the coordinates of three points on this plane.
4. a. A plane is determined by a normal, $\vec{n} = (15, 75, -105)$, and passes through the origin. Write the Cartesian equation of this plane, where the normal is in reduced form.
   b. A plane has a normal of $\vec{n} = \left(-\frac{1}{2}, \frac{3}{4}, \frac{7}{16}\right)$ and passes through the origin. Determine the Cartesian equation of this plane.

PART B

5. A plane is determined by a normal, $\vec{n} = (1, 7, 5)$, and contains the point $P(-3, 3, 5)$. Determine a Cartesian equation for this plane using the *two* methods shown in Example 1.

K 6. The three noncollinear points $P(-1, 2, 1)$, $Q(3, 1, 4)$, and $R(-2, 3, 5)$ lie on a plane.

a. Using $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ as direction vectors and the point $R(-2, 3, 5)$, determine the Cartesian equation of this plane.

b. Using $\overrightarrow{QP}$ and $\overrightarrow{PR}$ as direction vectors and the point $P(-1, 2, 1)$, determine the Cartesian equation of this plane.

c. Explain why the two equations must be the same.

7. Determine the Cartesian equation of the plane that contains the points $A(-2, 3, 1)$, $B(3, 4, 5)$, and $C(1, 1, 0)$.

8. The line with vector equation $\vec{r} = (2, 0, 1) + s(-4, 5, 5)$, $s \in \mathbf{R}$, lies on the plane π, as does the point $P(1, 3, 0)$. Determine the Cartesian equation of π.

9. Determine unit vectors that are normal to each of the following planes:

a. $2x + 2y - z - 1 = 0$

b. $4x - 3y + z - 3 = 0$

c. $3x - 4y + 12z - 1 = 0$

10. A plane contains the point $A(2, 2, -1)$ and the line $\vec{r} = (1, 1, 5) + s(2, 1, 3)$, $s \in \mathbf{R}$. Determine the Cartesian equation of this plane.

A 11. Determine the Cartesian equation of the plane containing the point $(-1, 1, 0)$ and perpendicular to the line joining the points $(1, 2, 1)$ and $(3, -2, 0)$.

C 12. a. Explain the process you would use to determine the angle formed between two intersecting planes.

b. Determine the angle between the planes $x - z + 7 = 0$ and $2x + y - z + 8 = 0$.

13. a. Determine the angle between the planes $x + 2y - 3z - 4 = 0$ and $x + 2y - 1 = 0$.

b. Determine the Cartesian equation of the plane that passes through the point $P(1, 2, 1)$ and is perpendicular to the line $\frac{x - 3}{-2} = \frac{y + 1}{3} = \frac{z + 4}{1}$.

14. a. What is the value of k that makes the planes $4x + ky - 2z + 1 = 0$ and $2x + 4y - z + 4 = 0$ parallel?

b. What is the value of k that makes these two planes perpendicular?

c. Can these two planes ever be coincident? Explain.

T 15. Determine the Cartesian equation of the plane that passes through the points $(1, 4, 5)$ and $(3, 2, 1)$ and is perpendicular to the plane $2x - y + z - 1 = 0$.

PART C

16. Determine an equation of the plane that is perpendicular to the plane $x + 2y + 4 = 0$, contains the origin, and has a normal that makes an angle of $30°$ with the z-axis.

17. Determine the equation of the plane that lies between the points $(-1, 2, 4)$ and $(3, 1, -4)$ and is equidistant from them.

Section 8.6—Sketching Planes in R^3

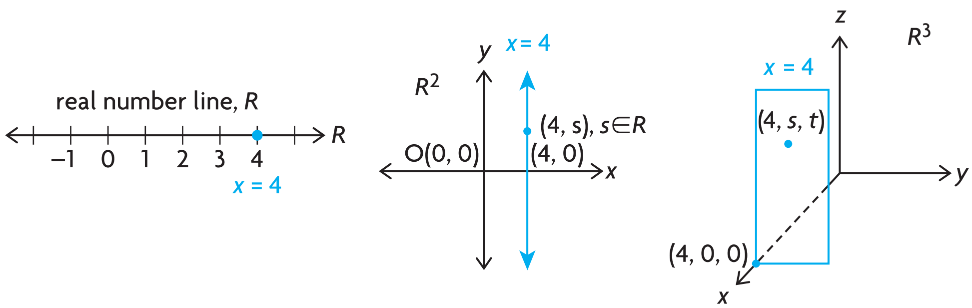
In previous sections, we developed methods for finding the equation of planes in both vector and Cartesian form. In this section, we examine how to sketch a plane if the equation is given in Cartesian form. When graphing planes in R^3, many of the same methods used for graphing a line in R^2 will be used.

An important first observation is that, if we are given an equation such as $x = 4$ and are asked to find its related graph, then a different graph is produced depending on the dimension in which we are working.

1. On the real number line, this equation refers to a point at $x = 4$.
2. In R^2, this equation represents a line parallel to the y-axis and 4 units to its right.
3. In R^3, the equation $x = 4$ represents a plane that intersects the x-axis at $(4, 0, 0)$ and is 4 units in front of the plane formed by the y- and z-axes.

We can see that the equation $x = 4$ results in a different graph depending on whether it is drawn on the number line, in R^2, or in R^3.

Different interpretations of the graph with equation $x = 4$



Varying the Coefficients in the Cartesian Equation

In the following situations, the graph of $Ax + By + Cz + D = 0$ in R^3 is considered for different cases.

Case 1– The equation contains one variable

Case 1a: Two of A, B, or C equal zero, and D equals zero.

In this case, the resulting equation will be of the form $x = 0$, $y = 0$, or $z = 0$. If $x = 0$, for example, this equation represents the yz-plane, since every point on this plane has an x-value equal to 0. Similarly, $y = 0$ represents the xz-plane, and $z = 0$ represents the xy-plane.

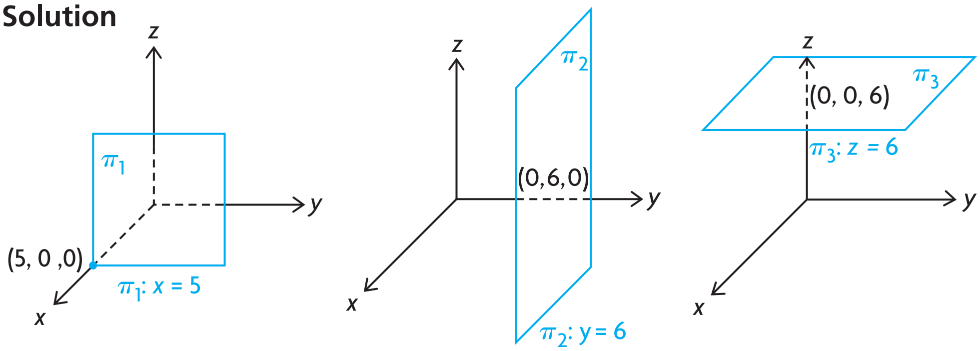
Case 1b: Two of A, B, or C equal zero, and D does not equal zero.

If two of the three coefficients are equal to zero, the resulting equation will be of the form $x = a$, $y = b$, or $z = c$. The following examples show that these equations represent planes parallel to the yz-, xz-, and xy-planes, respectively.

EXAMPLE 1 Representing the graphs of planes in R^3 whose Cartesian equations involve one variable

Draw the planes with equations $\pi_1 : x = 5$, $\pi_2 : y = 6$, and $\pi_3 : z = 6$.

Solution



Descriptions of the planes in Example 1 are given in the following table.

Plane	Description	Generalization
π_1: $x = 5$	A plane parallel to the yz-plane crosses the x-axis at (5, 0, 0). This plane has an x-intercept of 5.	A plane with equation $x = a$ is parallel to the yz-plane and crosses the x-axis at the point (a, 0, 0). The plane $x = 0$ is the yz-plane.
π_2: $y = 6$	A plane parallel to the xz-plane crosses the y-axis at (0, 6, 0). This plane has a y-intercept of 6.	A plane with equation $y = b$ is parallel to the xz-plane and crosses the y-axis at the point (0, b, 0). The plane $y = 0$ is the xz-plane.
π_3: $z = 6$	A plane parallel to the xy-plane crosses the x-axis at the point (0, 0, 6). This plane has a z-intercept of 6.	A plane with equation $z = c$ is parallel to the xy-plane and crosses the z-axis at the point (0, 0, c). The plane $z = 0$ is the xy-plane.

Case 2– The equation contains two variables

Case 2a: One of A, B, or C equals zero, and D equals zero.

In this case, the resulting equation will have the form $Ax + By = 0$, $Ax + Cz = 0$, or $By + Cz = 0$. The following example demonstrates how to graph an equation of this type.

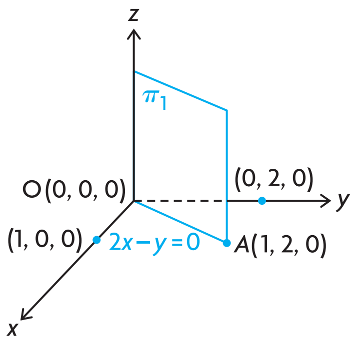
EXAMPLE 2 Representing the graph of a plane in R^3 whose Cartesian equation involves two variables, $D = 0$

Sketch the plane defined by the equation $\pi_1: 2x - y = 0$.

Solution

For $\pi_1: 2x - y = 0$, we note that the origin $O(0, 0, 0)$ lies on the plane, and it also contains the z-axis. We can see that π_1 contains the z-axis because, if it is written in the form $2x - y + 0z = 0$, $(0, 0, t)$ is on the plane because $2(0) - 0 + 0(t) = 0$. Since $(0, 0, t)$, $t \in \mathbf{R}$, represents any point on the z-axis, this means that the plane contains the z-axis.

In addition, the plane cuts the xy-plane along the line $2x - y = 0$. All that is necessary to graph this line is to select a point on the xy-plane that satisfies the equation and join that point to the origin. Since the point with coordinates $A(1, 2, 0)$ satisfies the equation, we draw the parallelogram determined by the z-axis and the line joining $O(0, 0, 0)$ to $A(1, 2, 0)$, and we have a sketch of the plane $2x - y = 0$.



EXAMPLE 3 Describing planes whose Cartesian equations involve two variables, $D = 0$

Write descriptions of the planes $\pi_1: 2x - z = 0$ and $\pi_2: y + 2z = 0$.

Solution

These equations can be written as $\pi_1: 2x + 0y - z = 0$ and $\pi_2: 0x + y + 2z = 0$.

Using the same reasoning as above, this implies that π_1 contains the origin and the y-axis, and cuts the xz-plane along the line with equation $2x - z = 0$. Similarly, π_2 contains the origin and the x-axis, and cuts the yz-plane along the line with equation $y + 2z = 0$.

Case 2b: One of A, B, or C equals zero, and D does not equal zero.

If one of the coefficients equals zero and $D \neq 0$, the resulting equations can be written as $Ax + By + D = 0$, $Ax + Cz + D = 0$, or $By + Cz + D = 0$.

EXAMPLE 4 Graphing planes whose Cartesian equations involve two variables, $D \neq 0$

Sketch the plane defined by the equation $\pi_1: 2x - 5y - 10 = 0$.

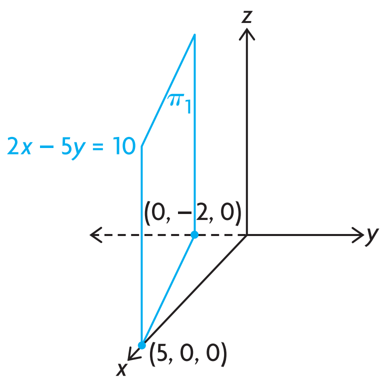
Solution

It is best to write this equation as $2x - 5y = 10$ so that we can easily calculate the intercepts.

x-intercept: We calculate the *x*-intercept for this plane in exactly the same way that we would calculate the *x*-intercept for the line $2x - 5y = 10$. If we let $y = 0$, then $2x - 5(0) = 10$ or $x = 5$. This means that the plane has an *x*-intercept of 5 and that it crosses the *x*-axis at $(5, 0, 0)$.

y-intercept: To calculate the *y*-intercept, we let $x = 0$. Thus, $-5y = 10, y = -2$. This means that the plane has a *y*-intercept of -2 and it crosses the *y*-axis at $(0, -2, 0)$.

To complete the analysis for the plane, we write the equation as $2x - 5y + 0z = 10$. If the plane did cross the *z*-axis, it would do so at a point where $x = y = 0$. Substituting these values into the equation, we obtain $2(0) - 5(0) + 0z = 10$ or $0z = 10$. This implies that the plane has no *z*-intercept because there is no value of *z* that will satisfy the equation. Thus, the plane passes through the points (5, 0, 0) and $(0, -2, 0)$ and is parallel to the *z*-axis. The plane is sketched in the diagram below. Possible direction vectors for the plane are $\overrightarrow{m_1} = (5 - 0, 0 - (-2), 0 - 0) = (5, 2, 0)$ and $\overrightarrow{m_2} = (0, 0, 1)$.



Using the same line of reasoning as above, if *A*, *C* and *D* are nonzero when $B = 0$, the resulting plane is parallel to the *y*-axis. If *B*, *C* and *D* are nonzero when $A = 0$, the resulting plane is parallel to the *x*-axis.

Case 3– The equation contains three variables

Case 3b: *A*, *B*, and *C* do not equal zero, and *D* equals zero.

This represents an equation of the form, $Ax + By + Cz = 0$, which is a plane most easily sketched using the fact that a plane is uniquely determined by three points. The following example illustrates this approach.

EXAMPLE 5 Graphing planes whose Cartesian equations involve three variables, $D = 0$

Sketch the plane $\pi_1 : x + 3y - z = 0$.

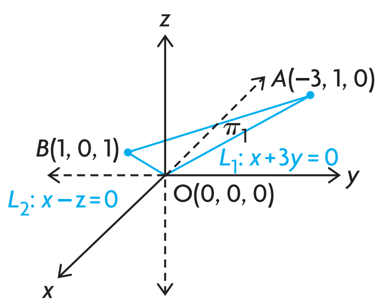
Solution

Since there is no constant in the equation, the point $(0, 0, 0)$ is on the plane. To sketch the plane, we require two other points. We first find a point on the xy-plane and a second point on the xz-plane.

Point on xy-plane: Any point on the xy-plane has $z = 0$. If we first substitute $z = 0$ into $x + 3y - z = 0$, then $x + 3y - 0 = 0$, or $x + 3y = 0$, which means that the given plane cuts the xy-plane along the line $x + 3y = 0$. Using this equation, we can now select convenient values for x and y to obtain the coordinates of a point on this line. The easiest values are $x = -3$ and $y = 1$, implying that the point $A(-3, 1, 0)$ is on the plane.

Point on xz-plane: Any point on the xz-plane has $y = 0$. If we substitute $y = 0$ into $x + 3y - z = 0$, then $x + 3(0) - z = 0$, or $x = z$, which means that the given plane cuts the xz-plane along the line $x = z$. As before, we choose convenient values for x and z. The easiest values are $x = z = 1$, implying that $B(1, 0, 1)$ is a point on the plane.

Since three points determine a plane, we locate these points in R^3 and form the related triangle. This triangle, OAB, represents part of the required plane.



Case 3b: A, B, and C do not equal zero, and D does not equal zero.

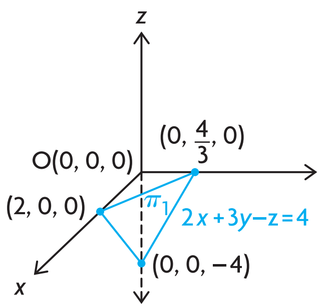
This represents the plane with equation $Ax + By + Cz + D = 0$, which is most easily sketched by finding its intercepts. Since we know that three noncollinear points determine a plane, knowing these three intercepts will allow us to graph the plane.

EXAMPLE 6 Graphing planes whose Cartesian equations involve three variables, $D \neq 0$

Sketch the plane defined by the equation $\pi_1 : 2x + 3y - z = 4$.

Solution

To sketch the plane, we calculate the coordinates of the points where the plane intersects each of the three coordinate axes (that is, we determine the three intercepts for the plane). This is accomplished by setting 2 of the 3 variables equal to zero and solving for the remaining variable. The x-, y-, and z-intercepts are 2, $\frac{4}{3}$, and -4, respectively. These three points form a triangle that forms part of the required plane.



EXAMPLE 7 Reasoning about direction vectors of planes

Determine two direction vectors for the planes $\pi_1 : 3x + 4y = 12$ and $\pi_2 : x - y - 5z = 0$.

Solution

The plane $\pi_1 : 3x + 4y = 12$ crosses the x-axis at the point $(4, 0, 0)$ and the y-axis at the point $(0, 3, 0)$. One direction vector is thus $\overrightarrow{m_1} = (4 - 0, 0 - 3, 0 - 0) = (4, -3, 0)$. Since the plane can be written as $3x + 4y + 0z = 12$, this implies that it does not intersect the z-axis, and therefore has $\overrightarrow{m_2} = (0, 0, 1)$ as a second direction vector.

The plane $\pi_2 : x - y - 5z = 0$ passes through $O(0, 0, 0)$ and cuts the xz-plane along the line $x - 5z = 0$. Convenient choices for x and z are 5 and 1, respectively. This means that $A(5, 0, 1)$ is on π_2. Similarly, the given plane cuts the xy-plane along the line $x - y = 0$. Convenient values for x and y are 1 and 1. This means that $B(1, 1, 0)$ is on π_2.

Possible direction vectors for π_2 are $\overrightarrow{m_1} = (5 - 0, 0 - 0, 1 - 0) = (5, 0, 1)$ and $\overrightarrow{m_2} = (1, 1, 0)$.

IN SUMMARY

Key Idea

- A sketch of a plane in R^3 can be created by using a combination of points and lines that help to define the plane.

Need to Know

- To sketch the graph of a plane, consider each of the following cases as it relates to the Cartesian equation $Ax + By + Cz + D = 0$:

  Case 1: The equation contains one variable.

  a. Two of the coefficients (two of A, B, or C) equal zero, and D equals zero.

     These are the three coordinate planes—xy-plane, xz-plane, and yz-plane.

  b. Two of the coefficients (two of A, B, or C) equal zero, and D does not equal zero.

     These are parallel to the three coordinate planes.

  Case 2: The equation contains two variables.

  a. One of the coefficients (one of A, B, or C) equals zero, and D equals zero.

     Find a point with missing variable set equal to 0. Join this point to (0, 0, 0), and draw a plane containing the missing variable axis and this point.

  b. One of the coefficients (one of A, B, or C) equals zero, and D does not equal zero.

     Find the two intercepts, and draw a plane parallel to the missing variable axis.

  Case 3: The equation contains three variables.

  a. None of the coefficients (none of A, B, or C) equals zero, and D equals zero.

     Determine two points in addition to (0, 0, 0), and draw the plane through these points.

  b. None of the coefficients (none of A, B, or C) equals zero, and D does not equal zero.

     Find the three intercepts, and draw a plane through these three points.

Exercise 8.6

PART A

C 1. Describe each of the following planes in words:

a. $x = -2$ b. $y = 3$ c. $z = 4$

2. For the three planes given in question 1, what are coordinates of their point of intersection?

3. On which of the planes π_1: $x = 5$ or π_2: $y = 6$ could the point $P(5, -3, -3)$ lie? Explain.

PART B

A 4. Given that $x^2 - 1 = (x - 1)(x + 1)$, sketch the two graphs associated with $x^2 - 1 = 0$ in R^2 and R^3.

5. a. State the x-, y-, and z-intercepts for each of the following three planes:
   i. $\pi_1 : 2x + 3y = 18$
   ii. $\pi_2 : 3x - 4y + 5z = 120$
   iii. $\pi_3 : 13y - z = 39$

   b. State two direction vectors for each plane.

6. a. For the plane with equation $\pi : 2x - y + 5z = 0$, determine
   i. the coordinates of three points on this plane
   ii. the equation of the line where this plane intersects the xy-plane

   b. Sketch this plane.

7. Name the three planes that the equation $xyz = 0$ represents in R^3.

K 8. For each of the following equations, sketch the corresponding plane:

   a. $\pi_1 : 4x - y = 0$
   b. $\pi_2 : 2x + y - z = 4$
   c. $\pi_3 : z = 4$
   d. $\pi_4 : y - z = 4$

9. a. Write the expression $xy + 2y = 0$ in factored form.
   b. Sketch the lines corresponding to this expression in R^2.
   c. Sketch the planes corresponding to this expression in R^3.

10. For each given equation, sketch the corresponding plane.
   a. $2x + 2y + z - 4 = 0$
   b. $3x - 4z = 12$
   c. $5y - 15 = 0$

PART C

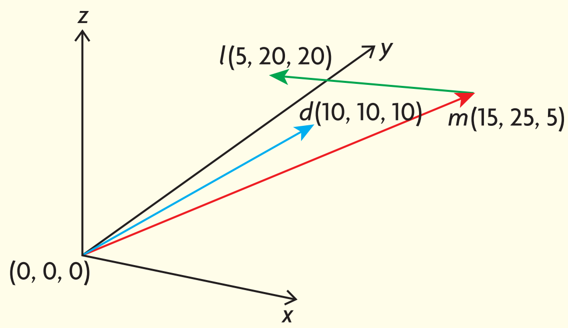
T 11. It is sometimes useful to be able to write an equation of a plane in terms of its intercepts. If *a, b,* and *c* represent the x-, y-, and z-intercepts, respectively, then the resulting equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

   a. Determine the equation of the plane that has x-, y-, and z-intercepts of 3, 4, and 6, respectively.
   b. Determine the equation of the plane that has x- and z-intercepts of 5 and –7, respectively, and is parallel to the y-axis.
   c. Determine the equation of the plane that has no x- or y-intercept, but has a z-intercept of 8.

CAREER LINK WRAP-UP Investigate and Apply

CHAPTER 8: COMPUTER PROGRAMMING WITH VECTORS

A computer programmer is designing a 3-D space game. She wants to have an asteroid fly past a spaceship along the path of vector m, collide with another asteroid, and be deflected along vector path $\overrightarrow{ml}$. The spaceship is treated as the origin and is travelling along vector d.



a. Determine the vector and parametric equations for the line determined by vector $\overrightarrow{ml}$ in its current position.

b. Determine the vector and parametric equations for the line determined by vector $\vec{d}$ in its current position.

c. By using the previous parts, can you determine if the asteroid and spaceship could possibly collide as they travel along their respective trajectories? Explain in detail all that would have to take place for this collision to occur (if, indeed, a collision is even possible).

Key Concepts Review

In Chapter 8, you examined how the algebraic description of a straight line could be represented using vectors in both two and three dimensions. The form of the vector equation of a line, $\vec{r} = \vec{r_0} + t\vec{d}$, is the same whether the line lies in a two-dimensional plane or a three-dimensional space. The following table summarizes the various forms of the equation of a line, where the coordinates of a point on the line are known as well as a direction vector. t is a parameter where $t \in \mathbf{R}$.

Form	R^2	R^3
Vector equation	$(x, y) = (x_0, y_0) + t(a, b)$	$(x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$
Parametric equations	$x = x_0 + at, y = y_0 + bt$	$x = x_0 + at, y = y_0 + bt, z = z_0 + ct$
Symmetric equations	$\frac{x - x_0}{a} = \frac{y - y_0}{b}$	$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$
Cartesian equation	$Ax + BY + C = 0$: $\vec{n} = (A, B)$	not applicable

This concept was then extended to planes in R^3. The following table summarizes the various forms of the equation of a plane, where the coordinates of a point on the plane are known as well as two direction vectors. s and t are parameters where $s, t \in \mathbf{R}$.

Form	R^3
Vector equation	$(x, y, z) = (x_0, y_0, z_0) + s(a_1, a_2, a_3) + t(b_1, b_2, b_3)$
Parametric equations	$x = x_0 + sa_1 + tb_1, y = y_0 + sa_2 + tb_2, z = z_0 + sa_3 + tb_3$
Cartesian equation	$Ax + BY + Cz + D = 0$: $\vec{n} = (A, B, C)$

You also saw that when lines and planes intersect, angles are formed between them. Both lines and planes have normals, which are vectors that run perpendicular to the respective line or plane. The size of an angle can be determined using the normal vectors and the following formula:

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$$

Sketching the graph of a plane in R^3 can be accomplished by examining the Cartesian equation of the plane. Determine whether the equation contains one, two, or three variables and whether it contains a constant. This information helps to narrow down which case you need to consider to sketch the graph. Once you have determined the specific case (see the In Summary table in Section 8.6), you can determine the appropriate points and lines to help you sketch a representation of the plane.

Review Exercise

1. Determine vector and parametric equations of the plane that contains the points $A(1, 2, -1)$, $B(2, 1, 1)$, and $C(3, 1, 4)$.

2. In question 1, there are a variety of different answers possible, depending on the points and direction vectors chosen. Determine two Cartesian equations for this plane using two different vector equations, and verify that these two equations are identical.

3. a. Determine the vector, parametric, and symmetric equations of the line passing through points $A(-3, 2, 8)$ and $B(4, 3, 9)$.
   b. Determine the vector and parametric equations of the plane containing the points $A(-3, 2, 8)$, $B(4, 3, 9)$, and $C(-2, -1, 3)$.
   c. Explain why a symmetric equation cannot exist for a plane.

4. Determine the vector, parametric, and symmetric equations of the line passing through the point $A(7, 1, -2)$ and perpendicular to the plane with equation $2x - 3y + z - 1 = 0$.

5. Determine the Cartesian equation of each of the following planes:
   a. through the point $P(0, 1, -2)$, with normal $\vec{n} = (-1, 3, 3)$
   b. through the points $(3, 0, 1)$ and $(0, 1, -1)$, and perpendicular to the plane with equation $x - y - z + 1 = 0$
   c. through the points $(1, 2, 1)$ and $(2, 1, 4)$, and parallel to the x-axis

6. Determine the Cartesian equation of the plane that passes through the origin and contains the line $\vec{r} = (3, 7, 1) + t(2, 2, 3)$, $t \in \mathbf{R}$.

7. Find the vector and parametric equations of the plane that is parallel to the yz-plane and contains the point $A(-1, 2, 1)$.

8. Determine the Cartesian equation of the plane that contains the line $\vec{r} = (2, 3, 2) + t(1, 1, 4)$, $t \in \mathbf{R}$, and the point $(4, -3, 2)$.

9. Determine the Cartesian equation of the plane that contains the following lines:
   $L_1: \vec{r} = (4, 4, 5) + t(5, -4, 6)$, $t \in \mathbf{R}$, and
   $L_2: \vec{r} = (4, 4, 5) + s(2, -3, -4)$, $s \in \mathbf{R}$

10. Determine an equation for the line that is perpendicular to the plane $3x - 2y + z = 1$ passing through $(2, 3, -3)$. Give your answer in vector, parametric, and symmetric form.

11. A plane has $3x + 2y - z + 6 = 0$ as its Cartesian equation. Determine the vector and parametric equations of this plane.

12. Determine an equation for the line that has the same x- and z-intercepts as the plane with equation $2x + 5y - z + 7 = 0$. Give your answer in vector, parametric, and symmetric form.
13. Determine the vector, parametric, and Cartesian forms of the equation of the plane containing the lines $L_1: \vec{r} = (3, -4, 1) + s(1, -3, -5),\ s \in \mathbf{R}$, and $L_2: \vec{r_2} = (7, -1, 0) + t(2, -6, -10),\ t \in \mathbf{R}$.
14. Sketch each of the following planes:
    a. $\pi_1: 2x + 3y - 6z - 12 = 0$
    b. $\pi_2: 2x + 3y - 12 = 0$
    c. $\pi_3: x - 3z - 6 = 0$
    d. $\pi_4: y - 2z - 4 = 0$
    e. $\pi_5: 2x + 3y - 6z = 0$
15. Determine the vector, parametric, and Cartesian equations of each of the following planes:
    a. passing through the points $P(1, -2, 5)$ and $Q(3, 1, 2)$ and parallel to the line with equation $L: \vec{r} = 2t\vec{i} + (4t + 3)\vec{j} + (t + 1)\vec{k},\ t \in \mathbf{R}$
    b. containing the point $A(1, 1, 2)$ and perpendicular to the line joining the points $B(2, 1, -6)$ and $C(-2, 1, 5)$
    c. passing through the points $(4, 1, -1)$ and $(5, -2, 4)$ and parallel to the z-axis
    d. passing through the points $(1, 3, -5)$, $(2, 6, 4)$, and $(3, -3, 3)$
16. Show that $L_1: \vec{r} = (1, 2, 3) + s(-3, 5, 21) + t(0, 1, 3)$, s, $t \in \mathbf{R}$, and $L_2: \vec{r} = (1, -1, -6) + u(1, 1, 1) + v(2, 5, 11)$, $u, v \in \mathbf{R}$, are equations for the same plane.
17. The two lines $L_1: \vec{r} = (-1, 1, 0) + s(2, 1, -1),\ s \in \mathbf{R}$, and $L_2: \vec{r} = (2, 1, 2) + t(2, 1, -1),\ t \in \mathbf{R}$, are parallel but do not coincide. The point $A(5, 4, -3)$ is on L_1. Determine the coordinates of a point B on L_2 such that $\overrightarrow{AB}$ is perpendicular to L_2.
18. Write a brief description of each plane.
    a. $\pi_1: 2x - 3y = 6$
    b. $\pi_2: x - 3z = 6$
    c. $\pi_3: 2y - z = 6$
19. a. Which of the following points lies on the line $x = 2t$, $y = 3 + t$, $z = 1 + t$?
    $A(2, 4, 2)$ $B(-2, 2, 1)$ $C(4, 5, 2)$ $D(6, 6, 2)$
    b. If the point $(a, b, -3)$ lies on the line, determine the values of a and b.

20. Calculate the acute angle that is formed by the intersection of each pair of lines.

a. $L_1: \frac{x-1}{1} = \frac{y-3}{5}$ and $L_2: \frac{x-2}{2} = \frac{1-y}{3}$

b. $y = 4x + 2$ and $y = -x + 3$

c. $L_1: x = -1 + 3t, y = 1 + 4t, z = -2t$ and
$L_2: x = -1 + 2s, y = 3s, z = -7 + s$

d. $L_1: (x, y, z) = (4, 7, -1) + t(4, 8, -4)$ and
$L_2: (x, y, z) = (1, 5, 4) + s(-1, 2, 3)$

21. Calculate the acute angle that is formed by the intersection of each pair of planes.

a. $2x + 3y - z + 9 = 0$ and $x + 2y + 4 = 0$

b. $x - y - z - 1 = 0$ and $2x + 3y - z + 4 = 0$

22. a. Which of the following lines is parallel to the plane $4x + y - z - 10 = 0$?

i. $\vec{r} = (3, 0, 2) + t(1, -2, 2)$

ii. $x = -3t, y = -5 + 2t, z = -10t$

iii. $\frac{x-1}{4} = \frac{y+6}{-1} = \frac{z}{1}$

b. Do any of these lines lie in the plane in part a.?

23. Does the point (4, 5, 6) lie in the plane
$(x, y, z) = (4, 1, 6) + p(3, -2, 1) + q(-6, 6, -1)$?
Support your answer with the appropriate calculations.

24. Determine the parametric equations of the plane that contains the following two parallel lines:

$L_1: (x, y, z) = (2, 4, 1) + t(3, -1, 1)$ and

$L_2: (x, y, z) = (1, 4, 4) + t(-6, 2, -2)$

25. Explain why the vector equation of a plane has two parameters, while the vector equation of a line has only one.

26. Explain why any plane with a vector equation of the form
$(x, y, z) = (a, b, c) + s(d, e, f) + t(a, b, c)$ will always pass through the origin.

27. a. Explain why the three points $(2, 3, -1)$, $(8, 5, -5)$ and $(-1, 2, 1)$ do not determine a plane.

b. Explain why the line $\vec{r} = (4, 9, -3) + t(1, -4, 2)$ and the point $(8, -7, 5)$ do not determine a plane.

28. Find a formula for the scalar equation of a plane in terms of a, b, and c, where a, b, and c are the x-intercept, the y-intercept, and the z-intercept of a plane, respectively. Assume that all intercepts are nonzero.

29. Determine the Cartesian equation of the plane that has normal vector $(6, -5, 12)$ and passes through the point $(5, 8, -3)$.

30. A plane passes through the points $A(1, -3, 2)$, $B(-2, 4, -2)$, and $C(3, 2, 1)$.
    a. Determine a vector equation of the plane.
    b. Determine a set of parametric equations of the plane.
    c. Determine the Cartesian equation of the plane.
    d. Determine if the point $(3, 5, -4)$ lies on the plane.

31. Determine the Cartesian equation of the plane that is parallel to the plane $4x - 2y + 5z - 10 = 0$ and passes through each point below.
    a. $(0, 0, 0)$
    b. $(-1, 5, -1)$
    c. $(2, -2, 2)$

32. Show that the following pairs of lines intersect. Determine the coordinates of the point of intersection and the angles formed by the lines.
    a. $L_1: x = 5 + 2t,\ y = -3 + t$ and $L_2: x = 23 - 2s,\ y = 6 - s$
    b. $L_1: \dfrac{x + 3}{3} = \dfrac{y + 1}{4}$ and $L_2: \dfrac{x - 6}{3} = \dfrac{y - 2}{-2}$

33. Determine the vector equation, parametric equations, and, if possible, symmetric equation of the line that passes through the point $P(1, 3, 5)$ and
    a. has direction vector $(-2, -4, -10)$
    b. also passes through the point $Q(-7, 9, 3)$
    c. is parallel to the line that passes through $R(4, 8, -5)$ and $S(-2, -5, 9)$
    d. is parallel to the x-axis
    e. is perpendicular to the line $(x, y, z) = (1, 0, 5) + t(-3, 4, -6)$
    f. is perpendicular to the plane determined by the points $A(4, 2, 1)$, $B(3, -4, 2)$, and $C(-3, 2, 1)$

34. Determine the Cartesian equation of the plane that
    a. contains the point $P(-2, 6, 1)$ and has normal vector $(2, -4, 5)$
    b. contains the point $P(-2, 0, 6)$ and the line $\frac{x - 4}{3} = \frac{y + 2}{-5} = \frac{z - 1}{2}$
    c. contains the point $P(3, 3, 3)$ and is parallel to the xy-plane
    d. contains the point $P(-4, 2, 4)$ and is parallel to the plane $3x + y - 4z + 8 = 0$
    e. is perpendicular to the yz-plane and has y-intercept 4 and z-intercept -2
    f. is perpendicular to the plane $x - 2y + z = 6$ and contains the line $(x, y, z) = (2, -1, -1) + t(3, 1, 2)$

Chapter 8 Test

1. a. Given the points $A(1, 2, 4)$, $B(2, 0, 3)$, and $C(4, 4, 4)$,
   i. determine the vector and parametric equations of the plane that contains these three points
   ii. determine the corresponding Cartesian equation of the plane that contains these three points
   b. Does the point with coordinates $\left(1, -1, -\frac{1}{2}\right)$ lie on this plane?
2. The plane π intersects the coordinate axes at $(2, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 4)$.
   a. Write an equation for this plane, expressing it in the form $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.
   b. Determine the coordinates of a normal to this plane.
3. a. Determine a vector equation for the plane containing the origin and the line with equation $\vec{r} = (2, 1, 3) + t(1, 2, 5), t \in \mathbf{R}$.
   b. Determine the corresponding Cartesian equation of this plane.
4. a. Determine a vector equation for the plane that contains the following two lines:
   $L_1 : \vec{r} = (4, -3, 5) + t(2, 0, -3), t \in \mathbf{R}$, and
   $L_2 : \vec{r} = (4, -3, 5) + s(5, 1, -1), s \in \mathbf{R}$
   b. Determine the corresponding Cartesian equation of this plane.
5. a. A line has $\frac{x-2}{4} = \frac{y-4}{-2} = z$ as its symmetric equations. Determine the coordinates of the point where this line intersects the yz-plane.
   b. Write a second symmetric equation for this line using the point you found in part a.
6. a. Determine the angle between π_1 and π_2 where the two planes are defined as $\pi_1 : x + y - z = 0$ and $\pi_2 : x - y + z = 0$.
   b. Given the planes $\pi_3 : 2x - y + kz = 5$ and $\pi_4 : kx - 2y + 8z = 9$,
   i. determine a value of k if these planes are parallel
   ii. determine a value of k if these planes are perpendicular
   c. Explain why the two given equations that contain the parameter k in part b cannot represent two identical planes.
7. a. Using a set of coordinate axes in R^2, sketch the line $x + 2y = 0$.
   b. Using a set of coordinate axes in R^3, sketch the plane $x + 2y = 0$.
   c. The equation $Ax + By = 0$, $A, B \neq 0$, represents an equation of a plane in R^3. Explain why this plane must always contain the z-axis.

Chapter 9

RELATIONSHIPS BETWEEN POINTS, LINES, AND PLANES

In this chapter, we will introduce perhaps the most important idea associated with vectors, the solution of systems of equations. In previous chapters, the solution of systems of equations was introduced in situations dealing with two equations in two unknowns. Geometrically, the solution of two equations in two unknowns is the point of intersection between two lines on the *xy*-plane. In this chapter, we are going to extend these ideas and consider systems of equations in R^3 and interpret their meaning. We will be working with systems of up to three equations in three unknowns, and we will demonstrate techniques for solving these systems.

CHAPTER EXPECTATIONS

In this chapter, you will

- determine the intersection between a line and a plane and between two lines in three-dimensional space, **Section 9.1**
- algebraically solve systems of equations involving up to three equations in three unknowns, **Section 9.2**
- determine the intersection of two or three planes, **Sections 9.3, 9.4**
- determine the distance from a point to a line in two- and three-dimensional space, **Section 9.5**
- determine the distance from a point to a plane, **Section 9.6**
- solve distance problems relating to lines and planes in three-dimensional space and interpret the results geometrically, **Sections 9.5, 9.6**

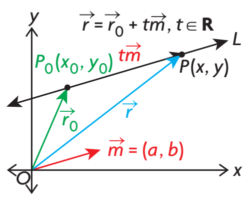
Review of Prerequisite Skills

In this chapter, you will examine how lines can intersect with other lines and planes, and how planes can intersect with other planes. Intersection problems are geometric models of linear systems. Before beginning, you may wish to review some equations of lines and planes.

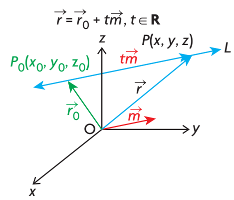
Type of Equation	Lines	Planes
Vector equation	$\vec{r} = \vec{r}_0 + t\vec{m}$	$\vec{r} = \vec{r}_0 + s\vec{a} + t\vec{b}$
Parametric equation	$x = x_0 + ta$ $y = y_0 + tb$ $z = z_0 + tc$	$x = x_0 + sa_1 + tb_1$ $y = y_0 + sa_2 + tb_2$ $z = z_0 + sa_3 + tb_3$
Cartesian equation	$Ax + By + C = 0$	$Ax + By + Cz + D = 0$ in three-dimensional space

In the table above,

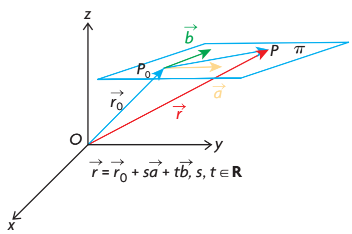
- $\vec{r}_0$ is the position vector whose tail is located at the origin and whose head is located at the point (x_0, y_0) in R^2 and (x_0, y_0, z_0) in R^3
- $\vec{m}$ is a direction vector whose components are (a, b) in R^2 and (a, b, c) in R^3
- $\vec{a}$ and $\vec{b}$ are noncollinear direction vectors whose components are (a_1, a_2, a_3) and (b_1, b_2, b_3) respectively in R^3
- s and t are parameters where $s \in \mathbf{R}$ and $t \in \mathbf{R}$
- (A, B) is a normal to the line defined by $Ax + By + C = 0$ in R^2
- (A, B, C) is a normal to the plane defined by $Ax + By + Cz + D = 0$ in R^3



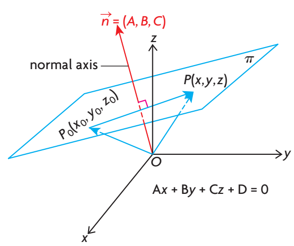
Vector Equation of a Line in R^2



Vector Equation of a Line in R^2



Vector Equation of a Plane in R^3



Scalar Equation of a Plane in R^3

Exercise

1. Determine if the point P_0 is on the given line.

a. $P_0(2, -5), \vec{r} = (10, -12) + t(8, -7), t \in \mathbf{R}$

b. $P_0(1, 2), 12x + 5y - 13 = 0$

c. $P_0(7, -3, 8), \vec{r} = (1, 0, -4) + t(2, -1, 4), t \in \mathbf{R}$

d. $P_0(1, 0, 5), \vec{r} = (2, 1, -2) + t(4, -1, 2), t \in \mathbf{R}$

2. Determine the vector and parametric equations of the line that passes through each of the following pairs of points:

a. $P_1(2, 5), P_2(7, 3)$

b. $P_1(-3, 7), P_2(4, -7)$

c. $P_1(-1, 0), P_2(-3, -11)$

d. $P_1(1, 3, 5), P_2(6, -7, 0)$

e. $P_1(2, 0, -1), P_2(-1, 5, 2)$

f. $P_1(2, 5, -1), P_2(12, -5, -7)$

3. Determine the Cartesian equation of the plane passing through point P_0 and perpendicular to $\vec{n}$.

a. $P_0(4, 1, -3), \vec{n} = (2, 6, -1)$

b. $P_0(-2, 0, 5), \vec{n} = (0, 7, 0)$

c. $P_0(3, -1, -2), \vec{n} = (4, -3, 0)$

d. $P_0(0, 0, 0), \vec{n} = (6, -5, 3)$

e. $P_0(4, 1, 8), \vec{n} = (11, -6, 0)$

f. $P_0(2, 5, 1), \vec{n} = (1, 1, -1)$

4. Determine the Cartesian equation of the plane that has the vector equation $\vec{r} = (2, 1, 0) + s(1, -1, 3) + t(2, 0, -5), s, t \in \mathbf{R}$.

5. Which of the following lines is parallel to the plane $4x + y - z = 10$? Do any of the lines lie on this plane?

L_1: $\vec{r} = (3, 0, 2) + t(1, -2, 2), t \in \mathbf{R}$

L_2: $x = -3t, y = -5 + 2t, z = -10t, t \in \mathbf{R}$

L_3: $\dfrac{x - 1}{4} = \dfrac{y + 6}{-1} = \dfrac{z}{1}$

6. Determine the Cartesian equations of the planes that contain the following sets of points:

a. $A(1, 0, -1), B(2, 0, 0), C(6, -1, 5)$

b. $P(4, 1, -2), Q(6, 4, 0), R(0, 0, -3)$

7. Determine the vector and Cartesian equations of the plane containing $P(1, -4, 3)$ and $Q(2, -1, 6)$ and parallel to the y-axis.

8. Determine the Cartesian equation of the plane that passes through $A(-1, 3, 4)$ and is perpendicular to $2x - y + 3z - 1 = 0$ and $5x + y - 3z + 6 = 0$.

CAREER LINK Investigate

CHAPTER 9: RELATIONSHIPS BETWEEN POINTS, LINES, AND PLANES

Much of the world's reserves of fossil fuels are found in places that are not accessible to water for shipment. Due to the enormous volumes of oil that are currently being extracted from the ground in places such as northern Alberta, Alaska, and Russia, shipment by trucks would be very costly. Instead, pipelines are built to move the fuel to a place where it can be processed or loaded onto a large sea tanker for shipment. The construction of the pipelines is a costly undertaking, but, once completed, pipelines save vast amounts of time, energy, and money.

A team of pipeline construction engineers is needed to design a pipeline. The engineers have to study surveys of the land that the pipeline will cross and choose the best path. Often the least difficult path is above ground, but engineers will choose to have the pipeline go below ground. In plotting the course for the pipeline, vectors can be used to determine if the intended path of the pipeline will cross an obstruction or to determine where two different pipelines will meet.

Case Study—Pipeline Construction Engineer

New pipelines must be a certain distance away from existing pipelines and buildings, depending on the type of product that the pipeline is carrying. To calculate the distance between the two closest points on two pipelines, the lines are treated as skew lines on two different planes. (Skew lines are lines that never intersect because they lie on parallel planes.)

Suppose that an engineer wants to lay a pipeline according to the line L_1: $r = (0, 2, 1) + s(2, -1, 1)$, $s \in \mathbf{R}$. There is an existing pipeline that has a pathway determined by L_2: $r = (1, 0, 1) + t(1, -2, 0)$, $t \in \mathbf{R}$. Determine whether the proposed pathway for the new pipeline is less than 2 units away from the existing pipeline.

DISCUSSION QUESTIONS

1. Construct two parallel planes, π_1 and π_2. The first plane contains L_1 and a second intersecting line that has a direction vector of $\vec{a} = (1, -2, 0)$, the same direction vector as L_2. The second plane contains L_2 and a second intersecting line that has a direction vector of $\vec{b} = (2, -1, 1)$, the same direction vector as L_1.

2. Find the distance between π_1 and π_2.

3. Write the equation of L_1 and L_2 in parametric form.

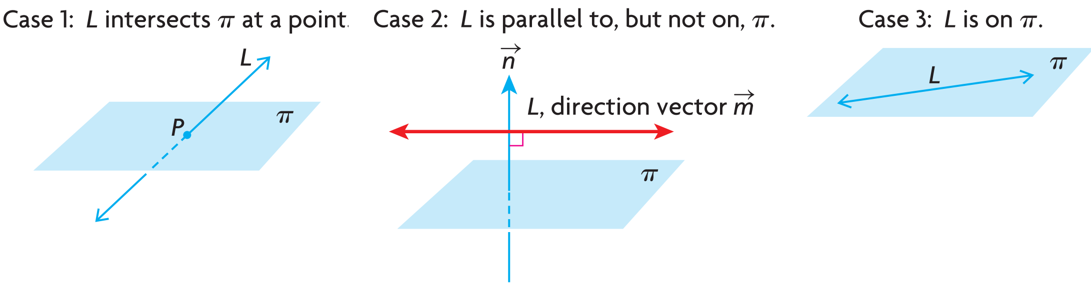
4. Determine the point on each of the two lines in problem 3 that produces the minimal distance.

Section 9.1—The Intersection of a Line with a Plane and the Intersection of Two Lines

We start by considering the intersection of a line with a plane.

Intersection between a Line and a Plane

Before considering mathematical techniques for the solution to this problem, we consider the three cases for the intersection of a line with a plane.



Case 1: The line L intersects the plane π at exactly one point, P.
Case 2: The line L does *not* intersect the plane so it is parallel to the plane. There are no points of intersection.
Case 3: The line L lies on the plane π. Every point on L intersects the plane. There are an infinite number of points of intersection.

For the intersection of a line with a plane, there are three different possibilities, which correspond to zero, one, or an infinite number of intersection points. It is not possible to have a finite number of intersection points, other than zero or one. These three possible intersections are considered in the following examples.

EXAMPLE 1

Selecting a strategy to determine the point of intersection between a line and a plane

Determine points of intersection between the line $L: \vec{r} = (3, 1, 2) + s(1, -4, -8), s \in \mathbf{R}$, and the plane $\pi: 4x + 2y - z - 8 = 0$, if any exist.

Solution

To determine the required point of intersection, first convert the line from its vector form to its corresponding parametric form. The parametric form is $x = 3 + s$, $y = 1 - 4s$, $z = 2 - 8s$. Using parametric equations allows for direct substitution into π.

$$4(3 + s) + 2(1 - 4s) - (2 - 8s) - 8 = 0 \quad \text{(Use substitution)}$$
$$12 + 4s + 2 - 8s - 2 + 8s - 8 = 0 \quad \text{(Isolate } s\text{)}$$
$$4s = -4$$
$$s = -1$$

This means that the point where L meets π corresponds to a single point on the line with a parameter value of $s = -1$. To obtain the coordinates of the required point, $s = -1$ is substituted into the parametric equations of L. The point of intersection is

$$x = 3 + (-1) = 2$$
$$y = 1 - 4(-1) = 5$$
$$z = 2 - 8(-1) = 10$$

Check (by substitution):
The point lies on the plane because
$4(2) + 2(5) - 10 - 8 = 8 + 10 - 10 - 8 = 0.$

The point that satisfies the equation of the plane and the line is $(2, 5, 10)$. Now we consider the situation in which the line does not intersect the plane.

EXAMPLE 2 Connecting the algebraic representation to the situation with no points of intersection

Determine points of intersection between the line
$L: x = 2 + t, y = 2 + 2t, z = 9 + 8t, t \in \mathbf{R}$,
and the plane $\pi: 2x - 5y + z - 6 = 0$, if any exist.

Solution

Method 1:
Because the line L is already in parametric form, we substitute the parametric equations into the equation for π.

$$2(2 + t) - 5(2 + 2t) + (9 + 8t) - 6 = 0 \quad \text{(Use substitution)}$$
$$4 + 2t - 10 - 10t + 9 + 8t - 6 = 0 \quad \text{(Isolate } t\text{)}$$
$$0t = -3$$

Since there is no value of t that, when multiplied by zero, gives -3, there is no solution to this equation. Because there is no solution to this equation, there is no point of intersection. Thus, L and π do not intersect. L is a line that lies on a plane that is parallel to π.

Method 2:
It is also possible to show that the given line and plane do not intersect by first considering $\vec{n} = (2, -5, 1)$, which is the normal for the plane, and $\vec{m} = (1, 2, 8)$, which is the direction vector for the line, and calculating their dot product. If the dot product is zero, this implies that the line is either on the plane or parallel to the plane.

$$\begin{aligned}\vec{n} \times \vec{m} &= (2, -5, 1) \cdot (1, 2, 8) \\ &= 2(1) - 5(2) + 1(8) \\ &= 0\end{aligned}$$ (Definition of dot product)

We can prove that the line does not lie on the plane by showing that the point $(2, 2, 9)$, which we know is on the line, is not on the plane.

Substituting $(2, 2, 9)$ into the equation of the plane, we get
$2(2) - 5(2) + 9 - 6 = -3 \neq 0$.

Since the point does not satisfy the equation of the plane, the point is not on the plane. The line and the plane are parallel and do not intersect.

Next, we examine the intersection of a line and a plane where the line lies on the plane.

EXAMPLE 3 **Connecting the algebraic representation to the situation with infinite points of intersection**

Determine points of intersection of the line $L: \vec{r} = (3, -2, 1) + s(14, -5, -3)$, $s \in \mathbf{R}$, and the plane $x + y + 3z - 4 = 0$, if any exist.

Solution

Method 1:
As before, we convert the equation of the line to its parametric form. Doing so, we obtain the equations $x = 3 + 14s$, $y = -2 - 5s$, and $z = 1 - 3s$.

$$(3 + 14s) + (-2 - 5s) + 3(1 - 3s) - 4 = 0$$ (Use substitution)

$$3 + 14s - 2 - 5s + 3 - 9s - 4 = 0$$ (Isolate s)

$$0s = 0$$

Since any real value of s will satisfy this equation, there are an infinite number of solutions to this equation, each corresponding to a real value of s. Since any value will work for s, every point on L will be a point on the plane. Therefore, the given line lies on the plane.

Method 2:
Again, this result can be achieved by following the same procedure as in the previous example. If $\vec{n} = (1, 1, 3)$ and $\vec{m} = (14, -5, -3)$, then $\vec{n} \times \vec{m} = 1(14) + 1(-5) + 3(-3) = 0$, implying that the line and plane are parallel. We substitute the coordinates $(3, -2, 1)$, which is a point on the line, into the equation of the plane and find that $3 + (-2) + 3(1) - 4 = 0$. So this point lies on the plane as well. Since the line and plane are parallel, and $(3, -2, 1)$ lies on the plane, the entire line lies on the plane.

Next, we consider the intersection of a line with a plane parallel to a coordinate plane.

EXAMPLE 4 **Reasoning about the intersection between a line and the *yz*-plane**

Determine points where $L: x = 2 - s$, $y = -1 + 3s$, $z = 4 - 2s$, $s \in \mathbf{R}$, and $\pi: x = -3$ intersect, if any exist.

Solution

At the point of intersection, the x-values for the line and the plane will be equal.

Equating the two gives $2 - s = -3$, or $s = 5$. The y- and z-values for the point of intersection can now be found by substituting $s = 5$ into the other two parametric equations. Thus, $y = -1 + 3s = -1 + 3(5) = 14$ and $z = 4 - 2s = 4 - 2(5) = -6$. The point of intersection between L and π is $(-3, 14, -6)$.

Intersection between Two Lines

Thus far, we have discussed the possible intersections between a line and a plane. Next, we consider the possible intersection between two lines.

There are four cases to consider for the intersection of two lines in R^3.

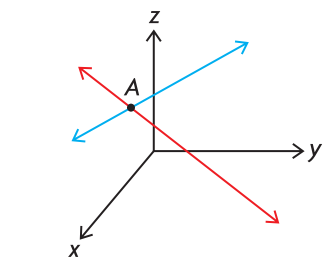
Intersecting Lines
Case 1: The lines are not parallel and intersect at a single point.
Case 2: The lines are coincident, meaning that the two given lines are identical. There are an infinite number of points of intersection.

Non-intersecting Lines
Case 3: The two lines are parallel, and there is no intersection.
Case 4: The two lines are not parallel, and there is no intersection. The lines in this case are called **skew lines**. (Skew lines do not exist in R^2, only in R^3.)

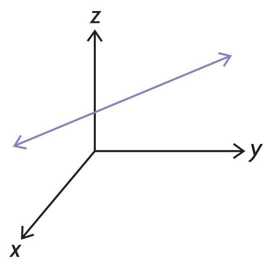
These four cases are shown in the diagram below.

Intersecting Lines

Case 1: Intersecting Lines at a Point

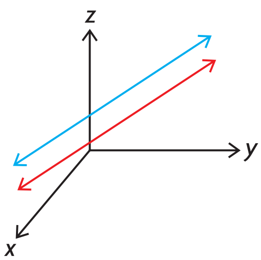


Case 2: Coincident Lines

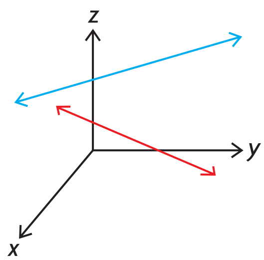


Non-intersecting Lines

Case 3: Parallel Lines



Case 4: Skew Lines



EXAMPLE 5 Selecting a strategy to determine the intersection of two lines in R^3

For $L_1: \vec{r} = (-3, 1, 4) + s(-1, 1, 4), s \in \mathbf{R}$, and $L_2: \vec{r} = (1, 4, 6) + t(-6, -1, 6), t \in \mathbf{R}$, determine points of intersection, if any exist.

Solution

Before calculating the coordinates of points of intersection between the two lines, we note that these lines are not parallel to each other because their direction vectors are not scalar multiples of each other—that is, $(-1, 1, 4) \neq k(-6, -1, 6)$. This indicates that these lines either intersect each other exactly once or are skew lines. If these lines intersect, there must be a single point that is on both lines. To use this idea, the vector equations for L_1 and L_2 must be converted to parametric form.

L_1	L_2
$x = -3 - s$	$x = 1 - 6t$
$y = 1 + s$	$y = 4 - t$
$z = 4 + 4s$	$z = 6 + 6t$

We can now select any two of the three equations from each line and equate them. Comparing the x and y components gives $-3 - s = 1 - 6t$ and $1 + s = 4 - t$. Rearranging and simplifying gives

① $s - 6t = -4$

② $s + t = 3$

Subtracting ② from ① yields the following:

$$-7t = -7$$
$$t = 1$$

Substituting $t = 1$ into equation ①,

$$s - 6(1) = -4$$
$$s = 2$$

We find $s = 2$ and $t = 1$. These two values can now be substituted into the parametric equations to find the corresponding values of x, y, and z.

L_1	L_2
$x = -3 - 2 = -5$	$x = 1 - 6(1) = -5$
$y = 1 + 2 = 3$	$y = 4 - 1 = 3$
$z = 4 + 4(2) = 12$	$z = 6 + 6(1) = 12$

Since we found that substituting $s = 2$ and $t = 1$ into the corresponding parametric equations gives the same values of x, y, and z, the point of intersection is $(-5, 3, 12)$.

It is important to understand that when finding the points of intersection between any pair of lines, the parametric values must be substituted back into the original equations to check that a consistent result is obtained. In other words, $s = 2$ and $t = 1$ must give the same point for each line. In this case, there were consistent values, and so we can be certain that the point of intersection is $(-5, 3, 12)$.

In the next example, we will demonstrate the importance of checking for consistency to find the possible point of intersection.

EXAMPLE 6 Connecting the solution to a system of equations to the case of skew lines

For $L_1 : x = -1 + s, y = 3 + 4s, z = 6 + 5s, s \in \mathbf{R}$, and $L_2 : x = 4 - t, y = 17 + 2t, z = 30 - 5t, t \in \mathbf{R}$, determine points of intersection, if any exist.

Solution

We use the same approach as in the previous example. In this example, we'll start by equating corresponding y- and z-coordinates.

L_1	L_2
$x = -1 + s$	$x = 4 - t$
$y = 3 + 4s$	$y = 17 + 2t$
$z = 6 + 5s$	$z = 30 - 5t$

Comparing y- and z-values, we get $3 + 4s = 17 + 2t$ and $6 + 5s = 30 - 5t$. Rearranging and simplifying gives

① $4s - 2t = 14$

② $5s + 5t = 24$

③ $10s - 5t = 35$ $\quad \frac{5}{2} \times$ ①

$15s = 59$ $\quad$ ② + ③

$$s = \frac{59}{15}$$

If $s = \frac{59}{15}$ is substituted into either equation ① or equation ②, we obtain the value of t.

Substituting into equation ①,

$$4\left(\frac{59}{15}\right) - 2t = 14$$

$$\frac{236}{15} - \frac{210}{15} = 2t$$

$$t = \frac{13}{15}$$

We found that $s = \frac{59}{15}$ and $t = \frac{13}{15}$. These two values can now be substituted back into the parametric equations to find the values of x, y, and z.

L_1	L_2
$x = -1 + \frac{59}{15} = \frac{44}{15}$	$x = 4 - \frac{13}{15} = \frac{47}{15}$
$y = 3 + 4\left(\frac{59}{15}\right) = \frac{281}{15}$	$y = 17 + 2\left(\frac{13}{15}\right) = \frac{281}{15}$
$z = 6 + 5\left(\frac{59}{15}\right) = \frac{77}{3}$	$z = 30 - 5\left(\frac{13}{15}\right) = \frac{77}{3}$

For these lines to intersect at a point, we must obtain equal values for each coordinate. From observation, we can see that the x-coordinates are different, which implies that these lines do not intersect. Since the two given lines do not intersect and have different direction vectors, they must be skew lines.

IN SUMMARY

Key Ideas

- Line and plane intersections can occur in three different ways.

  Case 1: The line L intersects the plane π at exactly one point, P.

  Case 2: The line L does *not* intersect the plane and is parallel to the plane π. In this case, there are no points of intersection and solving the system of equations results in an equation that has no solution ($0 \times$ variable $=$ a nonzero number).

  Case 3: The line L lies on the plane π. In this case, there are an infinite number of points of intersection between the line and the plane, and solving the system of equations results in an equation with an infinite number of solutions ($0 \times$ variable $= 0$).

- Line and line intersections can occur in four different ways.

  Case 1: The lines intersect at a single point.

  Case 2: The two lines are parallel, and there is no intersection.

  Case 3: The two lines are not parallel and do not intersect. The lines in this case are called *skew lines*.

  Case 4: The two lines are parallel and coincident. They are the same line.

Exercise 9.1

PART A

1. Tiffany is given the parametric equations for a line L and the Cartesian equation for a plane π and is trying to determine their point of intersection. She makes a substitution and gets $(1 + 5s) - 2(2 + s) - 3(-3 + s) - 6 = 0$.

   a. Give a possible equation for both the line and the plane.

   b. Finish the calculation, and describe the nature of the intersection between the line and the plane.

2. a. If a line and a plane intersect, in how many different ways can this occur? Describe each case.

   b. It is only possible to have zero, one, or an infinite number of intersections between a line and a plane. Explain why it is not possible to have a finite number of intersections, other than zero or one, between a line and a plane.

C 3. A line has the equation $\vec{r} = s(1, 0, 0)$, $s \in \mathbf{R}$, and a plane has the equation $y = 1$.

   a. Describe the line.

   b. Describe the plane.

c. Sketch the line and the plane.

d. Describe the nature of the intersection between the line and the plane.

PART B

4. For each of the following, show that the line lies on the plane with the given equation. Explain how the equation that results implies this conclusion.

   a. $L: x = -2 + t, y = 1 - t, z = 2 + 3t, t \in \mathbf{R}; \pi: x + 4y + z - 4 = 0$

   b. $L: \vec{r} = (1, 5, 6) + t(1, -2, -2), t \in \mathbf{R}; \pi: 2x - 3y + 4z - 11 = 0$

5. For each of the following, show that the given line and plane do not intersect. Explain how the equation that results implies there is no intersection.

   a. $L: \vec{r} = (-1, 1, 0) + s(-1, 2, 2), s \in \mathbf{R}; \pi: 2x - 2y + 3z - 1 = 0$

   b. $L: x = 1 + 2t, y = -2 + 5t, z = 1 + 4t, t \in \mathbf{R};$
   $\pi: 2x - 4y + 4z - 13 = 0$

6. Verify your results for question 5 by showing that the direction vector of the line and the normal for the plane meet at right angles, and the given point on the line does not lie on the plane.

7. For the following, determine points of intersection between the given line and plane, if any exist:

   a. $L: \vec{r} = (-1, 3, 4) + p(6, 1, -2), p \in \mathbf{R}; \pi: x + 2y - z + 29 = 0$

   b. $L: \dfrac{x - 1}{4} = \dfrac{y + 2}{-1} = z - 3; \pi: 2x + 7y + z + 15 = 0$

8. Determine points of intersection between the following pairs of lines, if any exist:

   a. $L_1: \vec{r} = (3, 1, 5) + s(4, -1, 2), s \in \mathbf{R};$
   $L_2: x = 4 + 13t, y = 1 - 5t, z = 5t, t \in \mathbf{R}$

   b. $L_3: \vec{r} = (3, 7, 2) + m(1, -6, 0), m \in \mathbf{R};$
   $L_4: \vec{r} = (-3, 2, 8) + s(7, -1, -6), s \in \mathbf{R}$

K

9. Determine which of the following pairs of lines are skew lines:

   a. $\vec{r} = (-2, 3, 4) + p(6, -2, 3), p \in \mathbf{R};$
   $\vec{r} = (-2, 3, -4) + q(6, -2, 11), q \in \mathbf{R}$

   b. $\vec{r} = (4, 1, 6) + t(1, 0, 4), t \in \mathbf{R}; \vec{r} = (2, 1, -8) + s(1, 0, 5), s \in \mathbf{R}$

   c. $\vec{r} = (2, 2, 1) + m(1, 1, 1), m \in \mathbf{R};$
   $\vec{r} = (-2, 2, 1) + p(3, -1, -1), p \in \mathbf{R}$

   d. $\vec{r} = (9, 1, 2) + m(5, 0, 4), m \in \mathbf{R}; \vec{r} = (8, 2, 3) + s(4, 1, -2), s \in \mathbf{R}$

10. The line with the equation $\vec{r} = (-3, 2, 1) + s(3, -2, 7), s \in \mathbf{R}$, intersects the z-axis at the point $Q(0, 0, q)$. Determine the value of q.

11. a. Show that the lines $L_1 : \vec{r} = (-2, 3, 4) + s(7, -2, 2)$, $s \in \mathbf{R}$, and $L_2 : \vec{r} = (-30, 11, -4) + t(7, -2, 2)$, $t \in \mathbf{R}$, are coincident by writing each line in parametric form and comparing components

b. Show that the point $(-2, 3, 4)$ lies on L_2. How does this show that the lines are coincident?

12. The lines $\vec{r} = (-3, 8, 1) + s(1, -1, 1)$, $s \in \mathbf{R}$, and $\vec{r} = (1, 4, 2) + t(-3, k, 8)$, $t \in \mathbf{R}$, intersect at a point.

a. Determine the value of k.

b. What are the coordinates of the point of intersection?

A 13. The line $\vec{r} = (-8, -6, -1) + s(2, 2, 1)$, $s \in \mathbf{R}$, intersects the xz- and yz-coordinate planes at the points A and B, respectively. Determine the length of line segment AB.

14. The lines $\vec{r} = (2, 1, 1) + p(4, 0, -1)$, $p \in \mathbf{R}$, and $\vec{r} = (3, -1, 1) + q(9, -2, -2)$, $q \in \mathbf{R}$, intersect at the point A.

a. Determine the coordinates of point A.

b. What is the distance from point A to the xy-plane?

T 15. The lines $\vec{r} = (-1, 3, 2) + s(5, -2, 10)$, $s \in \mathbf{R}$, and $\vec{r} = (4, -1, 1) + t(0, 2, 11)$, $t \in \mathbf{R}$, intersect at point A.

a. Determine the coordinates of point A.

b. Determine the vector equation for the line that is perpendicular to the two given lines and passes through point A.

16. a. Sketch the lines $L_1 : \vec{r} = p(0, 1, 0)$, $p \in \mathbf{R}$, and $L_2 : \vec{r} = q(0, 1, 1)$, $q \in \mathbf{R}$.

b. At what point do these lines intersect?

c. Verify your conclusion for part b. algebraically.

PART C

17. a. Show that the lines $\frac{x}{1} = \frac{y - 7}{-8} = \frac{z - 1}{2}$ and $\frac{x - 4}{3} = \frac{z - 1}{-2}$, $y = -1$, lie on the plane with equation $2x + y + 3z - 10 = 0$.

b. Determine the point of intersection of these two lines.

18. A line passing through point $P(-4, 0, -3)$ intersects the two lines with equations $L_1 : \vec{r} = (1, 1, -1) + s(1, 1, 0)$, $s \in \mathbf{R}$, and $L_2 : \vec{r} = (0, 1, 3) + t(-2, 1, 3)$, $t \in \mathbf{R}$. Determine a vector equation for this line.

Section 9.2—Systems of Equations

To solve problems in real-life situations, we often need to solve systems of linear equations. Thus far, we have seen systems of linear equations in a variety of different contexts dealing with lines and planes. The following is a typical example of a system of two equations in two unknowns:

① $2x + y = -9$

② $x + 2y = -6$

Each of the equations in this system is a linear equation. A linear equation is an equation of the form $a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n = b$, where $a_1, a_2, a_3, \ldots, a_n$ and b are real numbers with the variables $x_1, x_2, x_3, \ldots, x_n$ being the unknowns. Typical examples of linear equations are $y = 2x - 3$, $x + 4y = 9$, and $x + 3y - 2z - 2 = 0$. All the variables in each of these equations are raised to the first power only (degree of one). Linear equations do not include any products or powers of variables, and there are no trigonometric, logarithmic, or exponential functions making up part of the equation. Typical examples of nonlinear equations are $x - 3y^2 = 3$, $2x - xyz = 4$, and $y = \sin 2x$.

A system of linear equations is a set of one or more linear equations. When we solve a system of linear equations, we are trying to find values that will simultaneously satisfy the unknowns in each of the equations. In the following example, we consider a system of two equations in two unknowns and possible solutions for this system.

EXAMPLE 1 Reasoning about the solutions to a system of two equations in two unknowns

The number of solutions to the following system of equations depends on the value(s) of a and b. Determine values of a and b for which this system has no solutions, an infinite number of solutions, and one solution.

① $x + 4y = a$

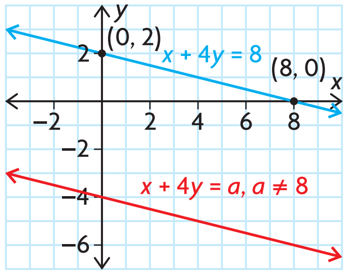
② $x + by = 8$

Solution

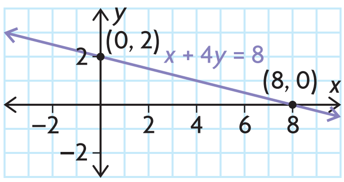
Each of the equations in this system represents a line in R^2. For these two lines, there are three cases to consider, each depending on the values of a and b.

Case 1: These equations represent two parallel and non-coincident lines. If these lines are parallel, they must have the same slope, implying that $b = 4$.

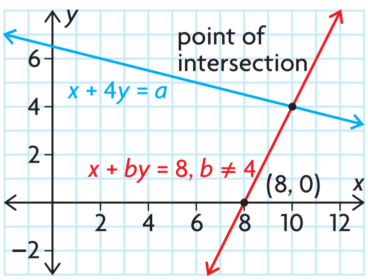
This means that the second equation is $x + 4y = 8$, and the slope of each line is $-\frac{1}{4}$. If $a \neq 8$, this implies that the two lines are parallel and have different equations. Since the lines would be parallel and not intersect, there is no solution to this system when $b = 4$ and $a \neq 8$.



Case 2: These equations represent two parallel and coincident lines. This means that the two equations must be equivalent. If $a = 8$ and $b = 4$, then both equations are identical and this system would be reduced to finding values of x and y that satisfy the equation $x + 4y = 8$. Since there are an infinite number of points that satisfy this equation, the original system will have an infinite number of solutions.



Case 3: These two equations represent two intersecting, non-coincident lines. The third possibility for these two lines is that they intersect at a single point in R^2. These lines will intersect at a single point if they are not parallel—that is, if $b \neq 4$. In this case, the solution is the point of intersection of these lines.



This system of linear equations is typical in that it can only have zero, one, or an infinite number of solutions. In general, it is not possible for any system of linear equations to have a finite number of solutions greater than one.

Number of Solutions to a Linear System of Equations

A linear system of equations can have zero, one, or an infinite number of solutions.

In Example 2, the idea of equivalent systems is introduced as a way of understanding how to solve a system of equations. Equivalent systems of equations are defined in the following way:

Definition of Equivalent Systems

Two systems of equations are defined as equivalent if every solution to one system is also a solution to the second system of equations, and vice versa.

The idea of equivalent systems is important because, when solving a system of equations, what we are attempting to do is create a system of equations that is easier to solve than the previous system. To construct an equivalent system of equations, the new system is obtained in a series of steps using a set of well-defined operations. These operations are referred to as **elementary operations**.

Elementary Operations Used to Create Equivalent Systems

1. Multiply an equation by a nonzero constant.
2. Interchange any pair of equations.
3. Add a nonzero multiple of one equation to a second equation to replace the second equation.

In previous courses, when we solved systems of equations, we often multiplied two equations by different constants and then added or subtracted to eliminate variables. Although these kinds of operations can be used algebraically to solve systems, elementary operations are used because of their applicability in higher-level mathematics.

The use of elementary operations to create equivalent systems is illustrated in the following example.

EXAMPLE 2 Using elementary operations to solve a system of two equations in two unknowns

Solve the following system of equations:

① $2x + y = -9$

② $x + 2y = -6$

Solution

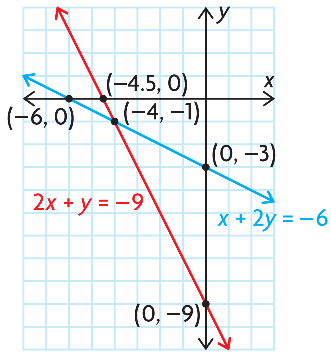
1: Interchange equations ① and ②.

① $x + 2y = -6$

② $2x + y = -9$

The equations have been interchanged to make the coefficient of x in the first equation equal to 1. This is always a good strategy when solving systems of linear equations.

This original system of equations is illustrated in the following diagram.



After Step 1

2: Multiply equation ① by -2, and then add equation ② to eliminate the variable x from the second equation to create equation ③. Note that the coefficient of the x-term in the new equation is 0.

① $x + 2y = -6$

③ $0x - 3y = 3 \qquad -2 \times ① + ②$

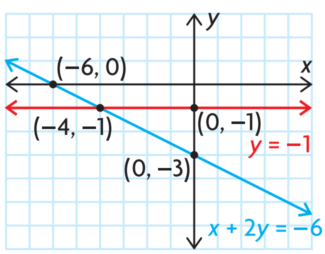
When solving a system of equations, the elementary operations that are used are specified beside the newly created equation.

3: Multiply each side of equation ③ by $-\frac{1}{3}$ to obtain a new equation that is labelled equation ④.

① $x + 2y = -6$

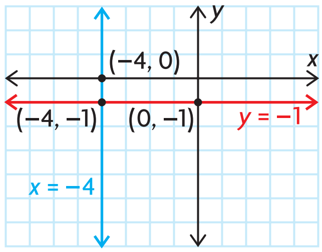
④ $0x + y = -1 \qquad -\dfrac{1}{3} \times ③$

This new system of equations is illustrated in the diagram below.



After Step 3

The new system of equations that we produced is easier to solve than the original system. If we substitute $y = -1$ into equation ①, we obtain $x + 2(-1) = -6$ or $x = -4$.
The solution to this system of equations is $x = -4$, $y = -1$, which is shown in the graph below.



Final system

The equivalence of the systems of equations was illustrated geometrically at different points in the calculations. As each elementary operation is applied, we create an equivalent system such that the two lines always have the point $(-4, -1)$ in common. When we create the various equivalent systems, the solution to each set of equations remains the same. This is what we mean when we use elementary operations to create equivalent systems.

The solution to the original system of equations is $x = -4$ and $y = -1$. This means that these two values of x and y must satisfy each of the given equations. It is easy to verify that these values satisfy each of the given equations. For the first equation, $2(-4) + (-1) = -9$. For the second equation, $-4 + 2(-1) = -6$.

A solution to a system of equations must satisfy each equation in the system for it to be a solution to the overall system. This is demonstrated in the following example.

EXAMPLE 3 Reasoning about the solution to a system of two equations in three unknowns

Determine whether $x = -3$, $y = 5$, and $z = 6$ is a solution to the following system:

① $2x + 3y - 5z = -21$

② $x - 6y + 6z = 8$

Solution

For the given values to be a solution to this system of equations, they must satisfy both equations.

Substituting into the first equation,

$2(-3) + 3(5) - 5(6) = -6 + 15 - 30 = -21$

Substituting into the second equation,

$-3 - 6(5) + 6(6) = -3 - 30 + 36 = 3 \neq 8$

Since the values of x, y, and z do not satisfy both equations, they are not a solution to this system.

If a system of equations has no solutions, it is said to be **inconsistent**. If a system has at least one solution, it is said to be **consistent**.

Consistent and Inconsistent Systems of Equations

A system of equations is consistent if it has either one solution or an infinite number of solutions. A system is inconsistent if it has no solutions.

In the next example, we show how to use elementary operations to solve a system of three equations in three unknowns.

EXAMPLE 4 Using elementary operations to solve a system of three equations in three unknowns

Solve the following system of equations for x, y, and z using elementary operations:

① $x - y + z = 1$

② $2x + y - z = 11$

③ $3x + y + 2z = 12$

Solution

1: Use equation ① to eliminate x from equations ② and ③.

① $x - y + z = 1$

④ $0x + 3y - 3z = 9$ $\quad -2 \times$ ① $+$ ②

⑤ $0x + 4y - z = 9$ $\quad -3 \times$ ① $+$ ③

2: Use equations ④ and ⑤ to eliminate y from equation ⑤, and then scale equation ④.

① $x - y + z = 1$

⑥ $0x + y - z = 3$ $\quad \frac{1}{3} \times$ ④

⑦ $0x + 0y + 3z = -3$ $\quad -\frac{4}{3} \times$ ④ $+$ ⑤

We can now solve this system by using a method known as **back substitution**. Start by solving for z in equation ⑦, and use this value to solve for y in equation ⑥. From there, we use the values for y and z to solve for x in equation ①.

From equation 7, $3z = -3$

$$z = -1$$

If we then substitute into equation 6,

$$y - (-1) = 3$$

$$y = 2$$

If $y = 2$ and $z = -1$, these values can now be substituted into equation 1 to obtain $x - 2 + (-1) = 1$, or $x = 4$.

Therefore, the solution to this system is $(4, 2, -1)$.

Check:
These values should be substituted into each of the original equations and checked to see that they satisfy each equation.

To solve this system of equations, we used elementary operations and ended up with a triangle of zeros in the lower left part:

$$ax + by + cz = d$$
$$0x + ey + fz = g$$
$$0x + 0y + hz = i$$

The use of elementary operations to create the lower triangle of zeros is our objective when solving systems of equations. Large systems of equations are solved using computers and elementary operations to eliminate unknowns. This is by far the most efficient and cost-effective method for their solution.

In the following example, we consider a system of equations with different possibilities for its solution.

EXAMPLE 5

Connecting the value of a parameter to the nature of the intersection between two lines in R^2

Consider the following system of equations:

1 $x + ky = 4$
2 $kx + 4y = 8$

Determine the value(s) of k for which this system of equations has

a. no solutions
b. one solution
c. an infinite number of solutions

Solution

Original System of Equations:

1 $x + ky = 4$
2 $kx + 4y = 8$

1: Multiply equation ① by $-k$, and add it to equation ② to eliminate x from equation ②.

① $x + ky = 4$

③ $0x - k^2y + 4y = -4k + 8, -k \times ① + ②$

Actual Solution to Problem:
To solve the problem, it is only necessary to deal with the equation $0x - k^2y + 4y = -4k + 8$ to determine the necessary conditions on k.

$$-k^2y + 4y = -4k + 8 \qquad \text{(Factor)}$$

$$y(-k^2 + 4) = -4(k - 2) \qquad \text{(Multiply by } -1\text{)}$$

$$y(k^2 - 4) = 4(k - 2)$$

$$(k - 2)(k + 2)y = 4(k - 2)$$

There are three different cases to consider.

Case 1: $k = 2$
If $k = 2$, this results in the equation $(2 - 2)(2 + 2)y = 4(2 - 2)$, or $0y = 0$.

Since this equation is true for all real values of y, we will have an infinite number of solutions. Substituting $k = 2$ into the original system of equations gives

① $x + 2y = 4$

② $2(x + 2y) = 2(4)$

This system can then be reduced to just a single equation, $x + 2y = 4$, which, as we have seen, has an infinite number of solutions.

Case 2: $k = -2$
If $k = -2$, this equation becomes $(-2 - 2)(-2 + 2)y = 4(-2 - 2)$, or $0y = -16$.

There are no solutions to this equation. Substituting $k = -2$ into the original system of equations gives

① $x - 2y = 4$

② $-2(x - 2y) = -2(-4)$

This system can be reduced to the two equations, $x - 2y = 4$ and $x - 2y = -4$, which are two parallel lines that do not intersect. Thus, there are no solutions.

Case 3: $k \neq \pm 2$
If $k \neq \pm 2$, we get an equation of the form $ay = b$, $a \neq 0$. This equation will always have a unique solution for y, which implies that the original system of equations will have exactly one solution, provided that $k \neq \pm 2$.

IN SUMMARY

Key Idea

- A system of two (linear) equations in two unknowns geometrically represents two lines in R^2. These lines may intersect at zero, one, or an infinite number of points, depending on how the lines are related to each other.

Need to Know

- Elementary operations can be used to solve a system of equations. The operations are defined as follows:
  1. Multiply an equation by a nonzero constant.
  2. Interchange any pair of equations.
  3. Add a multiple of one equation to a second equation to replace the second equation.

  As each elementary operation is applied, we create an equivalent system, which gets progressively easier to solve.
- The solution to a system of equations consists of the values of the variables that satisfy all the equations in the system simultaneously.
- A system of equations is consistent if it has either one solution or an infinite number of solutions. The system is inconsistent if it has no solutions.

Exercise 9.2

PART A

1. Given that k is a nonzero constant, which of the following are linear equations?

   a. $kx - \frac{1}{k}y = 3$

   b. $2 \sin x = kx$

   c. $2^k x + 3y - z = 0$

   d. $\frac{1}{x} - y = 3$

2. a. Create a system of three equations in three unknowns that has $x = -3$, $y = 4$, and $z = -8$ as its solution.

   b. Solve this system of equations using elementary operations.

3. Determine whether $x = -7$, $y = 5$, and $z = \frac{3}{4}$ is a solution to the following systems:

   a. ① $x - 3y + 4z = -19$
   ② $x - 8z = -13$
   ③ $x + 2y = 3$

   b. ① $3x - 2y + 16z = -19$
   ② $3x - 2y = -23$
   ③ $8x - y + 4z = -58$

PART B

4. Solve each system of equations, and state whether the systems given in parts a. and b. are equivalent or not. Explain.

a. ① $x = -2$
② $3y = -9$

b. ① $3x + 5y = -21$
② $\frac{1}{6}x - \frac{1}{2}y = \frac{7}{6}$

K 5. Solve each of the following systems using elementary operations:

a. ① $2x - y = 11$
② $x + 5y = 11$

b. ① $2x + 5y = 19$
② $3x + 4y = 11$

c. ① $-x + 2y = 10$
② $3x + 5y = 3$

C 6. Solve the following systems of equations, and explain the nature of each intersection:

a. ① $2x + y = 3$
② $2x + y = 4$

b. ① $7x - 3y = 9$
② $35x - 15y = 45$

7. Write a solution to each equation using parameters.

a. $2x - y = 3$

b. $x - 2y + z = 0$

8. a. Determine a linear equation that has $x = t$, $y = -2t - 11$, $t \in \mathbf{R}$, as its general solution.

b. Show that $x = 3t + 3$, $y = -6t - 17$, $t \in \mathbf{R}$, is also a general solution to the linear equation found in part a.

9. Determine the value(s) of the constant k for which the following system of equations has

a. no solutions
b. one solution
c. infinitely many solutions

① $x + y = 6$
② $2x + 2y = k$

10. For the equation $2x + 4y = 11$, determine

a. the number of solutions
b. a generalized parametric solution
c. an explanation as to why it will *not* have any integer solutions

11. a. Solve the following system of equations for x and y:

① $x + 3y = a$
② $2x + 3y = b$

b. Explain why this system of equations will always be consistent, irrespective of the values of a and b.

12. Solve each system of equations using elementary operations.

a. ① $x + y + z = 0$
② $x - y = 1$
③ $y - z = -5$

b. ① $2x - 3y + z = 6$
② $x + y + 2z = 31$
③ $x - 2y - z = -17$

c. ① $x + y = 10$
② $y + z = -2$
③ $x + z = -4$

d. ① $\frac{x}{3} + \frac{y}{4} + \frac{z}{5} = 14$
② $\frac{x}{4} + \frac{y}{5} + \frac{z}{3} = -21$
③ $\frac{x}{5} + \frac{y}{3} + \frac{z}{4} = 7$

e. ① $2x - y = 0$
② $2y - z = 7$
③ $2z - x = 0$

f. ① $x + y + 2z = 13$
② $2y - 3z = -12$
③ $x - y + 4z = 19$

A 13. A system of equations is given by the lines L_1: $ax + by = p$, L_2: $dx + ey = q$, and L_3: $gx + hy = r$. Sketch the lines under the following conditions:

a. when the system of equations represented by these lines has no solutions

b. when the system of equations represented by these lines has exactly one solution

c. when the system of equations represented by these lines has an infinite number of solutions

T 14. Determine the solution to the following system of equations:

① $x + y + z = a$
② $x + y = b$
③ $y + z = c$

PART C

15. Consider the following system of equations:

① $x + 2y = -1$
② $2x + k^2y = k$

Determine the values of k for which this system of equations has

a. no solutions

b. an infinite number of solutions

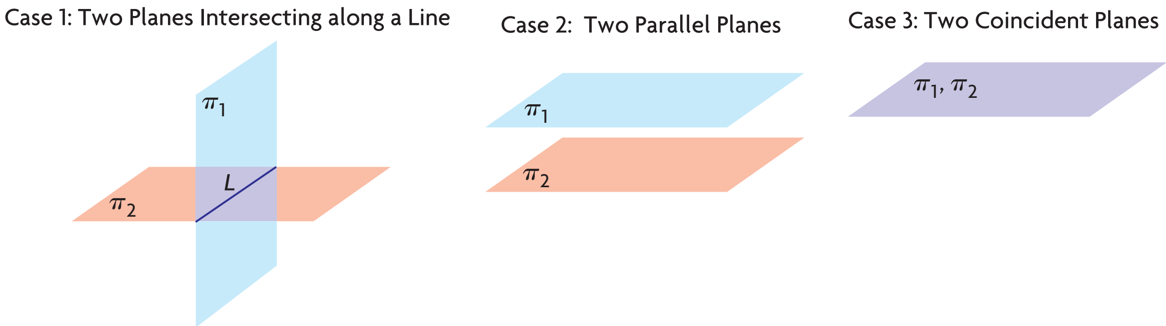
c. a unique solution

Section 9.3—The Intersection of Two Planes

In the previous section, we introduced elementary operations and their use in the solution of systems of equations. In this section, we will again examine systems of equations but will focus specifically on dealing with the intersections of two planes. Algebraically, these are typically represented by a system of two equations in three unknowns.

In our discussion on the intersections of two planes, there are three different cases to be considered, each of which is illustrated below.

Possible Intersections for Two Planes



Case 1: Two planes can intersect along a line. The corresponding system of equations will therefore have an infinite number of solutions.

Case 2: Two planes can be parallel and non-coincident. The corresponding system of equations will have no solutions.

Case 3: Two planes can be coincident and will have an infinite number of solutions.

Solutions for a System of Equations Representing Two Planes

The system of equations corresponding to the intersection of two planes will have either zero solutions or an infinite number of solutions.

It is not possible for two planes to intersect at a single point.

EXAMPLE 1

Reasoning about the nature of the intersection between two planes (Case 2)

Determine the solution to the system of equations $x - y + z = 4$ and $x - y + z = 5$. Discuss how these planes are related to each other.

Solution

Since the two planes have the same normals, $\vec{n_1} = \vec{n_2} = (1, -1, 1)$, this implies that the planes are parallel. Since the equations have different constants on the right side, the equations represent parallel and non-coincident planes. This indicates that there are no solutions to this system because the planes do not intersect.

The corresponding system of equations is

① $x - y + z = 4$

② $x - y + z = 5$

Using elementary operations, the following equivalent system of equations is obtained:

① $\quad x - y + z = 4$

③ $0x + 0y + 0z = 1 \qquad -1 \times$ ① + ②

Since there are no values that satisfy equation ③, there are no solutions to this system, confirming our earlier conclusion.

EXAMPLE 2

Reasoning about the nature of the intersection between two planes (Case 3)

Determine the solution to the following system of equations:

① $\quad x + 2y - 3z = -1$

② $4x + 8y - 12z = -4$

Solution

Since equation ② can be written as $4(x + 2y - 3z) = 4(-1)$, the two equations represent coincident planes. This means that there are an infinite number of values that satisfy the system of equations. The solution to the system of equations can be written using parameters in equation ①. If we let $y = s$ and $z = t$, then $x = -2s + 3t - 1$.

The solution to the system is $x = -2s + 3t - 1$, $y = s$, $z = t$, $s, t \in \mathbf{R}$. This is the equation of a plane, expressed in parametric form. Every point that lies on the plane is a solution to the given system of equations.

If we had solved the system using elementary operations, we would have arrived at the following equivalent system:

① $x + 2y - 3z = -1$

③ $0x + 0y + 0z = 0 \qquad -4 \times ① + ②$

There are an infinite number of ordered triples (x, y, z) that satisfy both equations ① and ③, confirming our earlier conclusion.

The normals of two planes give us important information about their intersection.

Intersection of Two Planes and their Normals

If the planes π_1 and π_2 have $\vec{n_1}$ and $\vec{n_2}$ as their respective normals, we know the following:

1. If $\vec{n_1} = k\vec{n_2}$ for some scalar, k, the planes are coincident or they are parallel and non-coincident. If they are coincident, there are an infinite number of points of intersection. If they are parallel and non-coincident, there are no points of intersection.
2. If $\vec{n_1} \neq k\vec{n_2}$, the two planes intersect in a line. This results in an infinite number of points of intersection.

EXAMPLE 3 Reasoning about the nature of the intersection between two planes (Case 1)

Determine the solution to the following system of equations:

① $x - y + z = 3$

② $2x + 2y - 2z = 3$

Solution

When solving a system involving two planes, it is useful to start by determining the normals for the two planes. The first plane has normal $\vec{n_1} = (1, -1, 1)$, and the second plane has $\vec{n_2} = (2, 2, -2)$. Since these vectors are not scalar multiples of each other, the normals are not parallel, which implies that the two planes intersect. Since the planes intersect and do not coincide, they intersect along a line.

We will use elementary operations to solve the system.

① $x - y + z = 3$

③ $0x + 4y - 4z = -3 \qquad -2 \times ① + ②$

To determine the equation of the line of intersection, a parameter must be introduced. From equation ③, which is written as $4y - 4z = -3$, we start by letting $z = s$. Substituting $z = s$ gives

$$\begin{aligned} 4y - 4s &= -3 \\ 4y &= 4s - 3 \\ y &= s - \frac{3}{4} \end{aligned}$$

Substituting $z = s$ and $y = s - \frac{3}{4}$ into equation ①, we obtain

$$\begin{aligned} x - \left(s - \frac{3}{4}\right) + s &= 3 \\ x &= \frac{9}{4} \end{aligned}$$

Therefore, the line of intersection expressed in parametric form is $x = \frac{9}{4}, y = s - \frac{3}{4}, z = s, s \in \mathbf{R}$.

Check:
To check, we'll substitute into each of the two original equations.

Substituting into equation ①,

$$x - y + z = \frac{9}{4} - \left(s - \frac{3}{4}\right) + s = \frac{9}{4} + \frac{3}{4} - s + s = 3$$

Substituting into equation ②,

$$2x + 2y - 2z = 2\left(\frac{9}{4}\right) + 2\left(s - \frac{3}{4}\right) - 2s = \frac{9}{2} - \frac{3}{2} = 3$$

This confirms our conclusion.

EXAMPLE 4 **Selecting the most efficient strategy to determine the intersection between two planes**

Determine the solution to the following system of equations:

① $2x - y + 3z = -2$

② $x - 3z = 1$

Solution

As in the first example, we note that the first plane has normal $\overrightarrow{n_1} = (2, -1, 3)$ and the second $\overrightarrow{n_2} = (1, 0, -3)$. These normals are not scalar multiples of each other, implying that the two planes have a line of intersection.

To find the line of intersection, it is not necessary to use elementary operations to reduce one of the equations. Since the second equation is missing a y-term, the best approach is to write the second equation using a parameter for z. If $z = s$, then $x = 3s + 1$. Now it is a matter of substituting these parametric values into the first equation and determining y in terms of s. Substituting gives

$$\begin{aligned} 2(3s + 1) - y + 3(s) &= -2 \\ 6s + 2 - y + 3s &= -2 \\ 9s + 4 &= y \end{aligned}$$

The line of intersection is given by the parametric equations $x = 3s + 1$, $y = 9s + 4$, and $z = s$, $s \in \mathbf{R}$.

Check:
Substituting into equation ①,
$2(3s + 1) - (9s + 4) + 3s = 6s + 2 - 9s - 4 + 3s = -2$
Substituting into equation ②, $(3s + 1) - 3s = 1$

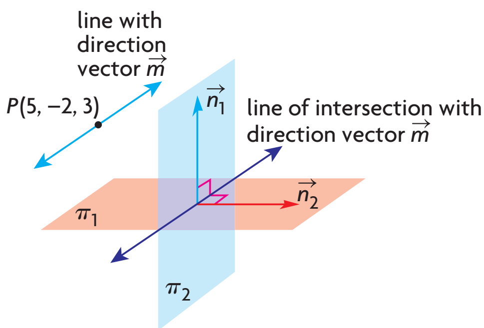
In the next example, we will demonstrate how a problem involving the intersection of two planes can be solved in more than one way.

EXAMPLE 5 Selecting a strategy to solve a problem involving two planes

Determine an equation of a line that passes through the point $P(5, -2, 3)$ and is parallel to the line of intersection of the planes π_1: $x + 2y - z = 6$ and π_2: $y + 2z = 1$.

Solution

Method 1:
Since the required line is parallel to the line of intersection of the planes, then the direction vectors for both of these lines must be parallel. Since the line of intersection is contained in both planes, its direction vector must then be perpendicular to the normals of each plane.



If $\vec{m}$ represents the direction vector of the required line, and it is perpendicular to $\vec{n_1} = (1, 2, -1)$ and $\vec{n_2} = (0, 1, 2)$, then we can choose $\vec{m} = \vec{n_1} \times \vec{n_2}$.

$$\begin{aligned} \text{Thus, } \vec{m} &= (1, 2, -1) \times (0, 1, 2) \\ &= (2(2) - (-1)(1), -1(0) - 1(2), 1(1) - 2(0)) \\ &= (5, -2, 1) \end{aligned}$$

Thus, the required line that passes through $P(5, -2, 3)$ and has direction vector $\vec{m} = (5, -2, 1)$ has parametric equations $x = 5 + 5t$, $y = -2 - 2t$, and $z = 3 + t$, $t \in \mathbf{R}$.

Method 2:
We start by finding the equation of the line of intersection between the two planes. In equation ②, if $z = t$, then $y = -2t + 1$ by substitution. Substituting these values into equation ① gives

$$
\begin{aligned}
x + 2(-2t + 1) - t &= 6 \\
x - 4t + 2 - t &= 6 \\
x &= 5t + 4
\end{aligned}
$$

The line of intersection has $x = 5t + 4$, $y = -2t + 1$, and $z = t$ as its parametric equations $t \in \mathbf{R}$. Since the direction vector for this line is $(5, -2, 1)$, we can choose the direction vector for the required line to also be $(5, -2, 1)$.

The equation for the required line is $x = 5 + 5t$, $y = -2 - 2t$, $z = 3 + t$, $t \in \mathbf{R}$.

IN SUMMARY

Key Ideas

- A system of two (linear) equations in three unknowns geometrically represents two planes in R^3. These planes may intersect at zero points or an infinite number of points, depending on how the planes are related to each other.

  Case 1: Two planes can intersect along a line and will therefore have an infinite number of points of intersection.

  Case 2: Two planes can be parallel and non-coincident. In this case, there are no points of intersection.

  Case 3: Two planes can be coincident and will have an infinite number of points of intersection.

Need to Know

- If the normals of two planes are known, examining how these are related to each other provides information about how the two planes are related.

  If planes π_1 and π_2 have $\vec{n_1}$ and $\vec{n_2}$ as their respective normals, we know the following:

  1. If $\vec{n_1} = k\vec{n_2}$ for some scalar k, the planes are either coincident or they are parallel and non-coincident. If they are coincident, there are an infinite number of points of intersection, and if they are parallel and non-coincident, there are no points of intersection.
  2. If $\vec{n_1} \neq k\vec{n_2}$ for some scalar k, the two planes intersect in a line. This results in an infinite number of points of intersection.

Exercise 9.3

PART A

1. A system of two equations in three unknowns has been manipulated, and, after correctly using elementary operations, a student arrives at the following equivalent system of equations:

   ① $x - y + z = 1$
   ③ $0x + 0y + 0z = 3$

   a. Explain what this equivalent system means.
   b. Give an example of a system of equations that might lead to this solution.

2. A system of two equations in three unknowns has been manipulated, and, after correctly using elementary operations, a student arrives at the following equivalent system of equations:

   ① $2x - y + 2z = 1$
   ③ $0x + 0y + 0z = 0$

   a. Write a solution to this system of equations, and explain what your solution means.
   b. Give an example of a system of equations that leads to your solution in part a.

C 3. A system of two equations in three unknowns has been manipulated, and, after correctly using elementary operations, a student arrives at the following equivalent system of equations:

   ① $x - y + z = -1$
   ③ $0x + 0y + 2z = -4$

   a. Write a solution to this system of equations, and explain what your solution means.
   b. Give an example of a system of equations that leads to your solution in part a.

PART B

K 4. Consider the following system of equations:

   ① $2x + y + 6z = p$
   ② $x + my + 3z = q$

   a. Determine values of *m, p,* and *q* such that the two planes are coincident. Are these values unique? Explain.
   b. Determine values of *m, p,* and *q* such that the two planes are parallel and not coincident. Are these values unique? Explain.
   c. A value of *m* such that the two planes intersect at right angles. Is this value unique? Explain.
   d. Determine values of *m, p,* and *q* such that the two planes intersect at right angles. Are these values unique? Explain.

5. Consider the following system of equations:

① $x + 2y - 3z = 0$

② $y + 3z = 0$

a. Solve this system of equations by letting $z = s$.

b. Solve this system of equations by letting $y = t$.

c. Show that the solution you found in part a. is the same as the solution you found in part b.

6. The following systems of equations involve two planes. State whether the planes intersect, and, if they do intersect, specify if their intersection is a line or a plane.

a. ① $x + y + z = 1$
② $2x + 2y + 2z = 2$

b. ① $2x - y + z + 1 = 0$
② $2x - y + z + 2 = 0$

c. ① $x - y + 2z = 2$
② $x + y + 2z = -2$

d. ① $x + y + 2z = 4$
② $x - y = 6$

e. ① $2x - y + 2z = 2$
② $-x + 2y + z = 1$

f. ① $x - y + 2z = 0$
② $z = 4$

7. Determine the solution to each system of equations in question 6.

8. A system of equations is given as follows:

① $x + y + 2z = 1$

② $kx + 2y + 4z = k$

a. For what value of k does the system have an infinite number of solutions? Determine the solution to the system for this value of k.

b. Is there any value of k for which the system does not have a solution? Explain.

9. Determine the vector equation of the line that passes through $A(-2, 3, 6)$ and is parallel to the line of intersection of the planes π_1: $2x - y + z = 0$ and π_2: $y + 4z = 0$.

A 10. For the planes $2x - y + 2z = 0$ and $2x + y + 6z = 4$, show that their line of intersection lies on the plane with equation $5x + 3y + 16z - 11 = 0$.

T 11. The line of intersection of the planes π_1: $2x + y - 3z = 3$ and π_2: $x - 2y + z = -1$ is L.

a. Determine parametric equations for L.

b. If L meets the xy-plane at point A and the z-axis at point B, determine the length of line segment AB.

PART C

12. Determine the Cartesian equation of the plane that is parallel to the line with equation $x = -2y = 3z$ and that contains the line of intersection of the planes with equations $x - y + z = 1$ and $2y - z = 0$.

Mid-Chapter Review

1. Determine the point of intersection between the line $\vec{r} = (4, -3, 15) + t(2, -3, 5), t \in \mathbf{R}$, and each of the following planes:
   a. the xy-plane
   b. the xz-plane
   c. the yz-plane
2. $A(2, 1, 3)$, $B(3, -2, 5)$, and $C(-8, -5, 7)$ are three points in R^3 that form a triangle.
   a. Determine the parametric equations for any two of the three medians. (A median is a line drawn from one vertex to the midpoint of the opposite side.)
   b. Determine the point of intersection of the two medians you found in part a.
   c. Determine the equation of the third median for this triangle.
   d. Verify that the point of intersection you found in part b. is a point on the line you found in part c.
   e. State the coordinates of the point of intersection of the three medians.
3. a. Determine an equation for the line of intersection of the planes $5x + y + 2z + 15 = 0$ and $4x + y + 2z + 8 = 0$.
   b. Determine an equation for the line of intersection of the planes $4x + 3y + 3z - 2 = 0$ and $5x + 2y + 3z + 5 = 0$.
   c. Determine the point of intersection between the line you found in part a. and the line you found in part b.
4. a. Determine the line of intersection of the planes π_1: $3x + y + 7z + 3 = 0$ and π_2: $x - 13y - 3z - 38 = 0$.
   b. Determine the line of intersection of the planes π_3: $x - 3y + z + 11 = 0$ and π_4: $6x - 13y + 8z - 28 = 0$.
   c. Show that the lines you found in parts a. and b. do not intersect.
5. Consider the following system of equations:

   ① $x + ay = 9$

   ② $ax + 9y = -27$

   Determine the value(s) of a for which the system of equations has
   a. no solution
   b. an infinite number of solutions
   c. one solution

6. Show that $\frac{x - 11}{2} = \frac{y - 4}{-4} = \frac{z - 27}{5}$ and $x = 0, y = 1 - 3t, z = 3 + 2t$, $t \in \mathbf{R}$, are skew lines.
7. a. Determine the intersection of the lines
   $(x - 3, y - 20, z - 7) = t(2, -4, 5), t \in \mathbf{R}$, and $\frac{x - 5}{2} = y - 2 = \frac{z + 4}{-3}$.
   b. What conclusion can you make about these lines?
8. Determine the point of intersection between the lines
   $x = 1 + 2s, y = 4 - s, z = -3s, s \in \mathbf{R}$, and
   $x = -3, y = t + 3, z = 2t, t \in \mathbf{R}$.
9. Determine the point of intersection for each pair of lines.
   a. $\vec{r} = (5, 1, 7) + s(2, 0, 5), s \in \mathbf{R}$, and
      $\vec{r} = (-1, -1, 3) + t(4, 2, -1), t \in \mathbf{R}$
   b. $\vec{r} = (2, -1, 3) + s(5, -1, 6), s \in \mathbf{R}$, and
      $\vec{r} = (-8, 1, -9) + t(5, -1, 6), t \in \mathbf{R}$
10. You are given a pair of vector equations that both represent lines in R^3.
    a. Explain all the possible ways that these lines could be related to each other. Support your explanation with diagrams.
    b. Explain how you could use the equations you are given to help you identify which of the situations you described in part a. you are dealing with.
11. a. Explain when a line and a plane can have an infinite number of points of intersection.
    b. Give an example of a pair of vector equations (one for a line and one for a plane) that have an infinite number of points of intersection.
12. Use elementary operations to solve each system of equations.
    a. ① $2x + 3y = 30$
       ② $x - 2y = -13$
    b. ① $x + 4y - 3z + 6 = 0$
       ② $2x + 8y - 6z + 11 = 0$
    c. ① $x - 3y - 2z = -9$
       ② $2x - 5y + z = 3$
       ③ $-3x + 6y + 2z = 8$
13. For the system of equations given in parts a. and b. of question 12, describe the corresponding geometrical representation.
14. L is the line of intersection of planes $x - y = 1$ and $y + z = -3$, and L_1 is the line of intersection of the planes $y - z = 0$ and $x = -\frac{1}{2}$.
    a. Determine the point of intersection of L and L_1.
    b. Determine the angle between the lines of intersection.
    c. Determine the Cartesian equation of the plane that contains the point you found in part a. and the two lines of intersection.

Section 9.4—The Intersection of Three Planes

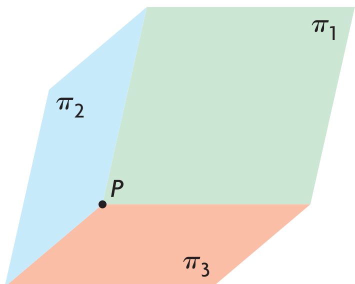
In the previous section, we discussed the intersection of two planes. In this section, we will extend these ideas and consider the intersection of three planes. Algebraically, the three planes are typically represented by a system of three linear equations in three unknowns.

First we will consider consistent systems. Later in the section, we will consider inconsistent systems.

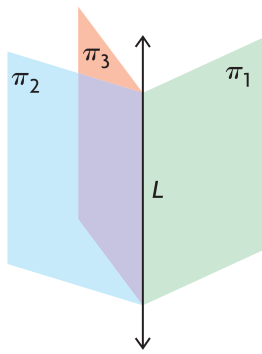
Consistent Systems for Three Equations Representing Three Planes

There are four cases that should be considered for the intersection of three planes. These four cases, which all result in one or more points of intersection between all three planes, are shown below.

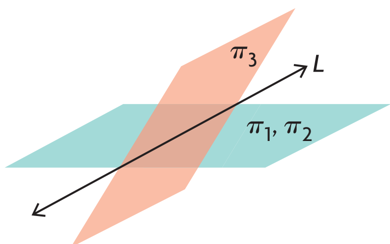
Case 1: P is the point of intersection of three planes.



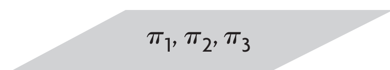
Case 2a: L is the line of intersection of three planes.



Case 2b: L is the line of intersection of two coincident planes and a third plane not parallel to the coincident planes.



Case 3: The plane of intersection of three coincident planes is the plane itself.



Possible Intersections for Three Planes

A description is given below for each situation represented in the diagram on the previous page.

Case 1: There is just one solution to the corresponding system of equations. This is a single point. The coordinates of the point of intersection will satisfy each of the three equations.

This case can be visualized by looking at the ceiling in a rectangular room. The point where the plane of the ceiling meets two walls represents the point of intersection of three planes. Although planes are not usually at right angles to each other and they extend infinitely far in all directions, this gives some idea of how planes can intersect at a point.

Case 2: There are an infinite number of solutions to the related system of equations. Geometrically, this corresponds to a line, and the solution is given in terms of one parameter. There are two sub-cases to consider.

Case 2a: The three planes intersect along a line and are mutually non-coincident.

Case 2b: Two planes are coincident, and the third plane cuts through these two planes intersecting along a line.

Case 3: Three planes are coincident, and there are an infinite number of solutions to the related system of equations. The number of solutions corresponds to the infinite number of points on a plane, and the solution is given in terms of two parameters. In this case, there are three coincident planes that have identical equations or can be reduced to three equivalent equations.

In the following examples, we use elementary operations to determine the solution of three equations in three unknowns.

EXAMPLE 1 **Using elementary operations to solve a system of three equations in three unknowns**

Determine the intersection of the three planes with the equations $x - y + z = -2$, $2x - y - 2z = -9$, and $3x + y - z = -2$.

Solution

For the intersection of the three planes, we must find the solution to the following system of equations:

① $x - y + z = -2$

② $2x - y - 2z = -9$

③ $3x + y - z = -2$

1: Create two new equations, ④ and ⑤, each containing an x-term with a coefficient of 0.

To determine the required intersection, we use elementary operations and solve the system of equations as shown earlier.

$$\begin{array}{lll} ① & x - y + z = -2 & \\ ④ & 0x + y - 4z = -5 & -2 \times ① + ② \\ ⑤ & 0x + 4y - 4z = 4 & -3 \times ① + ③ \end{array}$$

2: We create equation ⑥ by eliminating y from equation ⑤.

$$\begin{array}{lll} ① & x - y + z = -2 & \\ ④ & 0x + y - 4z = -5 & \\ ⑥ & 0x + 0y + 12z = 24 & -4 \times ④ + ⑤ \end{array}$$

This equivalent system can now be solved by first solving equation ⑥ for z.

Thus, $12z = 24$

$$z = 2$$

If we use the method of back substitution, we can substitute into equation ④ and solve for y.

Substituting into equation ④,

$$y - 4(2) = -5$$
$$y = 3$$

If we now substitute $y = 3$ and $z = 2$ into equation ①, we obtain the value of x.

$$x - 3 + 2 = -2$$
$$x = -1$$

Thus, the three planes intersect at the point with coordinates $(-1, 3, 2)$.

Check:
Substituting into equation ①, $x - y + z = -1 - 3 + 2 = -2$.

Substituting into equation ②, $2x - y - 2z = 2(-1) - 3 - 2(2) = -9$.

Substituting into equation ③, $3x + y - z = 3(-1) + 3 - 2 = -2$.

Checking each of the equations confirms the solution.

Two of the other possibilities involving consistent systems are demonstrated in the next two examples.

EXAMPLE 2 Selecting a strategy to determine the intersection of three planes

Determine the solution to the following system of equations:

① $2x - y + z = 1$

② $3x - 5y + 4z = 3$

③ $3x + 2y - z = 0$

Solution

In this situation, there is not a best way to solve the system. Because the coefficients of x for two of the equations are equal, however, the computation might be easier if we arrange them as follows (although, in situations like this, it is often a matter of individual preference).

① $3x + 2y - z = 0$ **(Interchange equations ① and ③)**

② $3x - 5y + 4z = 3$

③ $2x - y + z = 1$

Again, we try to create a zero for the coefficient of x in two of the equations.

Applying elementary operations gives the following system of equations:

① $3x + 2y - z = 0$

④ $0x - 7y + 5z = 3$ $\quad -1 \times$ ① + ②

⑤ $0x - \frac{7}{3}y + \frac{5}{3}z = 1$ $\quad -\frac{2}{3} \times$ ① + ③

Before proceeding with further computations, we should observe that equations ④ and ⑤ are scalar multiples of each other and that, if equation ⑤ is multiplied by 3, there will be two identical equations.

① $3x + 2y - z = 0$

④ $0x - 7y + 5z = 3$

⑥ $0x - 7y + 5z = 3$ $\quad 3 \times$ ⑤

By using elementary operations again, we create the following equivalent system:

① $3x + 2y - z = 0$

④ $0x - 7y + 5z = 3$

⑦ $0x + 0y + 0z = 0$ $\quad -1 \times$ ④ + ⑥

Equation ⑦, in conjunction with equations ① and ④, indicates that this system has an infinite number of solutions. To solve this system, we let $z = t$ and solve for y in equation ④.

Thus, $-7y = -5t + 3$. Dividing by -7, we get $y = \frac{5}{7}t - \frac{3}{7}$.

We determine the parametric equation for x by substituting in equation ①. Substituting $z = t$ and $y = \frac{5}{7}t - \frac{3}{7}$ into equation ① gives

$$3x + 2\left(\frac{5}{7}t - \frac{3}{7}\right) - t = 0$$

$$3x + \frac{3}{7}t - \frac{6}{7} = 0$$

$$x = -\frac{1}{7}t + \frac{2}{7}$$

Therefore, the solution to this system is $x = -\frac{1}{7}t + \frac{2}{7}$, $y = \frac{5}{7}t - \frac{3}{7}$, and $z = t$.

To help simplify the verification, we will remove the fractions from the *direction numbers* of this line by multiplying them by 7. (Recall that we cannot multiply the points by 7.)

In simplified form, the solution to the system of equations is $x = -t + \frac{2}{7}$, $y = 5t - \frac{3}{7}$, and $z = 7t$, $t \in \mathbf{R}$.

Check:
Substituting into equation ①,

$$3\left(-t + \frac{2}{7}\right) + 2\left(5t - \frac{3}{7}\right) - 7t = -3t + \frac{6}{7} + 10t - \frac{6}{7} - 7t = 0$$

Substituting into equation ②,

$$3\left(-t + \frac{2}{7}\right) - 5\left(5t - \frac{3}{7}\right) + 4(7t) = -3t + \frac{6}{7} - 25t + \frac{15}{7} + 28t = 3$$

Substituting into equation ③,

$$2\left(-t + \frac{2}{7}\right) - \left(5t - \frac{3}{7}\right) + 7t = -2t + \frac{4}{7} - 5t + \frac{3}{7} + 7t = 1$$

The solution to the system of equations is a line with parametric equations $x = -t + \frac{2}{7}$, $y = 5t - \frac{3}{7}$, and $z = 7t$, $t \in \mathbf{R}$. This is a line that has direction vector $\vec{m} = (-1, 5, 7)$ and passes through the point $\left(\frac{2}{7}, -\frac{3}{7}, 0\right)$.

It is useful to note that the normals for these three planes are $\vec{n_1} = (3, 2, -1)$, $\vec{n_2} = (3, -5, 4)$, and $\vec{n_3} = (2, -1, 1)$. Because none of these normals are collinear, this situation corresponds to Case 2a.

EXAMPLE 3 More on solving a consistent system of equations

Determine the solution to the following system of equations:

① $2x + y + z = 1$

② $4x - y - z = 5$

③ $8x - 2y - 2z = 10$

Solution

Again, using elementary operations,

① $2x + y + z = 1$

④ $0x - 3y - 3z = 3$ $\quad -2 \times ① + ②$

⑤ $0x - 6y - 6z = 6$ $\quad -4 \times ① + ③$

Continuing, we obtain

① $2x + y + z = 1$

④ $0x - 3y - 3z = 3$

⑥ $0x + 0y + 0z = 0$ $\quad -2 \times ④ + ⑤$

Equation ⑥ indicates that this system has an infinite number of solutions.

We can solve this system by using a parameter for either y or z.

Substituting $y = s$ in equation ④ gives $-3s - 3z = 3$ or $z = -s - 1$.

Substituting into equation ①, $2x + s + (-s - 1) = 1$ or $x = 1$.

Therefore, the solution to this system is $x = 1$, $y = s$, and $z = -s - 1$, $s \in \mathbf{R}$.

Check:
Substituting into equation ①, $2(1) + s + (-s - 1) = 2 + s - s - 1 = 1$.

Substituting into equation ②, $4(1) - s - (-s - 1) = 4 - s + s + 1 = 5$.

There is no need to check in our third equation since $2 \times ② = ③$. Equation ③ represents the same plane as equation ②.

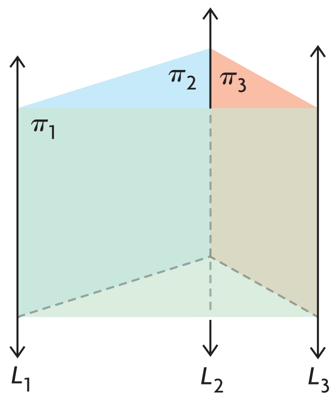
It is worth noting that the normals of the second and third planes, $\vec{n_2} = (4, -1, -1)$ and $\vec{n_3} = (8, -2, -2)$, are scalar multiples of each other, and that the constants on the right-hand side are related by the same factor. This indicates that the two equations represent the same plane. Since neither of these normals and the first plane's normal $\vec{n_1} = (2, 1, 1)$ are scalar multiples of each other, the first plane must intersect the two coincident planes along a line passing through the point $(1, 0, -1)$ with direction vector $\vec{m} = (0, 1, -1)$. This corresponds to Case 2b.

Inconsistent Systems for Three Equations Representing Three Planes

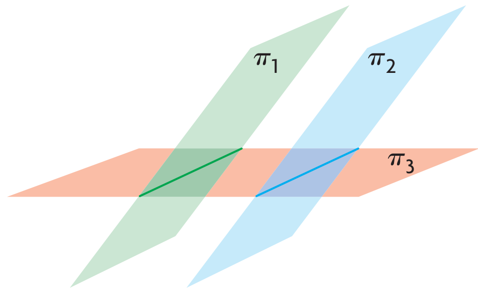
There are four cases to consider for inconsistent systems of equations that represent three planes.

The four cases, which all result in no points of intersection between all three planes, are shown below.

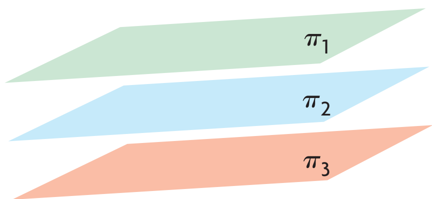
Case 1: A triangular prism is formed by three parallel lines.



Case 2: Two parallel planes intersect a third plane.



Case 3: Three planes are parallel; none are coincident.



Case 4: Two planes are coincident; a third plane is parallel to and non-coincident with the first two planes.

Non-intersections for Three Planes

A description of each case above is given below.

Case 1: Three planes (π_1, π_2, and π_3) form a triangular prism as shown. This means that, if you consider any two of the three planes, they intersect in a line and each of these three lines is parallel to the others. In the diagram, the lines L_1, L_2, and L_3 represent the lines of intersection between the three pairs of planes, and these lines have direction vectors that are identical to, or scalar multiples of, each other.

Even though the planes intersect in a pair-wise fashion, there is no common intersection between all three of the planes.

As well, the normals of the three planes are not scalar multiples of each other, and the system is inconsistent. The only geometric possibility is that the planes form a triangular prism. This idea is discussed in Example 4.

Case 2: We consider two parallel planes, each intersecting a third plane. Each of the parallel planes has a line of intersection with the third plane, but there is no intersection between all three planes.

Cases 3 and 4: In these two cases, which again implies that all three planes do not have any points of intersection.

EXAMPLE 4 Selecting a strategy to solve an inconsistent system of equations

Determine the solution to the following system of equations:

① $x - y + z = 1$

② $x + y + 2z = 2$

③ $x - 5y - z = 1$

Solution

Applying elementary operations to this system, the following system is obtained:

① $x - y + z = 1$

④ $0x + 2y + z = 1$ $\quad -1 \times$ ① + ②

⑤ $0x - 4y - 2z = 0$ $\quad -1 \times$ ① + ③

At this point, it can be observed that there is an inconsistency between equations ④ and ⑤. If equation ④ is multiplied by -2, it becomes $0x - 4y - 2z = -2$, which is inconsistent with equation ⑤ $(0x - 4y - 2z = 0)$. This implies that there is no solution to the system of equations. It is instructive, however, to continue using elementary operations and observe the results.

① $x - y + z = 1$

④ $0x + 2y + z = 1$

⑥ $0x + 0y + 0z = 2$ $\quad 2 \times$ ④ + ⑤

Equation ⑥, $0x + 0y + 0z = 2$, tells us there is no solution to the system, because there are no values of x, y, and z that satisfy this equation. The system is inconsistent.

If we use the normals for these three equations, we can calculate direction vectors for each pair of intersections. The normals for the three planes are $\overrightarrow{n_1} = (1, -1, 1)$, $\overrightarrow{n_2} = (1, 1, 2)$, and $\overrightarrow{n_3} = (1, -5, -1)$.
Let $\overrightarrow{m_1}$ be a direction vector for the line of intersection between π_1 and π_2.
Let $\overrightarrow{m_2}$ be a direction vector for the line of intersection between π_1 and π_3.
Let $\overrightarrow{m_3}$ be a direction vector for the line of intersection between π_2 and π_3.

Therefore, we can choose

$$\begin{aligned}\overrightarrow{m_1} &= \overrightarrow{n_1} \times \overrightarrow{n_2}\\ &= (1, -1, 1) \times (1, 1, 2)\\ &= (-1(2) - 1(1), 1(1) - 1(2), 1(1) - (-1)(1))\\ &= (-3, -1, 2)\end{aligned}$$

$$\begin{aligned}\overrightarrow{m_2} &= \overrightarrow{n_1} \times \overrightarrow{n_3}\\ &= (1, -1, 1) \times (1, -5, -1)\\ &= (-1(-1) - 1(-5), 1(1) - 1(-1), 1(-5) - (-1)(1))\\ &= -2(-3, -1, 2)\end{aligned}$$

$$\begin{aligned}\overrightarrow{m_3} &= \overrightarrow{n_2} \times \overrightarrow{n_3}\\ &= (1, 1, 2) \times (1, -5, -1)\\ &= (1(-1) - 2(-5), 2(1) - 1(-1), 1(-5) - 1(1))\\ &= -3(-3, -1, 2)\end{aligned}$$

We can see from our calculations that the system of equations corresponds to Case 1 for systems of inconsistent equations (triangular prism).

This conclusion could have been anticipated without doing any calculations. We have shown that the system of equations is inconsistent, and, because the normals are not scalar multiples of each other, we can reach the same conclusion.

In the following example, we deal with another inconsistent system.

EXAMPLE 5 Reasoning about an inconsistent system of equations

Determine the solution to the following system of equations:

① $x - y + 2z = -1$

② $x - y + 2z = 3$

③ $x - 3y + z = 0$

Solution

Using elementary operations,

① $x - y + 2z = -1$

④ $0x + 0y + 0z = 4$ $\quad -1 \times$ ① + ②

③ $x - 3y + z = 0$

It is only necessary to use elementary operations once, and we obtain equation ④. As before, we create an equivalent system of equations that does not have a solution, implying that the original system has no solution.

It should be noted that it is not necessary to use elementary operations in this example. Because equations ① and ② are the equations of non-coincident parallel planes, no intersection is possible. This corresponds to Case 2 for systems of inconsistent equations, since the third plane is not parallel to the first two.

EXAMPLE 6

Identifying coincident and parallel planes in an inconsistent system

Solve the following system of equations:

① $x + y + z = 5$

② $x + y + z = 4$

③ $x + y + z = 5$

Solution

It is clear, from observation, that this system of equations is inconsistent. Equations ① and ③ represent the same plane, and equation ② represents a plane that is parallel to, but different from, the other plane. This corresponds to Case 4 for systems of inconsistent equations, so there are no solutions.

IN SUMMARY

Key Idea

- A system of three (linear) equations in three unknowns geometrically represents three planes in R^3. These planes may intersect at zero, one, or an infinite number of points, depending on how the planes are related to each other.

Need to Know

- Consistent Systems for Three Equations Representing Three Planes

  Case 1 (one solution): There is a single point.

  Case 2 (infinite number of solutions): The solution uses one parameter.

  Case 2a: The three planes intersect along a line.

  Case 2b: Two planes are coincident, and the third plane cuts through these two planes.

  Case 3 (infinite number of solutions): The solution uses two parameters. There are three planes that have identical equations (after reducing the equations) that coincide with one another.

- Inconsistent Systems for Three Equations Representing Three Planes (No Intersection)

  Case 1: Three planes (π_1, π_2, and π_3) form a triangular prism.

  Case 2: Two non-coincident parallel planes each intersect a third plane.

  Case 3: The three planes are parallel and non-coincident.

  Case 4: Two planes are coincident and parallel to the third plane.

Exercise 9.4

PART A

1. A student is manipulating a system of equations and obtains the following equivalent system:

① $x - 3y + z = 2$

② $0x + y - z = -1$

③ $0x + 0y + 3z = -12$

a. Determine the solution to this system of equations.

b. How would your solution be interpreted geometrically?

2. When manipulating a system of equations, a student obtains the following equivalent system:

① $x - y + z = 4$

② $0x + 0y + 0z = 0$

③ $0x + 0y + 0z = 0$

a. Give a system of equations that would produce this equivalent system.

b. How would you interpret the solution to this system geometrically?

c. Write the solution to this system using parameters for x and y.

d. Write the solution to this system using parameters for y and z.

3. When manipulating a system of equations, a student obtains the following equivalent system:

① $2x - y + 3z = -2$

② $x - y + 4z = 3$

③ $0x + 0y + 0z = 1$

a. Give *two* systems of equations that could have produced this result.

b. What does this equivalent system tell you about possible solutions for the original system of equations?

4. When manipulating a system of equations, a student obtains the following equivalent system:

① $x + 2y - z = 4$

② $x + 0y - 2z = 0$

③ $2x + 0y + 0z = -6$

a. Without using any further elementary operations, determine the solution to this system.

b. How can the solution to this system be interpreted geometrically?

PART B

5. a. Without solving the following system, how can you deduce that these three planes must intersect in a line?

$$\begin{aligned} ①\quad 2x - y + z &= 1 \\ ②\quad x + y - z &= -1 \\ ③\quad -3x - 3y + 3z &= 3 \end{aligned}$$

   b. Find the solution to the given system using elementary operations.

C 6. Explain why there is no solution to the following system of equations:

$$\begin{aligned} ①\quad 2x + 3y - 4z &= -5 \\ ②\quad x - y + 3z &= -201 \\ ③\quad 5x - 5y + 15z &= -1004 \end{aligned}$$

7. Avery is solving a system of equations using elementary operations and derives, as one of the equations, $0x + 0y + 0z = 0$.

   a. Is it true that this equation will always have a solution? Explain.

   b. Construct your own system of equations in which the equation $0x + 0y + 0z = 0$ appears, but for which there is no solution to the constructed system of equations.

K 8. Solve the following systems of equations using elementary operations. Interpret your results geometrically.

   a. $$\begin{aligned} ①\quad 2x + y - z &= -3 \\ ②\quad x - y + 2z &= 0 \\ ③\quad 3x + 2y - z &= -5 \end{aligned}$$

   b. $$\begin{aligned} ①\quad \frac{x}{3} - \frac{y}{4} + z &= \frac{7}{8} \\ ②\quad 2x + 2y - 3z &= -20 \\ ③\quad x - 2y + 3z &= 2 \end{aligned}$$

   c. $$\begin{aligned} ①\quad x - y &= -199 \\ ②\quad x + z &= -200 \\ ③\quad y - z &= 201 \end{aligned}$$

   d. $$\begin{aligned} ①\quad x - y - z &= -1 \\ ②\quad y - 2 &= 0 \\ ③\quad x + 1 &= 5 \end{aligned}$$

9. Solve each system of equations using elementary operations. Interpret your results geometrically.

   a. ① $x - 2y + z = 3$
      ② $2x + 3y - z = -9$
      ③ $5x - 3y + 2z = 0$

   b. ① $x - 2y + z = 3$
      ② $x + y + z = 2$
      ③ $x - 3y + z = -6$

   c. ① $x - y + z = -2$
      ② $x + y + z = 2$
      ③ $x - 3y + z = -6$

10. Determine the solution to each system.

   a. ① $x - y + z = 2$
      ② $2x - 2y + 2z = 4$
      ③ $x + y - z = -2$

   b. ① $2x - y + 3z = 0$
      ② $4x - 2y + 6z = 0$
      ③ $-2x + y - 3z = 0$

A

11. a. Use elementary operations to show that the following system does not have a solution:

      ① $x + y + z = 1$
      ② $x - 2y + z = 0$
      ③ $x - y + z = 0$

   b. Calculate the direction vectors for the lines of intersection between each pair of planes, as shown in Example 4.

   c. Explain, in your own words, why the planes represented in this system of equations must correspond to a triangular prism.

   d. Explain how the same conclusion could have been reached without doing the calculations in part b.

12. Each of the following systems does not have a solution. Explain why.

   a. ① $x - y + 3z = 3$
      ② $x - y + 3z = 6$
      ③ $3x - 5z = 0$

   b. ① $5x - 2y + 3z = 1$
      ② $5x - 2y + 3z = -1$
      ③ $5x - 2y + 3z = 13$

   c. ① $x - y + z = 9$
      ② $2x - 2y + 2z = 18$
      ③ $2x - 2y + 2z = 17$

   d. ① $3x - 2y + z = 4$
      ② $9x - 6y + 3z = 12$
      ③ $6x - 4y + 2z = 5$

13. Determine the solution to each system of equations, if a solution exists.

a. ① $2x - y - z = 10$
② $x + y + 0z = 7$
③ $0x + y - z = 8$

b. ① $2x - y + z = -3$
② $x + y - 2z = 1$
③ $5x + 2y - 5z = 0$

c. ① $x + y - z = 0$
② $2x - y + z = 0$
③ $4x - 5y + 5z = 0$

d. ① $x - 10y + 13z = -4$
② $2x - 20y + 26z = -8$
③ $x - 10y + 13z = -8$

e. ① $x - y + z = -2$
② $x + y + z = 2$
③ $3x + y + 3z = 2$

f. ① $x + y + z = 0$
② $x - 2y + 3z = 0$
③ $2x - y + 3z = 0$

PART C

14. The following system of equations represents three planes that intersect in a line:

① $2x + y + z = 4$
② $x - y + z = p$
③ $4x + qy + z = 2$

a. Determine p and q.
b. Determine an equation in parametric form for the line of intersection.

T 15. Consider the following system of equations:

① $4x + 3y + 3z = -8$
② $2x + y + z = -4$
③ $3x - 2y + (m^2 - 6)z = m - 4$

Determine the value(s) of m for which this system of equations will have

a. no solution
b. one solution
c. an infinite number of solutions

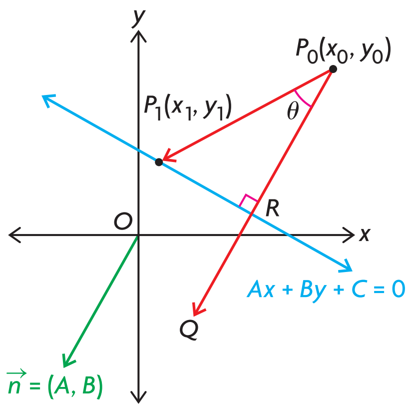
16. Determine the solution to the following system of equations:

① $\frac{1}{a} + \frac{1}{b} - \frac{1}{c} = 0$
② $\frac{2}{a} + \frac{3}{b} + \frac{2}{c} = \frac{13}{6}$
③ $\frac{4}{a} - \frac{2}{b} + \frac{3}{c} = \frac{5}{2}$

Section 9.5—The Distance from a Point to a Line in R^2 and R^3

In this section, we consider various approaches for determining the distance between a point and a line in R^2 and R^3.

Determining a Formula for the Distance between a Point and a Line in R^2



Consider a line in R^2 that has $Ax + By + C = 0$ as its general equation, as shown in the diagram above. The point $P_0(x_0, y_0)$ is a point not on the line and whose coordinates are known. A line from P_0 is drawn perpendicular to $Ax + By + C = 0$ and meets this line at R. The line from P_0 is extended to point Q. The point $P_1(x_1, y_1)$ represents a second point on the line different from R. We wish to determine a formula for $|\overrightarrow{P_0R}|$, the distance from P_0 to the line. (Note that when we are calculating the distance between a point and either a line or a plane, we are always calculating the perpendicular distance, which is always unique. In simple terms, this means that there is only one shortest distance that can be calculated between P_0 and $Ax + By + C = 0$.)

To determine the formula, we are going to take the scalar projection of $\overrightarrow{P_0P_1}$ on $\overrightarrow{P_0Q}$. Since $\overrightarrow{P_0Q}$ is perpendicular to $Ax + By + C = 0$, what we are doing is equivalent to taking the scalar projection of $\overrightarrow{P_0P_1}$ on the normal to the line, $n = (A, B)$, since n and P_0Q are parallel.

We know that $\overrightarrow{P_0P_1} = (x_1 - x_0, y_1 - y_0)$ and $\vec{n} = (A, B)$. The formula for the dot product is $\overrightarrow{P_0P_1} \times \vec{n} = |\overrightarrow{P_0P_1}||\vec{n}|\cos\theta$, where θ is the angle between $\overrightarrow{P_0P_1}$ and $\vec{n}$. Rearranging this formula gives

$$|\overrightarrow{P_0P_1}|\cos\theta = \frac{\overrightarrow{P_0P_1} \times \vec{n}}{|\vec{n}|} \qquad \text{(Equation 1)}$$

From triangle P_0RP_1

$$\cos\theta = \frac{|\overrightarrow{P_0R}|}{|\overrightarrow{P_0P_1}|}$$

$$|\overrightarrow{P_0P_1}|\cos\theta = |\overrightarrow{P_0R}|$$

Substituting $|\overrightarrow{P_0P_1}|\cos\theta = |\overrightarrow{P_0R}|$ into the dot product formula (equation 1 above) gives

$$|\overrightarrow{P_0R}| = \frac{\overrightarrow{P_0P_1} \times \vec{n}}{|\vec{n}|}$$

Since $\overrightarrow{P_0P_1} \times \vec{n} = (x_1 - x_0, y_1 - y_0)(A, B) = Ax_1 - Ax_0 + By_1 - By_0$ and $|\vec{n}| = \sqrt{A^2 + B^2}$, (by substitution) we obtain

$$|\overrightarrow{P_0R}| = \frac{Ax_1 + By_1 - Ax_0 - By_0}{\sqrt{A^2 + B^2}}$$

The point $P_1(x_1, y_1)$ is on the line $Ax + By + C = 0$, meaning that $Ax_1 + By_1 + C = 0$ or $Ax_1 + By_1 = -C$. Substituting this into the formula for $|\overrightarrow{P_0R}|$ gives

$$|\overrightarrow{P_0R}| = \frac{-C - Ax_0 - By_0}{\sqrt{A^2 + B^2}} = \frac{-(C + Ax_0 + By_0)}{\sqrt{A^2 + B^2}}$$

To ensure that this quantity is always positive, it is written as

$$|\overrightarrow{P_0R}| = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

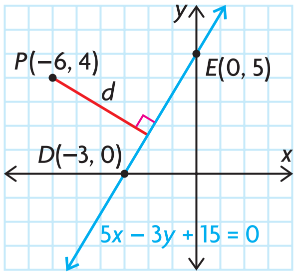
Distance from a Point $P_0(x_0, y_0)$ to the Line with Equation $Ax + By + C = 0$

$d = \dfrac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$, where d represents the distance between the point $P_0(x_0, y_0)$, and the line defined by $Ax + By + C = 0$, where the point does not lie on the line. But we don't really need this since the formula gives the correct value of 0 when the point *does* lie on the line.

EXAMPLE 1 Calculating the distance between a point and a line in R^2

Determine the distance from point $P(-6, 4)$ to the line with equation $5x - 3y + 15 = 0$.

Solution



Since $x_0 = -6$, $y_0 = 4$, $A = 5$, $B = -3$, and $C = 15$,

$$d = \frac{|5(-6) - 3(4) + 15|}{\sqrt{5^2 + (-3)^2}} = \frac{|-27|}{\sqrt{34}} \doteq 4.63$$

The distance from $P(-6, 4)$ to the line with equation $5x - 3y + 15 = 0$ is approximately 4.63.

It is not immediately possible to use the formula for the distance between a point and a line if the line is given in vector form. In the following example, we show how to find the required distance if the line is given in vector form.

EXAMPLE 2 Selecting a strategy to determine the distance between a point and a line in R^2

Determine the distance from point $P(15, -9)$ to the line with equation $\vec{r} = (-2, -1) + s(-4, 3)$, $s \in \mathbf{R}$.

Solution

To use the formula, it is necessary to convert the equation of the line in vector form to its corresponding Cartesian form. The given equation must first be written using parametric form. The parametric equations for this line are $x = -2 - 4s$ and $y = -1 + 3s$. Solving for the parameter s in each equation gives $\frac{x + 2}{-4} = s$ and $\frac{y + 1}{3} = s$. Therefore, $\frac{x + 2}{-4} = \frac{y + 1}{3} = s$. The required equation is $3(x + 2) = -4(y + 1)$ or $3x + 4y + 10 = 0$.

Therefore, $d = \dfrac{|3(15) + 4(-9) + 10|}{\sqrt{3^2 + 4^2}} = \dfrac{19}{5} = 3.80.$

The required distance is 3.80.

EXAMPLE 3 Selecting a strategy to determine the distance between two parallel lines

Calculate the distance between the two parallel lines $5x - 12y + 60 = 0$ and $5x - 12y - 60 = 0$.

Solution

To find the required distance, it is necessary to determine the coordinates of a point on one of the lines and then use the distance formula. For the line with equation $5x - 12y - 60 = 0$, we can determine the coordinates of the point where the line crosses either the x-axis or the y-axis. (This point was chosen because it is easy to calculate and it also makes the resulting computation simpler. In practice, however, any point on the chosen line is satisfactory.) If we let $x = 0$, then $5(0) - 12y - 60 = 0$, or $y = -5$. The line crosses the y-axis at $(0, -5)$. To find the required distance, d, it is necessary to find the distance from $(0, -5)$ to the line with equation $5x - 12y + 60 = 0$.

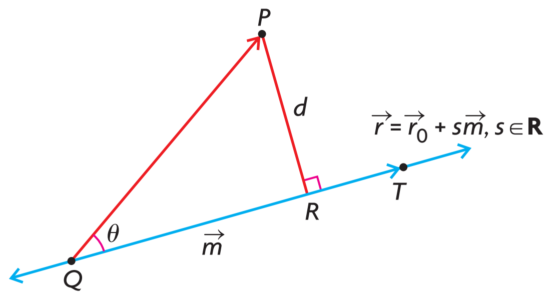
$$d = \frac{|5(0) - 12(-5) + 60|}{\sqrt{5^2 + (-12)^2}} = \frac{|120|}{13} = \frac{120}{13}$$

Therefore, the distance between the two parallel lines is $\frac{120}{13} \doteq 9.23$.

Determining the Distance between a Point and a Line in R^3

It is not possible to use the formula we just developed for finding the distance between a point and a line in R^3 because lines in R^3 are not of the form $Ax + By + C = 0$. We need to use a different approach.

The most efficient way to find the distance between a point and a line in R^3 is to use the cross product. In the following diagram, we would like to find d, which represents the distance between point P, whose coordinates are known, and a line with vector equation $\vec{r} = \vec{r}_0 + s\vec{m}$, $s \in \mathbf{R}$. Point Q is any point on the line whose coordinates are also known. Point T is the point on the line such that $\overrightarrow{QT}$ is a vector representing the direction $\vec{m}$, which is known.



The angle between $\overrightarrow{QP}$ and $\overrightarrow{QT}$ is θ. Note that, for computational purposes, it is possible to determine the coordinates of a position vector equivalent to either $\overrightarrow{QP}$ or $\overrightarrow{PQ}$.

In triangle PQR, $\sin\theta = \dfrac{d}{|\overrightarrow{QP}|}$

equivalently $d = |\overrightarrow{QP}|\sin\theta$

From our earlier discussion on cross products, we know that $|\vec{m} \times \overrightarrow{QP}| = |\vec{m}||\overrightarrow{QP}|\sin\theta$.

If we substitute $d = |\overrightarrow{QP}|\sin\theta$ into this formula, we find that $|\vec{m} \times \overrightarrow{QP}| = |\vec{m}|(d)$.

Solving for d gives $d = \dfrac{|\vec{m} \times \overrightarrow{QP}|}{|\vec{m}|}$.

Distance, d, from a Point, P, to the Line $\vec{r} = \vec{r}_0 + s\vec{m}$, $s \in \mathbf{R}$

In R^3, $d = \dfrac{|\vec{m} \times \overrightarrow{QP}|}{|\vec{m}|}$, where Q is a point on the line and P is any other point, both of whose coordinates are known, and $\vec{m}$ is the direction vector of the line.

EXAMPLE 4 Selecting a strategy to calculate the distance between a point and a line in R^3

Determine the distance from point $P(-1, 1, 6)$ to the line with equation $\vec{r} = (1, 2, -1) + t(0, 1, 1)$, $t \in \mathbf{R}$.

Solution

Method 1: Using the Formula

Since Q is $(1, 2, -1)$ and P is $(-1, 1, 6)$,
$\overrightarrow{QP} = (-1 - 1, 1 - 2, 6 - (-1)) = (-2, -1, 7)$.

From the equation of the line, we note that $\vec{m} = (0, 1, 1)$.

Thus, $d = \dfrac{|(0, 1, 1) \times (-2, -1, 7)|}{|(0, 1, 1)|}$.

Calculating,
$(0, 1, 1) \times (-2, -1, 7) = (7 - (-1), -2 - 0, 0 + 2) = (8, -2, 2)$
$|(8, -2, 2)| = \sqrt{8^2 + (-2)^2 + 2^2} = \sqrt{72} = 6\sqrt{2}$ and
$|(0, 1, 1)| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$

Therefore, the distance from the point to the line is $d = \dfrac{6\sqrt{2}}{\sqrt{2}} = 6$.

This calculation is efficient and gives the required answer quickly. (Note that the vector $(8, -2, 2)$ cannot be reduced by dividing by the common factor 2, or by any other factor.)

Method 2: Using the Dot Product

We start by writing the given equation of the line in parametric form. Doing so gives $x = 1$, $y = 2 + t$, and $z = -1 + t$. We construct a vector from a general point on the line to P and call this vector $\vec{a}$. Thus, $\vec{a} = (-1 - 1, 1 - (2 + t), 6 - (-1 + t)) = (-2, -1 - t, 7 - t)$. What we wish to find is the minimum distance between point $P(-1, 1, 6)$ and the given line. This occurs when $\vec{a}$ is perpendicular to the given line, or when $\vec{m} \cdot \vec{a} = 0$.

Calculating gives $(0, 1, 1) \cdot (-2, -1 - t, 7 - t) = 0$

$$0(-2) + 1(-1 - t) + 1(7 - t) = 0$$
$$-1 - t + 7 - t = 0$$
$$t = 3$$

This means that the minimal distance between $P(-1, 1, 6)$ and the line occurs when $t = 3$, which implies that the point corresponding to $t = 3$ produces the minimal distance between the point and the line. This point has coordinates $x = 1$, $y = 2 + 3 = 5$, and $z = -1 + 3 = 2$. In other words, the minimal distance between the point and the line is the distance between $P(-1, 1, 6)$ and the point, $(1, 5, 2)$. Thus,

$$d = \sqrt{(-1 - 1)^2 + (1 - 5)^2 + (6 - 2)^2} = \sqrt{4 + 16 + 16} = \sqrt{36} = 6$$

This gives the same answer that we found using Method 1. It has the advantage that it also allows us to find the coordinates of the point on the line that produces the minimal distance.

IN SUMMARY

Key Ideas

- In R^2, the distance from point $P_0(x_0, y_0)$ to the line with equation $Ax + By + C = 0$ is $d = \dfrac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$, where d represents the distance.
- In R^3, the formula for the distance d from point P to the line $\vec{r} = \vec{r_0} + s\vec{m}$, $s \in \mathbf{R}$, is $d = \dfrac{|\vec{m} \times \vec{QP}|}{|\vec{m}|}$, where Q is a point on the line whose coordinates are known.

Exercise 9.5

PART A

1. Determine the distance from $P(-4, 5)$ to each of the following lines:
   a. $3x + 4y - 5 = 0$
   b. $5x - 12y + 24 = 0$
   c. $9x - 40y = 0$

2. Determine the distance between the following parallel lines:
   a. $2x - y + 1 = 0, 2x - y + 6 = 0$
   b. $7x - 24y + 168 = 0, 7x - 24y - 336 = 0$

3. Determine the distance from $R(-2, 3)$ to each of the following lines:
   a. $\vec{r} = (-1, 2) + s(3, 4), s \in \mathbf{R}$
   b. $\vec{r} = (1, 0) + t(5, 12), t \in \mathbf{R}$
   c. $\vec{r} = (1, 3) + p(7, -24), p \in \mathbf{R}$

PART B

C 4. a. The formula for the distance from a point to a line is $d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$. Show that this formula can be modified so the distance from the origin, $O(0, 0)$, to the line $Ax + By + C = 0$ is given by the formula $d = \frac{|C|}{\sqrt{A^2 + B^2}}$.
   b. Determine the distance between $L_1: 3x - 4y - 12 = 0$ and $L_2: 3x - 4y + 12 = 0$ by first finding the distance from the origin to L_1 and then finding the distance from the origin to L_2.
   c. Find the distance between the two lines directly by first determining a point on one of the lines and then using the distance formula. How does this answer compare with the answer you found in part b.?

K 5. Calculate the distance between the following lines:
   a. $\vec{r} = (-2, 1) + s(3, 4); s \in \mathbf{R}; \vec{r} = (1, 0) + t(3, 4), t \in \mathbf{R}$
   b. $\frac{x - 1}{4} = \frac{y}{-3}, \frac{x}{4} = \frac{y + 1}{-3}$
   c. $2x - 3y + 1 = 0, 2x - 3y - 3 = 0$
   d. $5x + 12y = 120, 5x + 12y + 120 = 0$

6. Calculate the distance between point P and the given line.
   a. $P(1, 2, -1)$; $\vec{r} = (1, 0, 0) + s(2, -1, 2)$, $s \in \mathbf{R}$
   b. $P(0, -1, 0)$; $\vec{r} = (2, 1, 0) + t(-4, 5, 20)$, $t \in \mathbf{R}$
   c. $P(2, 3, 1)$; $\vec{r} = p(12, -3, 4)$, $p \in \mathbf{R}$

7. Calculate the distance between the following parallel lines.
   a. $\vec{r} = (1, 1, 0) + s(2, 1, 2)$, $s \in \mathbf{R}$; $\vec{r} = (-1, 1, 2) + t(2, 1, 2)$, $t \in \mathbf{R}$
   b. $\vec{r} = (3, 1, -2) + m(1, 1, 3)$, $m \in \mathbf{R}$; $\vec{r} = (1, 0, 1) + n(1, 1, 3)$, $n \in \mathbf{R}$

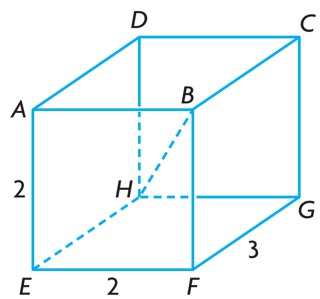
A 8. a. Determine the coordinates of the point on the line $\vec{r} = (1, -1, 2) + s(1, 3, -1)$, $s \in \mathbf{R}$, that produces the shortest distance between the line and a point with coordinates $(2, 1, 3)$.
   b. What is the distance between the given point and the line?

PART C

9. Two planes with equations $x - y + 2z = 2$ and $x + y - z = -2$ intersect along line L. Determine the distance from $P(-1, 2, -1)$ to L, and determine the coordinates of the point on L that gives this minimal distance.

10. The point $A(2, 4, -5)$ is reflected in the line with equation $\vec{r} = (0, 0, 1) + s(4, 2, 1)$, $s \in \mathbf{R}$, to give the point A'. Determine the coordinates of A'.

T 11. A rectangular box with an open top, measuring 2 by 2 by 3, is constructed. Its vertices are labelled as shown.



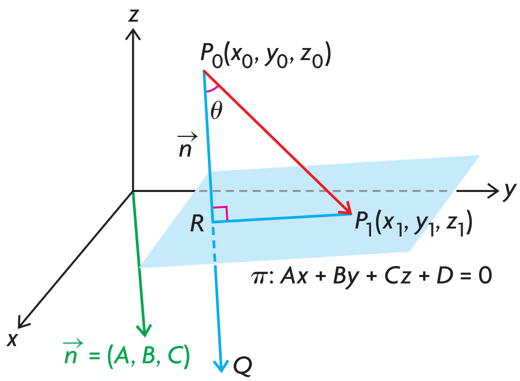
   a. Determine the distance from A to the line segment HB.
   b. What other vertices on the box will give the same distance to HB as the distance you found in part a.?
   c. Determine the area of the $\triangle AHB$.

Section 9.6—The Distance from a Point to a Plane

In the previous section, we developed a formula for finding the distance from a point $P_0(x_0, y_0)$ to the line $Ax + By + C = 0$ in R^2. In this section, we will use the same kind of approach to develop a formula for the distance from a point $P_0(x_0, y_0, z_0)$ to the plane with equation $Ax + By + \; + Cz + D = 0$ in R^3.

Determining a Formula for the Distance between a Point and a Plane in R^3

We start by considering a general plane in R^3 that has $Ax + By + Cz + D = 0$ as its equation. The point $P_0(x_0, y_0, z_0)$ is a point whose coordinates are known. A line from P_0 is drawn perpendicular to $Ax + By + Cz + D = 0$ and meets this plane at R. The point $P_1(x_1, y_1, z_1)$ is a point on the plane, with coordinates different from R, and Q is chosen so that $\overrightarrow{P_0Q} = n = (A, B, C)$ is the normal to the plane. The objective is to find a formula for $|\overrightarrow{PR}|$—the perpendicular distance from P_0 to the plane. To develop this formula, we are going to use the fact that $|\overrightarrow{PR}|$ is the scalar projection of $\overrightarrow{P_0P_1}$ on the normal $\vec{n}$.



In $\triangle P_0RP_1$, $\cos\theta = \dfrac{|\overrightarrow{PR}|}{|\overrightarrow{P_0P_1}|}$

$|\overrightarrow{PR}| = |\overrightarrow{P_0P_1}|\cos\theta$, where θ is the angle between the vectors $\vec{n}$ and $\overrightarrow{P_0P_1}$

Since $\vec{n} \cdot \overrightarrow{P_0P_1} = |\vec{n}||\overrightarrow{P_0P_1}|\cos\theta$,

$$|\overrightarrow{P_0P_1}|\cos\theta = \frac{\vec{n} \cdot \overrightarrow{P_0P_1}}{|\vec{n}|}$$

Therefore, $d = |\overrightarrow{PR}| = \dfrac{\vec{n} \cdot \overrightarrow{P_0P_1}}{|\vec{n}|}$, where d is the distance from the point to the plane.

Since $\vec{n} = (A, B, C)$, $\overrightarrow{P_0P_1} = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$ and

$$|\vec{n}| = \sqrt{A^2 + B^2 + C^2}$$

$$d = \left|\overrightarrow{PR}\right| = \frac{(A, B, C) \times (x_1 - x_0, y_1 - y_0, z_1 - z_0)}{\sqrt{A^2 + B^2 + C^2}}$$

or

$$d = \frac{Ax_1 - Ax_0 + By_1 - By_0 + Cz_1 - Cz_0}{\sqrt{A^2 + B^2 + C^2}}$$

Since $P_1(x_1, y_1, z_1)$ is a point on $Ax + By + Cz + D = 0$, $Ax_1 + By_1 + Cz_1 + D = 0$ and $Ax_1 + By_1 + Cz_1 = -D$.

Rearranging the formula,

$$d = \frac{-Ax_0 - By_0 - Cz_0 + Ax_1 + By_1 + Cz_1}{\sqrt{A^2 + B^2 + C^2}}$$

Therefore, $d = \dfrac{-Ax_0 - By_0 - Cz_0 - D}{\sqrt{A^2 + B^2 + C^2}}$

$$d = \frac{-(Ax_0 + By_0 + Cz_0 + D)}{\sqrt{A^2 + B^2 + C^2}}$$

Since the distance d is always positive, the formula is written as

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Distance from a Point $P_0(x_0, y_0, z_0)$ to the Plane with Equation $Ax + By + Cz + D = 0$

In R^3, $d = \dfrac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$, where d is the required distance between the point and the plane.

EXAMPLE 1 Calculating the distance from a point to a plane

Determine the distance from $S(-1, 2, -4)$ to the plane with equation $8x - 4y + 8z + 3 = 0$.

Solution

To determine the required distance, we substitute directly into the formula.

Therefore, $d = \dfrac{|8(-1) - 4(2) + 8(-4) + 3|}{\sqrt{8^2 + (-4)^2 + 8^2}} = \dfrac{|-45|}{12} = \dfrac{45}{12} = 3.75$

The distance between $S(-1, 2, -4)$ and the given plane is 3.75.

It is also possible to use the distance formula to find the distance between two parallel planes, as we show in the following example.

EXAMPLE 2 Selecting a strategy to determine the distance between two parallel planes

a. Determine the distance between the two planes π_1: $2x - y + 2z + 4 = 0$ and π_2: $2x - y + 2z + 16 = 0$.
b. Determine the equation of the plane that is equidistant from π_1 and π_2.

Solution

a. The two given planes are parallel because they each have the same normal, $\vec{n} = (2, -1, 2)$. To find the distance between π_1 and π_2, it is necessary to have a point on one of the planes and the equation of the second plane. If we consider π_1, we can determine the coordinates of its z-intercept by letting $x = y = 0$. Substituting, $2(0) - (0) + 2z + 4 = 0$, or $z = -2$. This means that point $X(0, 0, -2)$ lies on π_1. To find the required distance, apply the formula using $X(0, 0, -2)$ and π_2: $2x - y + 2z + 16 = 0$.

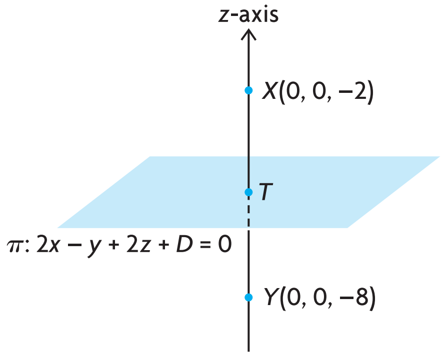
$$d = \frac{|2(0) - (0) + 2(-2) + 16|}{\sqrt{2^2 + (-1)^2 + 2^2}} = \frac{|-4 + 16|}{3} = \frac{12}{3} = 4$$

Therefore, the distance between π_1 and π_2 is 4.

When calculating the distance between these two planes, we used the coordinates of the z-intercept as our point on one of the planes. This point was chosen because it is easy to determine and leads to a simple calculation for the distance.

b. The plane that is equidistant from π_1 and π_2 is parallel to both planes and lies midway between them. Since the required plane is parallel to the two given planes, it must have the form π: $2x - y + 2z + D = 0$, with D to be determined. If we follow the same procedure for π_2 that we used in part a., we can find the coordinates of the point associated with its z-intercept. If we substitute $x = y = 0$ into π_2, $2(0) - (0) + 2z + 16 = 0$, or $z = -8$. This means the point $Y(0, 0, -8)$ is on π_2.

The situation can be visualized in the following way:



Point T is on the required plane π: $2x - y + 2z + D = 0$ and is the midpoint of the line segment joining $X(0, 0, -2)$ to $Y(0, 0, -8)$, meaning that point T has coordinates $(0, 0, -5)$. To find D, we substitute the coordinates of point T into π, which gives $2(0) - (0) + 2(-5) + D = 0$, or $D = 10$.

Thus, the required plane has the equation $2x - y + 2z + 10 = 0$. We should note, in this case, that the coordinates of its z-intercept are $(0, 0, -5)$.

We have shown how to use the formula to find the distance from a point to a plane. This formula can also be used to find the distance between two skew lines.

EXAMPLE 3 Selecting a strategy to determine the distance between skew lines

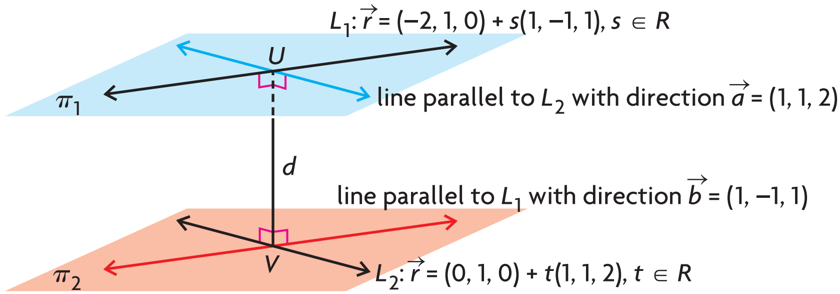
Determine the distance between L_1: $\vec{r} = (-2, 1, 0) + s(1, -1, 1)$, $s \in \mathbf{R}$, and L_2: $\vec{r} = (0, 1, 0) + t(1, 1, 2)$, $t \in \mathbf{R}$.

Solution

Method 1:

These two lines are skew lines because they are not parallel and do not intersect (you should verify this for yourself). To find the distance between the given lines, two parallel planes are constructed. The first plane is constructed so that L_1 lies on it, along with a second line that has direction $\vec{a} = (1, 1, 2)$, the direction vector for L_2.

In the same way, a second plane is constructed containing L_2, along with a second line that has direction $\vec{b} = (1, -1, 1)$, the direction of L_1.



The two constructed planes are parallel because they have two identical direction vectors. By constructing these planes, the problem of finding the distance between the two skew lines has been reduced to finding the distance between the planes π_1 and π_2. Finding the distance between the two planes means finding the Cartesian equation of one of the planes and using the distance formula with a point from the other plane. (This method of constructing planes will not work if the two given lines are parallel because the calculation of the normal for the planes would be (0, 0, 0).)

We first determine the equation of π_2. Since this plane has direction vectors $\overrightarrow{m_2} = (1, 1, 2)$ and $\vec{b} = (1, -1, 1)$, we can choose

$\vec{n} = \overrightarrow{m_2} \times \vec{b} = (1, 1, 2) \times (1, -1, 1)$.

Thus, $\vec{n} = (1(1) - 2(-1), 2(1) - 1(1), 1(-1) - 1(1)) = (3, 1, -2)$

We must now find D using the equation $3x + y - 2z + D = 0$. Since (0, 1, 0) is a point on this plane, $3(0) + 1 - 2(0) + D = 0$, or $D = -1$. This gives $3x + y - 2z - 1 = 0$ as the equation for π_2.

Using $3x + y - 2z - 1 = 0$ and the point $(-2, 1, 0)$ from the other plane π_1 the distance between the skew lines can be calculated.

$$d = \frac{|3(-2) + 1 - 2(0) - 1|}{\sqrt{3^2 + 1^2 + (-2)^2}} = \frac{6}{\sqrt{14}} = \frac{3}{7}\sqrt{14} \doteq 1.60$$

The distance between the two skew lines is approximately 1.60.

The first method gives one approach for finding the distance between two skew lines. In the second method, we will show how to determine the point on each of these two skew lines that produces this minimal distance.

Method 2:
In Method 1, we constructed two parallel planes and found the distance between them. Since the distance between the two planes is constant, our calculation also gave the distance between the two skew lines. There are points on each of these lines that will produce this minimal distance. Possible points, U and V, are shown

on the diagram for Method 1. To determine the coordinates of these points, we must use the fact that the vector found by joining the two points is perpendicular to the direction vector of each line.

We start by writing each line in parametric form.

For L_1, $x = -2 + s$, $y = 1 - s$, $z = s$, $\overrightarrow{m_1} = (1, -1, 1)$.

For L_2, $x = t$, $y = 1 + t$, $z = 2t$, $\overrightarrow{m_2} = (1, 1, 2)$.

The point with coordinates $U(-2 + s, 1 - s, s)$ represents a general point on L_1, and $V(t, 1 + t, 2t)$ represents a general point on L_2. We next calculate $\overrightarrow{UV}$.

$\overrightarrow{UV} = (t - (-2 + s), (1 + t) - (1 - s), 2t - s) = (t - s + 2, t + s, 2t - s)$

$\overrightarrow{UV}$ represents a general vector with its tail on L_1 and its head on L_2.

To find the points on each of the two lines that produce the minimal distance, we must use equations $\overrightarrow{m_1} \cdot \overrightarrow{UV} = 0$ and $\overrightarrow{m_2} \cdot \overrightarrow{UV} = 0$, since $\overrightarrow{UV}$ must be perpendicular to each of the two planes.

Therefore, $(1, -1, 1)(t - s + 2, t + s, 2t - s) = 0$

$$1(t - s + 2) - 1(t + s) + 1(2t - s) = 0$$

$$\text{or } 2t - 3s = -2 \qquad \text{(Equation 1)}$$

and $(1, 1, 2) \cdot (t - s + 2, t + s, 2t - s) = 0$

$$1(t - s + 2) + 1(t + s) + 2(2t - s) = 0$$

$$\text{or } 3t - s = -1 \qquad \text{(Equation 2)}$$

This gives the following system of equations:

① $2t - 3s = -2$

② $3t - s = -1$

$$-7t = 1 \qquad -3 \times ② + ①$$

$$t = -\frac{1}{7}$$

If we substitute $t = -\frac{1}{7}$ into equation ②, $3\left(-\frac{1}{7}\right) - s = -1$ or $s = \frac{4}{7}$.

We now substitute $s = \frac{4}{7}$ and $t = -\frac{1}{7}$ into the equations for each line to find the required points.

For L_1, $x = -2 + \frac{4}{7} = -\frac{10}{7}$, $y = 1 - \frac{4}{7} = \frac{3}{7}$, $z = \frac{4}{7}$.

Therefore, the required point on L_1 is $\left(-\frac{10}{7}, \frac{3}{7}, \frac{4}{7}\right)$.

For L_2, $x = -\frac{1}{7}$, $y = 1 + \left(\frac{-1}{7}\right) = \frac{6}{7}$, $z = 2\left(\frac{-1}{7}\right) = -\frac{2}{7}$.

Therefore, the required point on L_2 is $\left(-\frac{1}{7}, \frac{6}{7}, -\frac{2}{7}\right)$.

The required distance between the two lines is the distance between these two points. This distance is $\sqrt{\left(-\frac{10}{7}+\frac{1}{7}\right)^2+\left(\frac{3}{7}-\frac{6}{7}\right)^2+\left(\frac{4}{7}+\frac{2}{7}\right)^2}$

$$= \sqrt{\left(\frac{-9}{7}\right)^2+\left(\frac{-3}{7}\right)^2+\left(\frac{6}{7}\right)^2}$$

$$= \sqrt{\frac{81}{49}+\frac{9}{49}+\frac{36}{49}}$$

$$= \sqrt{\frac{126}{49}}$$

$$= \sqrt{\frac{9\times 14}{49}}$$

$$= \frac{3}{7}\sqrt{14}$$

Thus, the distance between the two skew lines is $\frac{3}{7}\sqrt{14}$, or approximately 1.60.

The two points that produce this distance are $\left(-\frac{10}{7}, \frac{3}{7}, \frac{4}{7}\right)$ on L_1 and $\left(-\frac{1}{7}, \frac{6}{7}, -\frac{2}{7}\right)$ on L_2.

INVESTIGATION

A. In this section, we showed that the formula for the distance d from a point $P_0(x_0, y_0, z_0)$ to the plane with equation $Ax + By + Cz + D = 0$ is
$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$.

By modifying this formula, show that a formula for finding the distance from $O(0, 0, 0)$ to the plane $Ax + By + Cz + D = 0$ is $d = \frac{|D|}{\sqrt{A^2 + B^2 + C^2}}$.

B. Determine the distance from $O(0, 0, 0)$ to the plane with equation $20x + 4y - 5z + 21 = 0$.

C. Determine the distance between the planes with equations π_1: $20x + 4y - 5z + 21 = 0$ and π_2: $20x + 4y - 5z + 105 = 0$.

D. Determine the coordinates of a point that is equidistant from π_1 and π_2.

E. Determine an equation for a plane that is equidistant from π_1 and π_2.

F. Determine two values of D if the plane with equation $20x + 4y - 5z + D = 0$ is 4 units away from the plane with equation $20x + 4y - 5z = 0$.

G. Determine the distance between the two planes π_3: $20x + 4y - 5z - 105 = 0$ and π_4: $20x + 4y - 5z + 147 = 0$.

H. Determine the distance between the following planes:

a. $2x - 2y + z - 6 = 0$ and $2x - 2y + z - 12 = 0$

b. $6x - 3y + 2z + 14 = 0$ and $6x - 3y + 2z + 35 = 0$

c. $12x + 3y + 4z - 26 = 0$ and $12x + 3y + 4z + 26 = 0$

I. If two planes have equations $Ax + By + Cz + D_1 = 0$ and $Ax + By + Cz + D_2 = 0$, explain why the formula for the distance d between these planes is $d = \frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}}$.

IN SUMMARY

Key Idea

- The distance from a point $P_0(x_0, y_0, z_0)$ to the plane with equation $Ax + By + Cz + D = 0$ is $d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$, where d is the required distance.

Need to Know

- The distance between skew lines can be calculated using two different methods.

  Method 1: To determine the distance between the given skew lines, two parallel planes are constructed that are the same distance apart as the skew lines. Determine the distance between the two planes.

  Method 2: To determine the coordinates of the points that produce the minimal distance, use the fact that the general vector found by joining the two points is perpendicular to the direction vector of each line.

Exercise 9.6

PART A

C 1. A student is calculating the distance d between point $A(-3, 2, 1)$ and the plane with equation $2x + y + 2z + 2 = 0$. The student obtains the following answer:

$$d = \frac{|2(-3) + 2 + 2(1) + 2|}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{0}{3} = 0$$

a. Has the student done the calculation correctly? Explain.

b. What is the significance of the answer 0? Explain.

PART B

K 2. Determine the following distances:

a. the distance from $A(3, 1, 0)$ to the plane with equation $20x - 4y + 5z + 7 = 0$

b. the distance from $B(0, -1, 0)$ to the plane with equation $2x + y + 2z - 8 = 0$

c. the distance from $C(5, 1, 4)$ to the plane with equation $3x - 4y - 1 = 0$

d. the distance from $D(1, 0, 0)$ to the plane with equation $5x - 12y = 0$

e. the distance from $E(-1, 0, 1)$ to the plane with equation $18x - 9y + 18z - 11 = 0$

3. For the planes π_1: $3x + 4y - 12z - 26 = 0$ and π_2: $3x + 4y - 12z + 39 = 0$, determine

a. the distance between π_1 and π_2

b. an equation for a plane midway between π_1 and π_2

c. the coordinates of a point that is equidistant from π_1 and π_2

4. Determine the following distances:

a. the distance from $P(1, 1, -3)$ to the plane with equation $y + 3 = 0$

b. the distance from $Q(-1, 1, 4)$ to the plane with equation $x - 3 = 0$

c. the distance from $R(1, 0, 1)$ to the plane with equation $z + 1 = 0$

A 5. Points $A(1, 2, 3)$, $B(-3, -1, 2)$, and $C(13, 4, -1)$ lie on the same plane. Determine the distance from $P(1, -1, 1)$ to the plane containing these three points.

T 6. The distance from $R(3, -3, 1)$ to the plane with equation $Ax - 2y + 6z = 0$ is 3. Determine all possible value(s) of A for which this is true.

7. Determine the distance between the lines $\vec{r} = (0, 1, -1) + s(3, 0, 1)$, $s \in \mathbf{R}$, and $\vec{r} = (0, 0, 1) + t(1, 1, 0)$, $t \in \mathbf{R}$.

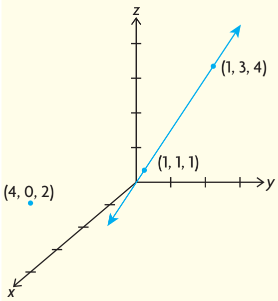
PART C

8. a. Calculate the distance between the lines L_1: $\vec{r} = (1, -2, 5) + s(0, 1, -1)$, $s \in \mathbf{R}$, and L_2: $\vec{r} = (1, -1, -2) + t(1, 0, -1)$, $t \in \mathbf{R}$.

b. Determine the coordinates of points on these lines that produce the minimal distance between L_1 and L_2.

CAREER LINK WRAP-UP Investigate and Apply

CHAPTER 9: RELATIONSHIPS BETWEEN POINTS, LINES, AND PLANES



A pipeline engineer needs to find the line that will allow a new pipeline to intersect and join an existing pipeline at a right angle. The existing line has a pathway determined by the equation L_2: $r = (1, 1, 1) + d(0, 2, 3)$, $d \in \mathbf{R}$. The new pipeline will also need to be exactly 2 units away from the point (4, 0, 2).

a. Determine the vector and parametric equations of L_3, the line that passes through (4, 0, 2) and is perpendicular to L_2.

b. Determine the vector and parametric equations of L_1, the line that is parallel to L_3 and 2 units away from (4, 0, 2). There will be exactly two lines that fulfill this condition.

c. Plot each line on the coordinate axes.

Key Concepts Review

In this chapter, you learned how to solve systems of linear equations using elementary operations. The number of equations and the number of variables in the system are directly related to the geometric interpretation that each system represents.

System of Equations	Geometric Interpretation	Possible Points of Intersection
Two equations and two unknowns	two lines in R^2	zero, one, or an infinite number
Two equations and three unknowns	two planes in R^3	zero or an infinite number
Three equations and three unknowns	three planes in R^3	zero, one, or an infinite number

To make a connection between the algebraic equations and the geometric position and orientation of lines or planes in space, draw graphs or diagrams and compare the direction vectors of the lines and the normals of the planes. This will help you decide whether the system is consistent or inconsistent and which case you are dealing with.

Distances between points, lines and planes can be determined using the formulas developed in this chapter.

Distance between a point and a line in R^2	$d = \dfrac{\lvert Ax_0 + By_0 + C\rvert}{\sqrt{A^2 + B^2}}$
Distance between a point and a line in R^3	$d = \dfrac{\lvert \vec{m} \times \overrightarrow{QP}\rvert}{\lvert \vec{m}\rvert}$
Distance between a point and a plane in R^3	$d = \dfrac{\lvert Ax_0 + By_0 + Cz_0 + D\rvert}{\sqrt{A^2 + B^2 + C^2}}$

Review Exercise

1. The lines $2x - y = 31$, $x + 8y = -34$, and $3x + ky = 38$ all pass through a common point. Determine the value of k.

2. Solve the following system of equations:

   ① $x - y = 13$

   ② $3x + 2y = -6$

   ③ $x + 2y = -19$

3. Solve each system of equations.

   a. ① $x - y + 2z = 3$

   ② $2x - 2y + 3z = 1$

   ③ $2x - 2y + z = 11$

   b. ① $x + y + z = 300$

   ② $x + y - z = 98$

   ③ $x - y + z = 100$

4. a. Show that the points $(1, 2, 6)$, $(7, -5, 1)$, $(1, 1, 4)$, and $(-3, 5, 6)$ all lie on the same plane.

   b. Determine the distance from the origin to the plane you found in part a.

5. Determine the following distances:

   a. the distance from $A(-1, 1, 2)$ to the plane with equation $3x - 4y - 12z - 8 = 0$

   b. the distance from $B(3, 1, -2)$ to the plane with equation $8x - 8y + 4z - 7 = 0$

6. Determine the intersection of the plane $3x - 4y - 5z = 0$ with $\vec{r} = (3, 1, 1) + t(2, -1, 2)$, $t \in \mathbf{R}$.

7. Solve the following systems of equations:

   a. ① $3x - 4y + 5z = 9$

   ② $6x - 9y + 10z = 9$

   ③ $9x - 12y + 15z = 9$

   b. ① $2x + 3y + 4z = 3$

   ② $4x + 6y + 8z = 4$

   ③ $5x + y - z = 1$

   c. ① $4x - 3y + 2z = 2$

   ② $8x - 6y + 4z = 4$

   ③ $12x - 9y + 6z = 1$

8. Solve each system of equations.

a. ① $3x + 4y + z = 4$
② $5x + 2y + 3z = 2$
③ $6x + 8y + 2z = 8$

b. ① $4x - 8y + 12z = 4$
② $2x + 4y + 6z = 4$
③ $x - 2y - 3z = 4$

c. ① $x - 3y + 3z = 7$
② $2x - 6y + 6z = 14$
③ $-x + 3y - 3z = -7$

9. Solve each of the following systems:

a. ① $3x - 5y + 2z = 4$
② $6x + 2y - z = 2$
③ $6x - 3y + 8z = 6$

b. ① $2x - 5y + 3z = 1$
② $4x + 2y + 5z = 5$
③ $2x + 7y + 2z = 4$

10. Determine the intersection of each set of planes, and show your answer geometrically.

a. $2x + y + z = 6, x - y - z = -9, 3x + y = 2$

b. $2x - y + 2z = 2, 3x + y - z = 1, x - 3y + 5z = 4$

c. $2x + y - z = 0, x - 2y + 3z = 0, \ 9x + 2y - z = 0$

11. The line $\vec{r} = (2, -1, -2) + s(1, 1, -2), s \in \mathbf{R}$, intersects the xz-plane at point P and the xy-plane at point Q. Calculate the length of the line segment PQ.

12. a. Given the line $\vec{r} = (3, 1, -5) + s(2, 1, 0), s \in \mathbf{R}$, and the plane $x - 2y + z + 4 = 0$, verify that the line lies on the plane.

b. Determine the point of intersection between the line $\vec{r} = (7, 5, -1) + t(4, 3, 2), t \in \mathbf{R}$, and the line given in part a.

c. Show that the point of intersection of the lines is a point on the plane given in part a.

d. Determine the Cartesian equation of the plane that contains the line $\vec{r} = (7, 5, -1) + t(4, 3, 2), t \in \mathbf{R}$ and is perpendicular to the plane given in part a.

13. a. Determine the distance from point $A(-2, 1, 1)$ to the line with equation $\vec{r} = (3, 0, -1) + t(1, 1, 2), t \in \mathbf{R}$.

b. What are the coordinates of the point on the line that produces this shortest distance?

14. You are given the lines $\vec{r} = (1, -1, 1) + t(3, 2, 1), t \in \mathbf{R}$, and $\vec{r} = (-2, -3, 0) + s(1, 2, 3), s \in \mathbf{R}$.

    a. Determine the coordinates of their point of intersection.

    b. Determine a vector equation for the line that is perpendicular to both of the given lines and passes through their point of intersection.

15. a. Determine the equation of the plane that contains $L: \vec{r} = (1, 2, -3) + s(1, 2, -1), s \in \mathbf{R}$, and point $K(3, -2, 4)$.

    b. Determine the distance from point $S(1, 1, -1)$ to the plane you found in part a.

16. Consider the following system of equations:

    ① $x + y - z = 1$

    ② $2x - 5y + z = -1$

    ③ $7x - 7y - z = k$

    a. Determine the value(s) of k for which the solution to this system is a line.

    b. Determine the vector equation of the line.

17. Determine the solution to each system of equations.

    a. ① $x + 2y + z = 1$

    ② $2x - 3y - z = 6$

    ③ $3x + 5y + 4z = 5$

    ④ $4x + y + z = 8$

    b. ① $x - 2y + z = 1$

    ② $2x - 5y + z = -1$

    ③ $3x - 7y + 2z = 0$

    ④ $6x - 14y + 4z = 0$

18. Solve the following system of equations for a, b, and c:

    ① $\dfrac{9a}{b} - 8b + \dfrac{3c}{b} = 4$

    ② $\dfrac{-3a}{b} + 4b + \dfrac{4c}{b} = 3$

    ③ $\dfrac{3a}{b} + 4b - \dfrac{4c}{b} = 3$

    $\left(\textit{Hint}\text{: Let } x = \frac{a}{b}, y = b, \text{ and } z = \frac{c}{b}.\right)$

19. Determine the point of intersection of the line $\frac{x+1}{-4} = \frac{y-2}{3} = \frac{z-1}{-2}$ and the plane with equation $x + 2y - 3z + 10 = 0$.

20. Point $A(1, 0, 4)$ is reflected in the plane with equation $x - y + z - 1 = 0$. Determine the coordinates of the image point.

21. The three planes with equations $3x + y + 7z + 3 = 0$, $4x - 12y + 4z - 24 = 0$, and $x + 2y + 3z - 4 = 0$ do not simultaneously intersect.

a. Considering the planes in pairs, determine the three lines of intersection.

b. Show that these three lines are parallel.

22. Solve for a, b, and c in the following system of equations:

$$① \quad \frac{2}{a^2} + \frac{5}{b^2} + \frac{3}{c^2} = 40$$

$$② \quad \frac{3}{a^2} - \frac{6}{b^2} - \frac{1}{c^2} = -3$$

$$③ \quad \frac{9}{a^2} - \frac{5}{b^2} + \frac{4}{c^2} = 67$$

23. Determine the equation of a parabola that has its axis parallel to the y-axis and passes through the points $(-1, 2)$, $(1, -1)$, and $(2, 1)$. (Note that the general form of the parabola that is parallel to the y-axis is $y = ax^2 + bx + c$.)

24. A perpendicular line is drawn from point $X(3, 2, -5)$ to the plane $4x - 5y + z - 9 = 0$ and meets the plane at point M. Determine the coordinates of M.

25. Determine the values of A, B, and C if the following is true:

$$\frac{11x^2 - 14x + 9}{(3x - 1)(x^2 + 1)} = \frac{A}{3x - 1} + \frac{Bx + C}{x^2 + 1}$$

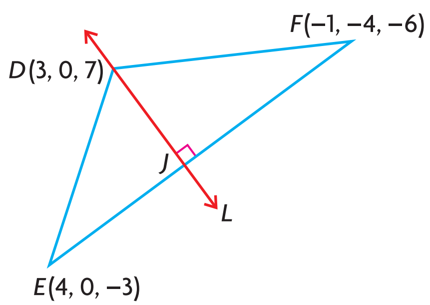
(*Hint*: Simplify the right side by combining fractions and comparing numerators.)

26. A line L is drawn through point D, perpendicular to the line segment EF, and meets EF at point J.

a. Determine an equation for the line containing the line segment EF.

b. Determine the coordinates of point J on EF.

c. Determine the area of $\triangle DEF$.



27. Determine the equation of the plane that passes through $(5, -5, 5)$ and is perpendicular to the line of intersection of the planes $3x - 2z + 1 = 0$ and $4x + 3y + 7 = 0$.

Chapter 9 Test

1. a. Determine the point of intersection for the lines having equations $\vec{r} = (4, 2, 6) + s(1, 3, 11), s \in \mathbf{R}$, and $\vec{r} = (5, -1, 4) + t(2, 0, 9), t \in \mathbf{R}$.

   b. Verify that the intersection point of these two lines is on the plane $x - y + z + 1 = 0$.

2. a. Determine the distance from point $A(3, 2, 3)$ to $\pi: 8x - 8y + 4z - 7 = 0$.

   b. Determine the distance between the planes $\pi_1: 2x - y + 2z - 16 = 0$ and $\pi_2: 2x - y + 2z + 24 = 0$.

3. a. Determine the equation of the line of intersection L between the planes $\pi_1: 2x + 3y - z = 3$ and $\pi_2: -x + y + z = 1$.

   b. Determine the point of intersection between L and the xz-plane.

4. a. Solve the following system of equations:

   ① $x - y + z = 10$

   ② $2x + 3y - 2z = -21$

   ③ $\frac{1}{2}x + \frac{2}{5}y + \frac{1}{4}z = -\frac{1}{2}$

   b. Explain what your solution means geometrically.

5. a. Solve the following system of equations:

   ① $x - y + z = -1$

   ② $2x + 2y - z = 0$

   ③ $x - 5y + 4z = -3$

   b. Explain what your solution means geometrically.

6. The three planes $x + y + z = 0$, $x + 2y + 2z = 1$, and $2x - y + mz = n$ intersect in a line.

   a. Determine the values of m and n for which this is true.

   b. What is the equation of the line?

7. Determine the distance between the skew lines with equations $L_1: \vec{r} = (-1, -3, 0) + s(1, 1, 1), s \in \mathbf{R}$, and $L_2: \vec{r} = (-5, 5, -8) + t(1, 2, 5), t \in \mathbf{R}$.

Cumulative Review of Vectors

1. For the vectors $\vec{a} = (2, -1, -2)$ and $\vec{b} = (3, -4, 12)$, determine the following:
   a. the angle between the two vectors
   b. the scalar and vector projections of $\vec{a}$ on $\vec{b}$
   c. the scalar and vector projections of $\vec{b}$ on $\vec{a}$

2. a. Determine the line of intersection between $\pi_1: 4x + 2y + 6z - 14 = 0$ and $\pi_2: x - y + z - 5 = 0$.
   b. Determine the angle between the two planes.

3. If $\vec{x}$ and $\vec{y}$ are unit vectors, and the angle between them is 60°, determine the value of each of the following:
   a. $|\vec{x} \cdot \vec{y}|$ b. $|2\vec{x} \cdot 3\vec{y}|$ c. $|(2\vec{x} - \vec{y}) \cdot (\vec{x} + 3\vec{y})|$

4. Expand and simplify each of the following, where $\vec{i}$, $\vec{j}$, and $\vec{k}$ represent the standard basis vectors in R^3:
   a. $2(\vec{i} - 2\vec{j} + 3\vec{k}) - 4(2\vec{i} + 4\vec{j} + 5\vec{k}) - (\vec{i} - \vec{j})$
   b. $-2(3\vec{i} - 4\vec{j} - 5\vec{k}) \cdot (2\vec{i} + 3\vec{k}) + 2\vec{i} \cdot (3\vec{j} - 2\vec{k})$

5. Determine the angle that the vector $\vec{a} = (4, -2, -3)$ makes with the positive x-axis, y-axis, and z-axis.

6. If $\vec{a} = (1, -2, 3)$, $\vec{b} = (-1, 1, 2)$, and $\vec{c} = (3, -4, -1)$, determine each of the following:
   a. $\vec{a} \times \vec{b}$
   b. $2\vec{a} \times 3\vec{b}$
   c. the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$
   d. $\vec{c} \cdot (\vec{b} \times \vec{a})$

7. Determine the coordinates of the unit vector that is perpendicular to $\vec{a} = (1, -1, 1)$ and $\vec{b} = (2, -2, 3)$.

8. a. Determine vector and parametric equations for the line that contains $A(2, -3, 1)$ and $B(1, 2, 3)$.
   b. Verify that $C(4, -13, -3)$ is on the line that contains A and B.

9. Show that the lines $L_1: \vec{r} = (2, 0, 9) + t(-1, 5, 2)$, $t \in \mathbf{R}$, and $L_2: x - 3 = \frac{y + 5}{-5} = \frac{z - 10}{-2}$ are parallel and distinct.

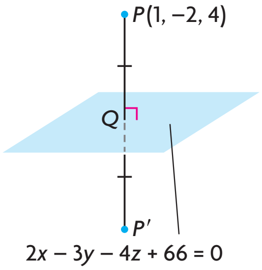
10. Determine vector and parametric equations for the line that passes through $(0, 0, 4)$ and is parallel to the line with parametric equations $x = 1$, $y = 2 + t$, and $z = -3 + t$, $t \in \mathbf{R}$.

11. Determine the value of c such that the plane with equation $2x + 3y + cz - 8 = 0$ is parallel to the line with equation $\frac{x - 1}{2} = \frac{y - 2}{3} = z + 1$.

12. Determine the intersection of the line $\frac{x-2}{3} = y + 5 = \frac{z-3}{5}$ with the plane $5x + y - 2z + 2 = 0$.

13. Sketch the following planes, and give two direction vectors for each.

a. $x + 2y + 2z - 6 = 0$ b. $2x - 3y = 0$ c. $3x - 2y + z = 0$



14. If $P(1, -2, 4)$ is reflected in the plane with equation $2x - 3y - 4z + 66 = 0$, determine the coordinates of its image point, P'. (Note that the plane $2x - 3y - 4z + 66 = 0$ is the right bisector of the line joining $P(1, -2, 4)$ with its image.)

15. Determine the equation of the line that passes through the point $A(1, 0, 2)$ and intersects the line $\vec{r} = (-2, 3, 4) + s(1, 1, 2)$, $s \in \mathbf{R}$, at a right angle.

16. a. Determine the equation of the plane that passes through the points $A(1, 2, 3)$, $B(-2, 0, 0)$, and $C(1, 4, 0)$.

b. Determine the distance from $O(0, 0, 0)$ to this plane.

17. Determine a Cartesian equation for each of the following planes:

a. the plane through the point $A(-1, 2, 5)$ with $\vec{n} = (3, -5, 4)$

b. the plane through the point $K(4, 1, 2)$ and perpendicular to the line joining the points $(2, 1, 8)$ and $(1, 2, -4)$

c. the plane through the point $(3, -1, 3)$ and perpendicular to the z-axis

d. the plane through the points $(3, 1, -2)$ and $(1, 3, -1)$ and parallel to the y-axis

18. An airplane heads due north with a velocity of 400 km/h and encounters a wind of 100 km/h from the northeast. Determine the resultant velocity of the airplane.

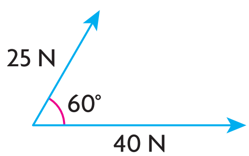
19. a. Determine a vector equation for the plane with Cartesian equation $3x - 2y + z - 6 = 0$, and verify that your vector equation is correct.

b. Using coordinate axes you construct yourself, sketch this plane.

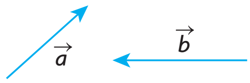
20. a. A line with equation $\vec{r} = (1, 0, -2) + s(2, -1, 2)$, $s \in \mathbf{R}$, intersects the plane $x + 2y + z = 2$ at an angle of θ degrees. Determine this angle to the nearest degree.

b. Show that the planes π_1: $2x - 3y + z - 1 = 0$ and π_2: $4x - 3y - 17z = 0$ are perpendicular.

c. Show that the planes π_3: $2x - 3y + 2z - 1 = 0$ and π_4: $2x - 3y + 2z - 3 = 0$ are parallel but not coincident.



21. Two forces, 25 N and 40 N, have an angle of 60° between them. Determine the resultant and equilibrant of these two vectors.



22. You are given the vectors $\vec{a}$ and $\vec{b}$, as shown at the left.

    a. Sketch $\vec{a} - \vec{b}$.

    b. Sketch $2\vec{a} + \frac{1}{2}\vec{b}$.

23. If $\vec{a} = (6, 2, -3)$, determine the following:

    a. the coordinates of a unit vector in the same direction as $\vec{a}$

    b. the coordinates of a unit vector in the opposite direction to $\vec{a}$

24. A parallelogram $OBCD$ has one vertex at $O(0, 0)$ and two of its remaining three vertices at $B(-1, 7)$ and $D(9, 2)$.

    a. Determine a vector that is equivalent to each of the two diagonals.

    b. Determine the angle between these diagonals.

    c. Determine the angle between $\overrightarrow{OB}$ and $\overrightarrow{OD}$.

25. Solve the following systems of equations:

    a. ① $x - y + z = 2$
    ② $-x + y + 2z = 1$
    ③ $x - y + 4z = 5$

    b. ① $-2x - 3y + z = -11$
    ② $x + 2y + z = 2$
    ③ $-x - y + 3z = -12$

    c. ① $2x - y + z = -1$
    ② $4x - 2y + 2z = -2$
    ③ $2x + y - z = 5$

    d. ① $x - y - 3z = 1$
    ② $2x - 2y - 6z = 2$
    ③ $-4x + 4y + 12z = -4$

26. State whether each of the following pairs of planes intersect. If the planes do intersect, determine the equation of their line of intersection.

    a. $x - y + z - 1 = 0$
    $x + 2y - 2z + 2 = 0$

    b. $x - 4y + 7z = 28$
    $2x - 8y + 14z = 60$

    c. $x - y + z - 2 = 0$
    $2x + y + z - 4 = 0$

27. Determine the angle between the line with symmetric equations $x = -y$, $z = 4$ and the plane $2x - 2z = 5$.

28. a. If $\vec{a}$ and $\vec{b}$ are unit vectors, and the angle between them is 60°, calculate $(6\vec{a} + \vec{b}) \cdot (\vec{a} - 2\vec{b})$.

    b. Calculate the dot product of $4\vec{x} - \vec{y}$ and $2\vec{x} + 3\vec{y}$ if $|\vec{x}| = 3$, $|\vec{y}| = 4$, and the angle between $\vec{x}$ and $\vec{y}$ is 60°.

29. A line that passes through the origin is perpendicular to a plane π and intersects the plane at $(-1, 3, 1)$. Determine an equation for this line and the cartesian equation of the plane.

30. The point $P(-1, 0, 1)$ is reflected in the plane $\pi: y - z = 0$ and has P' as its image. Determine the coordinates of the point P'.

31. A river is 2 km wide and flows at 4 km/h. A motorboat that has a speed of 10 km/h in still water heads out from one bank, which is perpendicular to the current. A marina lies directly across the river, on the opposite bank.

    a. How far downstream from the marina will the motorboat touch the other bank?

    b. How long will it take for the motorboat to reach the other bank?

32. a. Determine the equation of the line passing through $A(2, -1, 3)$ and $B(6, 3, 4)$.

    b. Does the line you found lie on the plane with equation $x - 2y + 4z - 16 = 0$? Justify your answer.

33. A sailboat is acted upon by a water current and the wind. The velocity of the wind is 16 km/h from the west, and the velocity of the current is 12 km/h from the south. Find the resultant of these two velocities.

34. A crate has a mass of 400 kg and is sitting on an inclined plane that makes an angle of 30° with the level ground. Determine the components of the *weight* of the mass, perpendicular and parallel to the plane. (Assume that a 1 kg mass exerts a force of 9.8 N.)

35. State whether each of the following is true or false. Justify your answer.

    a. Any two non-parallel lines in R^2 must always intersect at a point.

    b. Any two non-parallel planes in R^3 must always intersect on a line.

    c. The line with equation $x = y = z$ will always intersect the plane with equation $x - 2y + 2z = k$, regardless of the value of k.

    d. The lines $\frac{x}{2} = y - 1 = \frac{z + 1}{2}$ and $\frac{x - 1}{-4} = \frac{y - 1}{-2} = \frac{z + 1}{-2}$ are parallel.

36. Consider the lines $L_1: x = 2, \frac{y - 2}{3} = z$ and $L_2: x = y + k = \frac{z + 14}{k}$.

    a. Explain why these lines can never be parallel, regardless of the value of k.

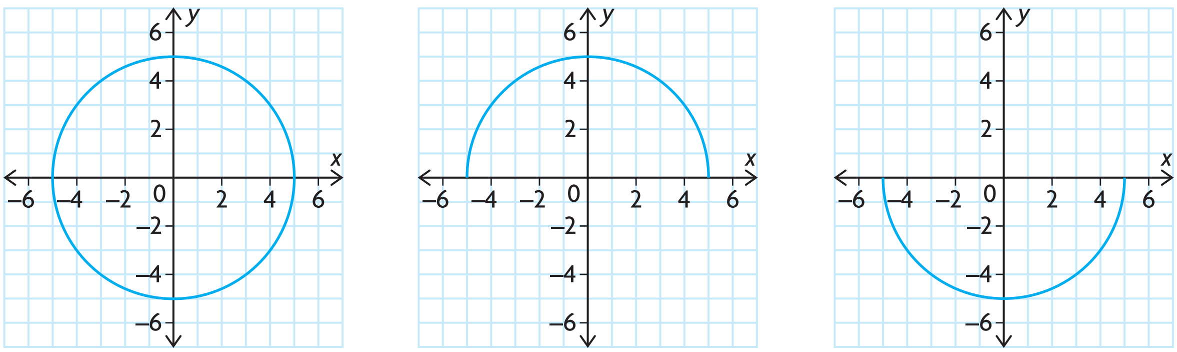
    b. Determine the value of k that makes these two lines intersect at a single point, and find the actual point of intersection.

Calculus Appendix

Implicit Differentiation

In Chapters 1 to 5, most functions were written in the form $y = f(x)$, in which y was defined explicitly as a function of x, such as $y = x^3 - 4x$ and $y = \frac{7}{x^2 + 1}$. In these equations, y is isolated on one side and is expressed explicitly as a function of x.

Functions can also be defined implicitly by relations, such as the circle $x^2 + y^2 = 25$. In this case, the dependent variable, y, is not isolated or explicitly defined in terms of the independent variable, x. Since there are x-values that correspond to two y-values, y is not a function of x on the entire circle. Solving for y gives $y = \pm\sqrt{25 - x^2}$, where $y = \sqrt{25 - x^2}$ represents the upper semicircle and $y = -\sqrt{25 - x^2}$ represents the lower semicircle. The given relation defines two different functions of x.



$x^2 + y^2 = 25$ $\qquad$ $y = \sqrt{25 - x^2}$ $\qquad$ $y = -\sqrt{25 - x^2}$

Consider the problem of determining the slope of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3, -4)$. Since this point lies on the lower semicircle, we could differentiate the function $y = -\sqrt{25 - x^2}$ and substitute $x = 3$. An alternative, which avoids having to solve for y explicitly in terms of x, is to use the method of **implicit differentiation**. Example 1 illustrates this method.

EXAMPLE 1 Selecting a strategy to differentiate an implicit relation

a. If $x^2 + y^2 = 25$, determine $\frac{dy}{dx}$.

b. Determine the slope of the tangent to the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution

a. Differentiate both sides of the equation with respect to x.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

To determine $\frac{d}{dx}(y^2)$, use the chain rule, since y is a function of x.

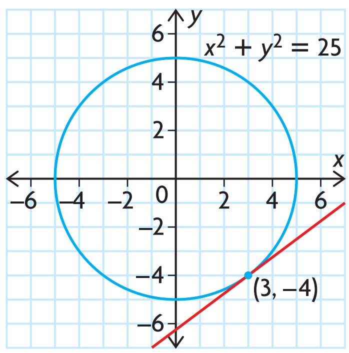
$$\begin{aligned}\frac{d}{dx}(y^2) &= \frac{d(y^2)}{dy} \times \frac{dy}{dx} \\ &= 2y\frac{dy}{dx}\end{aligned}$$

So, $\frac{d}{dx}(x^2) + \frac{d(y^2)}{dy} \times \frac{dy}{dx} = \frac{d}{dx}(25)$ (Substitute)

$$2x + 2y\frac{dy}{dx} = 0$$ $\left(\text{Solve for } \frac{dy}{dx}\right)$

$$\frac{dy}{dx} = -\frac{x}{y}$$

b. The derivative in part a. depends on both x and y. With the derivative in this form, we need to substitute values for both variables.
At the point $(3, -4)$, $x = 3$ and $y = -4$.
The slope of the tangent line to $x^2 + y^2 = 25$ at $(3, -4)$ is
$\frac{dy}{dx} = -\left(\frac{3}{-4}\right) = \frac{3}{4}$.



In Example 1, the derivative could be determined either by using implicit differentiation or by solving for y in terms of x and using one of the methods introduced earlier in the text. There are many situations in which solving for y in terms of x is very difficult and, in some cases, impossible. In such cases, implicit differentiation is the only algebraic method available to us.

EXAMPLE 2 Using implicit differentiation to determine the derivative

Determine $\frac{dy}{dx}$ for $2xy - y^3 = 4$.

Solution

Differentiate both sides of the equation with respect to x as follows:

$$\frac{d}{dx}(2xy) - \frac{d}{dx}(y^3) = \frac{d}{dx}(4)$$

Use the product rule to differentiate the first term and the chain rule to differentiate the second term.

$$\left[\left(\frac{d}{dx}(2x)\right)y + 2x\frac{dy}{dx}\right] - \frac{d(y^3)}{dy} \times \frac{dy}{dx} = \frac{d}{dx}(4)$$

$$2y + 2x\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0 \qquad \text{(Rearrange and factor)}$$

$$(2x - 3y^2)\frac{dy}{dx} = -2y \qquad \left(\text{Solve for } \frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = -\frac{2y}{2x - 3y^2}$$

Procedure for Implicit Differentiation

If an equation defines y implicitly as a differentiable function of x, determine $\frac{dy}{dx}$ as follows:

1: Differentiate both sides of the equation with respect to x. Remember to use the chain rule when differentiating terms containing y.

2: Solve for $\frac{dy}{dx}$.

Note that implicit differentiation leads to a derivative expression that usually includes terms with both x and y. The derivative is defined at a specific point on the original function if, after substituting the x and y coordinates of the point, the value of the denominator is nonzero.

Exercise

PART A

1. State the chain rule. Outline a procedure for implicit differentiation.
2. Determine $\frac{dy}{dx}$ for each of the following in terms of x and y, using implicit differentiation:
   a. $x^2 + y^2 = 36$
   b. $15y^2 = 2x^3$
   c. $3xy^2 + y^3 = 8$
   d. $9x^2 - 16y^2 = -144$
   e. $\frac{x^2}{16} + \frac{3y^2}{13} = 1$
   f. $x^2 + y^2 + 5y = 10$
3. For each relation, determine the equation of the tangent at the given point.
   a. $x^2 + y^2 = 13, (2, -3)$
   b. $x^2 + 4y^2 = 100, (-8, 3)$
   c. $\frac{x^2}{25} - \frac{y^2}{36} = -1, (5\sqrt{3}, -12)$
   d. $\frac{x^2}{81} - \frac{5y^2}{162} = 1, (-11, -4)$

PART B

4. At what point is the tangent to the curve $x + y^2 = 1$ parallel to the line $x + 2y = 0$?
5. The equation $5x^2 - 6xy + 5y^2 = 16$ represents an ellipse.
   a. Determine $\frac{dy}{dx}$ at $(1, -1)$.
   b. Determine two points on the ellipse at which the tangent is horizontal.
6. Determine the slope of the tangent to the ellipse $5x^2 + y^2 = 21$ at the point $A(-2, -1)$.
7. Determine the equation of the normal to the curve $x^3 + y^3 - 3xy = 17$ at the point (2, 3).
8. Determine the equation of the normal to $y^2 = \frac{x^3}{2 - x}$ at the point $(1, -1)$.
9. Determine $\frac{dy}{dx}$.
   a. $(x + y)^3 = 12x$
   b. $\sqrt{x + y} - 2x = 1$
10. The equation $4x^2y - 3y = x^3$ implicitly defines y as a function of x.
    a. Use implicit differentiation to determine $\frac{dy}{dx}$.
    b. Write y as an explicit function of x, and compute $\frac{dy}{dx}$ directly.
    c. Show that your results for parts a. and b. are equivalent.
11. Graph each relation using graphing technology. For each graph, determine the number of tangents that exist when $x = 1$.
    a. $y = \sqrt{3 - x}$
    b. $y = -\sqrt{5 - x}$
    c. $y = x^7 - x$
    d. $x^3 + 4x^2 + (x - 4)y^2 = 0$ (This curve is known as the strophoid.)

PART C

12. Show that $\frac{dy}{dx} = \frac{y}{x}$ for the relation $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = 10, x, y \neq 0$.
13. Determine the equations of the lines that are tangent to the ellipse $x^2 + 4y^2 = 16$ and also pass through the point (4, 6).
14. The angle between two intersecting curves is defined as the angle between their tangents at the point of intersection. If this angle is 90°, the two curves are said to be orthogonal at this point.
    Prove that the curves defined by $x^2 - y^2 = k$ and $xy = p$ intersect orthogonally for all values of the constants k and p. Illustrate your proof with a sketch.
15. Let l be any tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{k}$, where k is a constant. Show that the sum of the intercepts of l is k.
16. Two circles of radius $3\sqrt{2}$ are tangent to the graph $y^2 = 4x$ at the point (1, 2). Determine the equations of these two circles.

Related Rates

Oil that is spilled from a tanker spreads in a circle. The area of the circle increases at a constant rate of 6 km^2/h. How fast is the radius of the spill increasing when the area is 9π km^2? Knowing the rate of increase of the radius is important for planning the containment operation.

In this section, you will encounter some interesting problems that will help you understand the applications of derivatives and how they can be used to describe and predict the phenomena of change. In many practical applications, several quantities vary in relation to one another. The rates at which they vary are also related to one another. With calculus, we can describe and calculate such rates.

EXAMPLE 1 **Solving a related rate problem involving a circular model**

When a raindrop falls into a still puddle, it creates a circular ripple that spreads out from the point where the raindrop hit. The radius of the circle grows at a rate of 3 cm/s.

a. Determine the rate of increase of the circumference of the circle with respect to time.

b. Determine the rate of increase of the area of the circle when its area is 81π cm^2.

Solution

The radius, r, and the circumference of a circle, C, are related by the formula $C = 2\pi r$.

The radius, r, and the area of a circle, A, are related by the formula $A = \pi r^2$.

We are given $\frac{dr}{dt} = 3$ at any time t.

a. To determine $\frac{dC}{dt}$ at any time, it is necessary to differentiate the equation $C = 2\pi r$ with respect to t, using the chain rule.

$$\frac{dC}{dt} = \frac{dC}{dr}\frac{dr}{dt}$$

$$\frac{dC}{dt} = 2\pi\frac{dr}{dt}$$

At time t, since $\frac{dr}{dt} = 3$,

$$\frac{dC}{dt} = 2\pi(3)$$
$$= 6\pi$$

Therefore, the circumference is increasing at a constant rate of 6π cm/s.

b. To determine $\frac{dA}{dt}$, differentiate $A = \pi r^2$ with respect to t, using the chain rule.

$$\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

We know that $\frac{dr}{dt} = 3$, so we need to determine r.

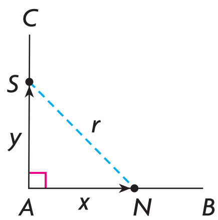
Since $A = 81\pi$ and $A = \pi r^2$,

$$\pi r^2 = 81\pi$$
$$r^2 = 81$$
$$r = 9,\ r > 0, \text{ and } \frac{dA}{dt} = 2\pi(9)(3)$$
$$= 54\pi$$

The area of the circle is increasing at a rate of 54π cm^2/s at the given instant.

Many related-rate problems involve right triangles and the Pythagorean theorem. In these problems, the lengths of the sides of the triangle vary with time. The lengths of the sides and the related rates can be represented quite simply on the Cartesian plane.

EXAMPLE 2 Solving a related rate problem involving a right triangle model



Natalie and Shannon start from point A and drive along perpendicular roads AB and AC, respectively, as shown. Natalie drives at a speed of 45 km/h, and Shannon travels at a speed of 40 km/h. If Shannon begins 1 h before Natalie, at what rate are their cars separating 3 h after Shannon leaves?

Solution

Let x represent the distance that Natalie's car has travelled along AB, and let y represent the distance that Shannon's car has travelled along AC.

Therefore, $\frac{dx}{dt} = 45$ and $\frac{dy}{dt} = 40$, where t is the time in hours. (Note that both of these rates of change are positive since both distances, x and y, are increasing with time.)

Let r represent the distance between the two cars at time t.

Therefore, $x^2 + y^2 = r^2$.

Differentiate both sides of the equation with respect to time.

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(r^2)$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt} \text{ or } x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt}$$

Natalie has travelled for 2 h, or $2 \times 45 = 90$ km.
Shannon has travelled for 3 h, or $3 \times 40 = 120$ km.
The distance between the cars is

$$90^2 + 120^2 = r^2$$
$$r = 150$$

Thus, $x = 90, \frac{dx}{dt} = 45, y = 120, \frac{dy}{dt} = 40$, and $r = 150$

$$\text{So, } 90 \times 45 + 120 \times 40 = 150\frac{dr}{dt}$$ (Substitute)

$$4050 + 4800 = 150\frac{dr}{dt}$$ $\left(\text{Solve for } \frac{dr}{dt}\right)$

$$59 = \frac{dr}{dt}.$$

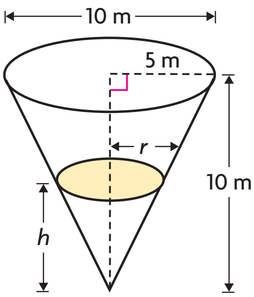
Therefore, the distance between Natalie's car and Shannon's car is increasing at a rate of 59 km/h, 3 h after Shannon leaves.

EXAMPLE 3 Solving a related rate problem involving a conical model

Water is pouring into an inverted right circular cone at a rate of π m^3/min. The height and the diameter of the base of the cone are both 10 m. How fast is the water level rising when the depth of the water is 8 m?

Solution

Let V represent the volume, r represent the radius, and h represent the height of the water in the cone at time t. The volume of the water in the cone, at any time, is $V = \frac{1}{3}\pi r^2 h$. Since we are given $\frac{dV}{dt}$ and we want to determine $\frac{dh}{dt}$ when $h = 8$, we solve for r in terms of h from the ratio determined from the similar triangles $\frac{r}{h} = \frac{5}{10}$ or $r = \frac{1}{2}h$. Therefore, we can simplify the volume formula so it involves only V and h.



Substituting into $V = \frac{1}{3}\pi r^2 h$, we get

$$V = \frac{1}{3}\pi\left(\frac{1}{4}h^2\right)h$$

$$V = \frac{1}{12}\pi h^3$$

Differentiating with respect to time, $\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$.

At a specific time, when $h = 8$ and $\frac{dV}{dt} = \pi$,

$$\pi = \frac{1}{4}\pi(8)^2\frac{dh}{dt}$$

$$\frac{1}{16} = \frac{dh}{dt}$$

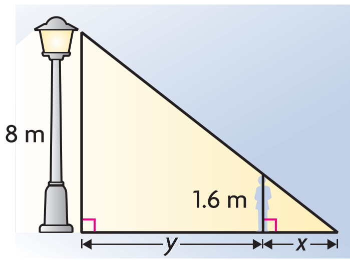
Therefore, at the moment when the depth of the water is 8 m, the level is rising at $\frac{1}{16}$ m/min.

EXAMPLE 4 Solving a related rate problem involving similar triangle models

A student who is 1.6 m tall walks directly away from a lamppost at a rate of 1.2 m/s. A light is situated 8 m above the ground on the lamppost. Show that the student's shadow is lengthening at a rate of 0.3 m/s when she is 20 m from the base of the lamppost.

Solution

Let x be the length of the student's shadow, and let y be her distance from the lamppost, in metres, as shown. Let t denote the time, in seconds.



We are given that $\frac{dy}{dt} = 1.2$ m/s, and we want to determine $\frac{dx}{dt}$ when $y = 20$ m.

To determine a relationship between x and y, use similar triangles.

$$\frac{x + y}{8} = \frac{x}{1.6}$$
$$1.6x + 1.6y = 8x$$
$$1.6y = 6.4x$$

Differentiating both sides with respect to t, $1.6\frac{dy}{dt} = 6.4\frac{dx}{dt}$.

When $y = 20$ and $\frac{dy}{dt} = 1.2$,

$$1.6(1.2) = 6.4\frac{dx}{dt}$$
$$\frac{dx}{dt} = 0.3$$

Therefore, the student's shadow is lengthening at 0.3 m/s. (Note that her shadow is lengthening at a constant rate, independent of her distance from the lamppost.)

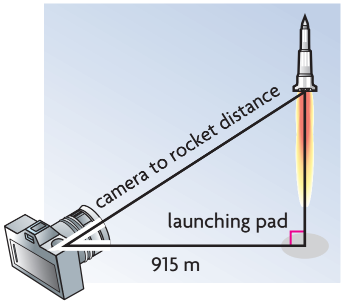
Exercise

PART A

1. Express the following statements in symbols:
   a. The area, A, of a circle is increasing at a rate of 4 m^2/s.
   b. The surface area, S, of a sphere is decreasing at a rate of 3 m^2/min.
   c. After travelling for 15 min, the speed of a car is 70 km/h.
   d. The x- and y-coordinates of a point are changing at equal rates.
   e. The head of a short-distance radar dish is revolving at three revolutions per minute.

PART B

2. The function $T(x) = \frac{200}{1 + x^2}$ represents the temperature, in degrees Celsius, perceived by a person standing x metres from a fire.
   a. If the person moves away from the fire at 2 m/s, how fast is the perceived temperature changing when the person is 5 m away?
   b. Using a graphing calculator, determine the distance from the fire when the perceived temperature is changing the fastest.
   c. What other calculus techniques could be used to check the result?
3. The side of a square is increasing at a rate of 5 cm/s. At what rate is the area changing when the side is 10 cm long? At what rate is the perimeter changing when the side is 10 cm long?
4. Each edge of a cube is expanding at a rate of 4 cm/s.
   a. How fast is the volume changing when each edge is 5 cm?
   b. At what rate is the surface area changing when each edge is 7 cm?
5. The width of a rectangle increases at 2 cm/s, while the length decreases at 3 cm/s. How fast is the area of the rectangle changing when the width equals 20 cm and the length equals 50 cm?
6. The area of a circle is decreasing at the rate of 5 m^2/s when its radius is 3 m.
   a. At what rate is the radius decreasing at that moment?
   b. At what rate is the diameter decreasing at that moment?
7. Oil that is spilled from a ruptured tanker spreads in a circle. The area of the circle increases at a constant rate of 6 km^2/h. How fast is the radius of the spill increasing when the area is 9π km^2?
8. The top of a 5 m wheeled ladder rests against a vertical wall. If the bottom of the ladder rolls away from the base of the wall at a rate of $\frac{1}{3}$ m/s, how fast is the top of the ladder sliding down the wall when it is 3 m above the base of the wall?
9. How fast must someone let out line if a kite is 30 m high, 40 m away horizontally, and continuing to move away horizontally at a rate of 10 m/min?
10. If the rocket shown below is rising vertically at 268 m/s when it is 1220 m up, how fast is the camera-to-rocket distance changing at that instant?



11. Two cyclists depart at the same time from a starting point along routes that make an angle of $\frac{\pi}{3}$ radians with each other. The first cyclist is travelling at 15 km/h, while the second cyclist is moving at 20 km/h. How fast are the two cyclists moving apart after 2 h?

12. A spherical balloon is being filled with helium at a rate of 8 cm^3/s. At what rate is its radius increasing at the following moments.

    a. when the radius is 12 cm

    b. when the volume is 1435 cm^3 (Your answer should be correct to the nearest hundredth.)

    c. when the balloon has been filling for 33.5 s

13. A cylindrical tank, with height 15 m and diameter 2 m, is being filled with gasoline at a rate of 500 L/min. At what rate is the fluid level in the tank rising? $(1 \text{ L} = 1000 \text{ cm}^3)$ About how long will it take to fill the tank?

14. If $V = \pi r^2 h$, determine $\frac{dV}{dt}$ if r and h are both variables that depend on t. In your journal, write three problems that involve the rate of change of the volume of a cylinder such that

    a. r is a variable and h is a constant

    b. r is a constant and h is a variable

    c. r and h are both variables

15. The trunk of a tree is approximately cylindrical in shape and has a diameter of 1 m when the height of the tree is 15 m. If the radius is increasing at 0.003 m per year and the height is increasing at 0.4 m per year, determine the rate of increase of the volume of the trunk at this moment.

16. A conical paper cup, with radius 5 cm and height 15 cm, is leaking water at a rate of 2 cm^3/min. At what rate is the water level decreasing when the water is 3 cm deep?

17. The cross-section of a water trough is an equilateral triangle with a horizontal top edge. If the trough is 5 m long and 25 cm deep, and water is flowing in at a rate of 0.25 m^3/min, how fast is the water level rising when the water is 10 cm deep at the deepest point?

18. The shadow cast by a man standing 1 m from a lamppost is 1.2 m long. If the man is 1.8 m tall and walks away from the lamppost at a speed of 120 m/min, at what rate is the shadow lengthening after 5 s?

PART C

19. A railroad bridge is 20 m above, and at right angles to, a river. A person in a train travelling at 60 km/h passes over the centre of the bridge at the same instant that a person in a motorboat travelling at 20 km/h passes under the centre of the bridge. How fast are the two people separating 10 s later?

20. Liquid is being poured into the top of a funnel at a steady rate of 200 cm^3/s. The funnel is in the shape of an inverted right circular cone, with a radius equal to its height. It has a small hole in the bottom, where the liquid is flowing out at a rate of 20 cm^3/s.

    a. How fast is the height of the liquid changing when the liquid in the funnel is 15 cm deep?

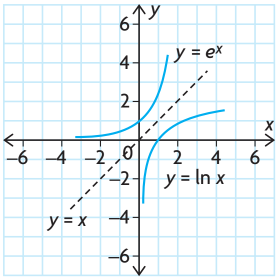
    b. At the instant when the height of the liquid is 25 cm, the funnel becomes clogged at the bottom and no more liquid flows out. How fast does the height of the liquid change just after this occurs?

21. A ladder of length l, standing on level ground, is leaning against a vertical wall. The base of the ladder begins to slide away from the wall. Introduce a coordinate system so that the wall lies along the positive y-axis, the ground is on the positive x-axis, and the base of the wall is the origin.

    a. What is the equation of the path followed by the midpoint of the ladder?

    b. What is the equation of the path followed by any point on the ladder? (*Hint:* Let k be the distance from the top of the ladder to the point on the ladder.)

The Natural Logarithm and its Derivative



The logarithmic function is the inverse of the exponential function. For the particular exponential function $y = e^x$, the inverse is $x = e^y$ or $y = \log_e x$, a logarithmic function where $e \doteq 2.718$. This logarithmic function is referred to as the "natural" logarithmic function and is usually written as $y = \ln x$.

The functions $y = e^x$ and $y = \ln x$ are inverses of each other. This means that the graphs of the functions are reflections of each other in the line $y = x$, as shown.

What is the derivative of the natural logarithmic function?

For $y = \ln x$, the definition of the derivative yields $\frac{dy}{dx} = \lim_{h \to 0} \frac{\ln(x + h) - \ln(x)}{h}$.

We can determine the derivative of the natural logarithmic function using the derivative of the exponential function that we developed earlier.

Given $y = \ln x$, we can rewrite this as $e^y = x$. Differentiating both sides of the equation with respect to x, and using implicit differentiation on the left side, yields

$$e^y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y}$$

$$= \frac{1}{x}$$

The Derivative of the Natural Logarithmic Function

The derivative of the natural logarithmic function $y = \ln x$ is $\frac{dy}{dx} = \frac{1}{x}$, $x > 0$.

This derivative makes sense when we consider the graph of $y = \ln x$. The function is defined only for $x > 0$, and the slopes are all positive. We see that, as $x \to \infty$, $\frac{dy}{dx} \to 0$. As x increases, the slope of the tangent decreases.

We can apply this new derivative, along with the product, quotient, and chain rules to determine derivatives of fairly complicated functions.

EXAMPLE 1 Selecting a strategy to determine the derivative of a function involving a natural logarithm

Determine $\frac{dy}{dx}$ for the following functions:

a. $y = \ln(5x)$ b. $y = \frac{\ln x}{x^3}$ c. $y = \ln(x^2 + e^x)$

Solution

a. $y = \ln(5x)$

Using the chain rule,

$$\frac{dy}{dx} = \frac{1}{5x}(5) = \frac{1}{x}$$

Using properties of logarithms,

$$y = \ln(5x) = \ln(5) + \ln(x)$$

$$\frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x}$$

b. $y = \dfrac{\ln x}{x^3}$

Using the quotient and power rules,

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(\ln(x))\left(x^3 - \ln(x)\frac{d}{dx}(x^3)\right)}{(x^3)^2}$$

$$= \frac{\frac{1}{x}x^3 - \ln(x) \times 3x^2}{x^6} \qquad \text{(Simplify)}$$

$$= \frac{x^2 - 3x^2\ln(x)}{x^6} \qquad \text{(Divide by } x^2\text{)}$$

$$= \frac{1 - 3\ln(x)}{x^4}$$

c. $y = \ln(x^2 + e^x)$

Using the chain rule,

$$\frac{dy}{dx} = \frac{1}{(x^2 + e^x)}\frac{d}{dx}(x^2 + e^x)$$

$$= \frac{2x + e^x}{(x^2 + e^x)}$$

The Derivative of a Composite Natural Logarithmic Function

If $f(x) = \ln(g(x))$, then $f'(x) = \frac{1}{g(x)}g'(x)$, by the chain rule.

EXAMPLE 2 Selecting a strategy to solve a tangent problem

Determine the equation of the line that is tangent to $y = \frac{\ln x^2}{3x}$ at the point where $x = 1$.

Solution

$\ln 1 = 0$, so $y = 0$ when $x = 1$, and the point of contact of the tangent is $(1, 0)$.

The slope of the tangent is given by $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{3x\left(\frac{1}{x^2}\right)2x - 3\ln x^2}{9x^2} \quad \text{(Quotient rule)}$$

$$= \frac{6 - 3\ln x^2}{9x^2}$$

When $x = 1$, $\frac{dy}{dx} = \frac{2}{3}$.

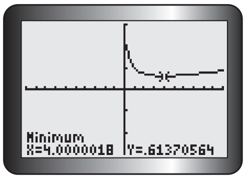
The equation of the tangent is $y - 0 = \frac{2}{3}(x - 1)$, or $2x - 3y - 2 = 0$.

EXAMPLE 3 Determining where the minimum value of a function occurs

a. For the function $f(x) = \sqrt{x} - \ln x$, $x > 0$, use your graphing calculator to determine the x-value that minimizes the value of the function.

b. Use calculus methods to determine the exact x-value where the minimum is attained.

Solution



a. The graph of $f(x) = \sqrt{x} - \ln x$ is shown.

Use the minimum value operation of your calculator to determine the minimum value of $f(x)$. The minimum value occurs at $x = 4$.

b. $f(x) = \sqrt{x} - \ln x$

To minimize $f(x)$, set the derivative equal to zero.

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{x}$$

$$\frac{1}{2\sqrt{x}} - \frac{1}{x} = 0$$

$$\frac{1}{2\sqrt{x}} = \frac{1}{x}$$

$$x = 2\sqrt{x}$$

$$x^2 = 4x$$

$x(x - 4) = 0$

$x = 4$ or $x = 0$

But $x = 0$ is not in the domain of the function, so $x = 4$.

Therefore, the minimum value of $f(x)$ occurs at $x = 4$.

We now look back at the derivative of the natural logarithmic function using the definition.

For the function $f(x) = \ln(x)$,

$$f'(x) = \lim_{h \to 0} \frac{\ln(x + h) - \ln(x)}{h}$$

and, specifically,

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{\ln(1 + h) - \ln(1)}{h} \\ &= \lim_{h \to 0} \frac{\ln(1 + h)}{h} \\ &= \lim_{h \to 0} \ln(1 + h)^{\frac{1}{h}}, \text{ since } \frac{1}{h}\ln(1 + h) = \ln(1 + h)^{\frac{1}{h}} \end{aligned}$$

However, we know that $f'(x) = \frac{1}{x}, f'(1) = 1$.

We conclude that $\lim_{h \to 0} \ln(1 + h)^{\frac{1}{h}} = 1$.

Since the natural logarithmic function is a continuous and one-to-one function (meaning that, for each function value, there is exactly one value of the independent variable that produces this function value), we can rewrite this as $\ln\left[\lim_{h \to 0}(1 + h)^{\frac{1}{h}}\right] = 1$.

Since $\ln e = 1$, $\ln\left[\lim_{h \to 0}(1 + h)^{\frac{1}{h}}\right] = \ln e$.

Therefore, $\lim_{h \to 0}(1 + h)^{\frac{1}{h}} = e$.

We now have a way to approximate the value of e using the above limit.

h	0.1	0.01	0.001	0.0001
$(1 + h)^{\frac{1}{h}}$	2.593 742 46	2.704 813 829	2.716 923 932	2.718 145 927

From the table, it appears that $e \doteq 2.718$ is a good approximation as h approaches zero.

Exercise

PART A

1. Distinguish between natural logarithms and common logarithms.

2. At the end of this section, we found that we could approximate the value of e (Euler's constant) using $e = \lim_{h\to 0}(1 + h)^{\frac{1}{h}}$. By substituting $h = \frac{1}{n}$, we can express e as $e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$. Justify this definition by evaluating the limit for increasing values of n.

3. Determine the derivative for each of the following:

   a. $y = \ln(5x + 8)$

   b. $y = \ln(x^2 + 1)$

   c. $s = 5\ln t^3$

   d. $y = \ln\sqrt{x + 1}$

   e. $s = \ln(t^3 - 2t^2 + 5)$

   f. $w = \ln\sqrt{z^2 + 3z}$

4. Differentiate each of the following:

   a. $f(x) = x\ln x$

   b. $y = e^{\ln x}$

   c. $v = e^t\ln t$

   d. $g(z) = \ln(e^{-z} + ze^{-z})$

   e. $s = \dfrac{e^t}{\ln t}$

   f. $h(u) = e^{\sqrt{u}}\ln\sqrt{u}$

5. a. If $g(x) = e^{2x-1}\ln(2x - 1)$, evaluate $g'(1)$.

   b. If $f(t) = \ln\left(\frac{t - 1}{3t + 5}\right)$, evaluate $f'(5)$.

   c. Check your calculations for parts a. and b. using either a calculator or a computer.

6. For each of the following functions, solve the equation $f'(x) = 0$:

   a. $f(x) = \ln(x^2 + 1)$

   b. $f(x) = (\ln x + 2x)^{\frac{1}{3}}$

   c. $f(x) = (x^2 + 1)^{-1}\ln(x^2 + 1)$

7. a. Determine the equation of the tangent to the curve defined by $f(x) = \dfrac{\ln\sqrt[3]{x}}{x}$ at the point where $x = 1$.

   b. Use technology to graph the function in part a. and then draw the tangent at the point where $x = 1$.

   c. Compare the equation you obtained in part a. with the equation you obtained in part b.

PART B

8. Determine the equation of the tangent to the curve defined by $y = \ln x - 1$ that is parallel to the straight line with equation $3x - 6y - 1 = 0$.

9. a. If $f(x) = (x\ln x)^2$, determine all the points at which the graph of $f(x)$ has a horizontal tangent line.

   b. Use graphing technology to check your work in part a.

   c. Comment on the efficiency of the two solutions.

10. Determine the equation of the tangent to the curve defined by $y = \ln(1 + e^{-x})$ at the point where $x = 0$.

11. The velocity, in kilometres per hour, of a car as it begins to slow down is given by the equation $v(t) = 90 - 30\ln(3t + 1)$, where t is in seconds.

    a. What is the velocity of the car as the driver begins to brake?

    b. What is the acceleration of the car?

    c. What is the acceleration at $t = 2$?

    d. How long does the car take to stop?

PART C

12. Use the definition of the derivative to evaluate $\lim_{h\to 0}\dfrac{\ln(2 + h) - \ln(2)}{h}$.

13. Consider $f(x) = \ln(\ln x)$.

    a. Determine $f'(x)$.

    b. State the domains of $f(x)$ and $f'(x)$.

The Derivatives of General Logarithmic Functions

In the previous section, we learned how to determine the derivative of the natural logarithmic function (base e). But what is the derivative of $y = \log_2 x$? The base of this function is 2, not e.

To differentiate the general logarithmic function $y = \log_a x$, $a > 0$, $a \neq 1$, we can use the properties of logarithms so that we can use the base e.

Let $y = \log_a x$.

Then $a^y = x$.

Take the logarithm of both sides using the base e.

$$\ln a^y = \ln x$$

$$y \ln a = \ln x$$

$$y = \frac{\ln x}{\ln a}$$

Differentiating both sides with respect to x, we obtain

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d\left(\frac{\ln x}{\ln a}\right)}{dx} \\
&= \frac{1}{\ln a} \times \frac{d(\ln x)}{dx} \quad (\ln a \text{ is a constant.}) \\
&= \frac{1}{\ln a} \times \frac{1}{x} \\
&= \frac{1}{x \ln a}
\end{aligned}$$

The Derivative of the Logarithmic Function $y = \log_a x$

If $y = \log_a x$, $a > 0$, $a \neq 1$, then $\frac{dy}{dx} = \frac{1}{x \ln a}$.

EXAMPLE 1 Solving a tangent problem involving a logarithmic function

Determine the equation of the tangent to $y = \log_2 x$ at $(8, 3)$.

Solution

The slope of the tangent is given by the derivative $\frac{dy}{dx}$, where $y = \log_2 x$.

$$\frac{dy}{dx} = \frac{1}{x \ln 2}$$

At $x = 8$, $\frac{dy}{dx} = \frac{1}{8 \ln 2}$.

The equation of the tangent is

$$y - 3 = \frac{1}{8 \ln 2}(x - 8)$$

$$y = \frac{1}{8 \ln 2}x + 3 - \frac{1}{\ln 2}$$

We can determine the derivatives of other logarithmic functions using the rule $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$, along with other derivative rules.

EXAMPLE 2 Selecting a strategy to differentiate a composite logarithmic function

Determine the derivative of $y = \log_4(2x + 3)^5$.

Solution

We can rewrite the logarithmic function as follows:

$$y = \log_4(2x + 3)^5$$

$$y = 5 \log_4(2x + 3) \qquad \text{(Property of logarithms)}$$

$$\frac{dy}{dx} = \frac{d}{dx}[5 \log_4(2x + 3)]$$

$$= 5\frac{d}{dx}[\log_4(2x + 3)]$$

$$= 5\frac{d[\log_4(2x + 3)]}{d[(2x + 3)]}\frac{d(2x + 3)}{dx} \qquad \text{(Chain rule)}$$

$$= 5\left(\frac{1}{(2x + 3) \ln 4}\right)(2) \qquad \text{(Simplify)}$$

$$= \frac{10}{(2x + 3) \ln 4}$$

The Derivative of a Composite Function Involving $y = \log_a x$

If $y = \log_a f(x)$, $a > 0$, $a \neq 1$, then $\frac{dy}{dx} = \frac{f'(x)}{f(x) \ln a}$.

Exercise

PART A

1. Determine $\frac{dy}{dx}$ for each function.
   a. $y = \log_5 x$
   b. $y = \log_3 x$
   c. $y = 2 \log_4 x$
   d. $y = -3 \log_7 x$
   e. $y = -(\log x)$
   f. $y = 3 \log_6 x$

2. Determine the derivative of each function.
   a. $y = \log_3(x + 2)$
   b. $y = \log_8(2x)$
   c. $y = -3 \log_3(2x + 3)$
   d. $y = \log_{10}(5 - 2x)$
   e. $y = \log_8(2x + 6)$
   f. $y = \log_7(x^2 + x + 1)$

PART B

3. a. If $f(t) = \log_2\left(\frac{t + 1}{2t + 7}\right)$, evaluate $f'(3)$.
   b. If $h(t) = \log_3(\log_2(t))$, determine $h'(8)$.

4. Differentiate.
   a. $y = \log_{10}\left(\frac{1 + x}{1 - x}\right)$
   b. $y = \log_2 \sqrt{x^2 + 3x}$
   c. $y = 2 \log_3(5^x) - \log_3(4^x)$
   d. $y = 3^x \log_3 x$
   e. $y = 2x \log_4 x$
   f. $y = \frac{\log_5(3x^2)}{\sqrt{x + 1}}$

5. Determine the equation of the tangent to the curve $y = x \log x$ at $x = 10$. Graph the function and the tangent.

6. Explain why the derivative of $y = \log_a kx$, $k > 0$, is $\frac{dy}{dx} = \frac{1}{x \ln a}$ for any constant k.

7. Determine the equation of the tangent to the curve $y = 10^{2x-9} \log_{10}(x^2 - 3x)$ at $x = 5$.

8. A particle's distance, in metres, from a fixed point at time, t, in seconds is given by $s(t) = t \log_6(t + 1)$, $t \geq 0$. Is the distance increasing or decreasing at $t = 15$? How do you know?

PART C

9. a. Determine the equation of the tangent to $y = \log_3 x$ at the point (9, 2).
   b. Graph the function and include any asymptotes.
   c. Will this tangent line intersect any asymptotes? Explain.

10. Determine the domain, critical numbers, and intervals of increase and decrease of $f(x) = \ln(x^2 - 4)$.

11. Do the graphs of either of these functions have points of inflection? Justify your answers with supporting calculations.
    a. $y = x \ln x$
    b. $y = 3 - 2 \log x$

12. Determine whether the slope of the graph of $y = 3^x$ at the point $(0, 1)$ is greater than the slope of the graph of $y = \log_3 x$ at the point $(1, 0)$. Include graphs with your solution.

Logarithmic Differentiation

The derivatives of most functions involving exponential and logarithmic expressions can be determined by using the methods that we have developed. A function such as $y = x^x$ poses new problems, however. The power rule cannot be used because the exponent is not a constant. The method of determining the derivative of an exponential function also cannot be used because the base is not a constant. What can be done?

It is frequently possible, with functions presenting special difficulties, to simplify the function by using the properties of logarithms. In such cases we say that we are using **logarithmic differentiation**.

EXAMPLE 1 **Determining the derivative of a function using logarithmic differentiation**

Determine $\frac{dy}{dx}$ for the function $y = x^x$, $x > 0$.

Solution

Take the natural logarithms of each side, and rewrite.

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

Differentiate both sides with respect to x, using implicit differentiation on the left side and the product rule on the right side.

$$\frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + \ln x$$

$$\frac{dy}{dx} = y(1 + \ln x)$$

$$= x^x(1 + \ln x)$$

This method of logarithmic differentiation also works well to help simplify a function with many factors and powers before the differentiation takes place.

We can use logarithmic differentiation to prove the power rule, $\frac{d}{dx}(x^n) = nx^{n-1}$, for all real values of n. (In a previous chapter, we proved this rule for positive integer values of n and we have been cheating a bit in using it for other values of n.)

Given the function $y = x^n$, for any real value of n where $x > 0$, how do we determine $\frac{dy}{dx}$?

To solve this, we take the natural logarithm of both sides of the expression and get $\ln y = \ln x^n = n \ln x$.

Differentiating both sides with respect to x, using implicit differentiation, and remembering that n is a constant, we get

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= n\frac{1}{x}\\ \frac{dy}{dx} &= ny\frac{1}{x}\\ &= nx^n\frac{1}{x}\\ &= nx^{n-1}\end{aligned}$$

Therefore, $\frac{d}{dx}(x^n) = nx^{n-1}$ for any real value of n.

EXAMPLE 2

Selecting a logarithmic differentiation strategy to determine a derivative

For $y = (x^2 + 3)^x$, determine $\frac{dy}{dx}$.

Solution

Take the natural logarithm of both sides of the equation.

$$\begin{aligned}y &= (x^2 + 3)^x\\ \ln y &= \ln (x^2 + 3)^x\\ \ln y &= x \ln (x^2 + 3)\end{aligned}$$

Differentiate both sides of the equation with respect to x, using implicit differentiation on the left side and the product and chain rules on the right side.

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= (1)\ln(x^2 + 3) + x\left(\frac{1}{x^2 + 3}(2x)\right)\\ \frac{dy}{dx} &= y\left[\ln(x^2 + 3) + x\left(\frac{2x}{x^2 + 3}\right)\right]\\ &= (x^2 + 3)^x\left[\ln(x^2 + 3) + \left(\frac{2x^2}{x^2 + 3}\right)\right]\end{aligned}$$

You will recognize logarithmic differentiation as the method used in the previous section, and its use makes memorization of many formulas unnecessary. It also allows complicated functions to be handled much more easily.

EXAMPLE 3 Using logarithmic differentiation

Given $y = \frac{(x^4 + 1)\sqrt{x + 2}}{(2x^2 + 2x + 1)}$, determine $\frac{dy}{dx}$ at $x = -1$.

Solution

While it is possible to determine $\frac{dy}{dx}$ using a combination of the product, quotient, and chain rules, this process is awkward and time-consuming. Instead, before differentiating, we take the natural logarithm of both sides of the equation.

Since $y = \frac{(x^4 + 1)\sqrt{x + 2}}{(2x^2 + 2x + 1)}$,

$$\begin{aligned}
\ln y &= \ln\left[\frac{(x^4 + 1)\sqrt{x + 2}}{(2x^2 + 2x + 1)}\right] \\
\ln y &= \ln(x^4 + 1) + \ln\sqrt{x + 2} - \ln(2x^2 + 2x + 1) \\
\ln y &= \ln(x^4 + 1) + \frac{1}{2}\ln(x + 2) - \ln(2x^2 + 2x + 1)
\end{aligned}$$

The right side of this equation looks much simpler. We can now differentiate both sides with respect to x, using implicit differentiation on the left side.

$$\begin{aligned}
\frac{1}{y}\frac{dy}{dx} &= \frac{1}{x^4 + 1}(4x^3) + \frac{1}{2}\frac{1}{x + 2} - \frac{1}{2x^2 + 2x + 1}(4x + 2) \\
\frac{dy}{dx} &= y\left[\frac{4x^3}{x^4 + 1} + \frac{1}{2(x + 2)} - \frac{4x + 2}{2x^2 + 2x + 1}\right] \\
&= \frac{(x^4 + 1)\sqrt{x + 2}}{(2x^2 + 2x + 1)}\left[\frac{4x^3}{x^4 + 1} + \frac{1}{2(x + 2)} - \frac{4x + 2}{2x^2 + 2x + 1}\right]
\end{aligned}$$

While this derivative is a very complicated function, the process of determining the derivative is straightforward, using only the derivative of the natural logarithmic function and the chain rule.

We do not need to simplify this in order to determine the value of the derivative at $x = -1$.

$$\begin{aligned}
\text{For } x = -1, \frac{dy}{dx} &= \frac{(1 + 1)\sqrt{1}}{(2 - 2 + 1)}\left[\frac{-4}{1 + 1} + \frac{1}{2(-1 + 2)} - \frac{-4 + 2}{2 - 2 + 1}\right] \\
&= 2\left[-2 + \frac{1}{2} + 2\right] \\
&= 1
\end{aligned}$$

Exercise

PART A

1. Differentiate each of the following:

   a. $y = x^{\sqrt{10}} - 3$

   b. $f(x) = 5x^{3\sqrt{2}}$

   c. $s = t^{\pi}$

   d. $f(x) = x^e + e^x$

2. Use the method of logarithmic differentiation to determine the derivative for each of the following functions:

   a. $y = x^{\ln x}$

   b. $y = \dfrac{(x + 1)(x - 3)^2}{(x + 2)^3}$

   c. $y = x^{\sqrt{x}}$

   d. $s = \left(\dfrac{1}{t}\right)^t$

3. a. If $y = f(x) = x^x$, evaluate $f'(e)$.

   b. If $s = e^t + t^e$, determine $\frac{ds}{dt}$ when $t = 2$.

   c. If $f(x) = \dfrac{(x - 3)^2\sqrt[3]{x + 1}}{(x - 4)^5}$, determine $f'(7)$.

4. Determine the equation of the tangent to the curve defined by $y = x^{(x^2)}$ at the point where $x = 2$.

PART B

5. If $y = \dfrac{1}{(x + 1)(x + 2)(x + 3)}$, determine the slope of the tangent to the curve at the point where $x = 0$.

6. Determine the points on the curve defined by $y = x^{\frac{1}{x}}$, $x > 0$, where the slope of the tangent is zero.

7. If tangents to the curve defined by $y = x^2 + 4 \ln x$ are parallel to the line defined by $y - 6x + 3 = 0$, determine the points where the tangents touch the curve.

8. The tangent to the curve defined by $y = x^{\sqrt{x}}$ at the point $A(4, 16)$ is extended to cut the x-axis at B and the y-axis at C. Determine the area of $\triangle OBC$, where O is the origin.

PART C

9. Determine the slope of the line that is tangent to the curve defined by $y = \dfrac{e^x\sqrt{x^2 + 1}}{(x^2 + 2)^3}$ at the point $\left(0, \frac{1}{8}\right)$.

10. Determine $f'(x)$ if $f(x) = \left(\dfrac{x \sin x}{x^2 - 1}\right)^2$.

11. Differentiate $y = x^{\cos x}$, $x > 0$.

12. Determine the equation of the line that is tangent to the curve $y = x^x$ at the point $(1, 1)$.

13. The position of a particle that moves on a straight line is given by $s(t) = t^{\frac{1}{t}}$ for $t > 0$.

    a. Determine the velocity and acceleration functions.

    b. At what time, t, is the velocity zero? What is the acceleration at this time?

14. Make a conjecture about which number is larger: e^{π} or π^e. Verify your work with a calculator.

Vector Appendix

Gaussian Elimination

In Chapter 9, we developed a method for solving systems of linear equations that uses elementary operations to eliminate unknowns. We will now introduce a method for solving equations that is known as Gaussian elimination. This method uses matrices and elementary operations to deal with larger systems of equations more easily.

Consider the following system of three equations in three unknowns:

① $x + 2y + 2z = 9$
② $x + y = 1$
③ $2x + 3y - z = 1$

To use Gaussian elimination to solve a system of equations, first write the system in matrix form—an abbreviated form that omits the variables and uses only the coefficients of each equation. Each row in the matrix represents an equation from the original system. For example, the second equation in the original system is represented by the second row in the matrix. The coefficients of the unknowns are entered in columns on the left side of the matrix, with a vertical line separating the coefficients from the numbers on the right side. When a system of equations is written in this form, the associated matrix is called its **augmented matrix**. The matrix below is the augmented matrix representing the original system of equations. Note that the 0 in row 2 of the augmented matrix represents the coefficient of z in the second equation, if the equation had been written as $1x + 1y + 0z = 1$. We have also included a second matrix, called the **coefficient matrix**. We make sure that in the coefficient matrix, each column corresponds to a single variable. For instance, in this case the first column corresponds to the coefficients for x. This matrix represents the coefficients of the unknowns in each of the equations.

Coefficient Matrix

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 2 & 3 & -1 \end{bmatrix}$$

Augmented Matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 1 & 1 & 0 & 1 \\ 2 & 3 & -1 & 1 \end{array}\right]$$

The benefit of using matrices (the plural of *matrix*) is that it allows us to introduce a method for solving systems of equations that is systematic, methodical, and useful for dealing with larger systems of equations.

For solving systems of equations using matrices, we introduce operations that are similar to the previously introduced elementary operations. When dealing with matrices, however, we call them **elementary row operations**. Notice the term *row operations*—they are only applied to the rows. Just as with elementary operations, elementary row operations are always used to reduce matrices into simpler form by eliminating unknowns. In Example 1, notice that the previously introduced elementary operations have been modified for use with matrices.

Elementary Row Operations for Matrices (Systems of Equations)

1. Multiply a row (equation) by a nonzero constant.
2. Interchange any pair of rows (equations).
3. Add a multiple of one row (equation) to a second row (equation) to replace the second row (equation).

EXAMPLE 1 Using elementary row operations to solve a system of equations

Solve the following system of equations using elementary row operations:

① $x + 2y + 2z = 9$
② $x + y = 1$
③ $2x + 3y - z = 1$

Solution

1: Start by writing the system of equations in its augmented matrix form.

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 1 & 1 & 0 & 1 \\ 2 & 3 & -1 & 1 \end{array}\right]$$

2: Multiply row 1 by -1, and add the result to row 2. This leaves the first row unchanged, and the second row is replaced with the result.

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -8 \\ 2 & 3 & -1 & 1 \end{array}\right] \quad -1\text{ (row 1)} + \text{row 2}$$

When solving a system of equations using elementary row operations, normally we try to make the first entry in the second row 0, because our ultimate objective is to have an augmented matrix in **row-echelon form.** In this form, the second row of the coefficient matrix can only have nonzero entries in the y- and z-columns, and the third row can only have a nonzero entry in the z-column (so that the coefficient matrix looks like an *upper triangle* of nonzero entries). Notice that we indicate the operations we use to reduce the matrix.

3: Multiply row 1 by -2, and add the result to row 3.

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -8 \\ 0 & -1 & -5 & -17 \end{array}\right] \quad -2\text{ (row 1)} + \text{row 3}$$

We have now produced a new row 3, with 0 as the coefficient of x.

4: Multiply row 2 by -1, and add the result to row 3.

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -8 \\ 0 & 0 & -3 & -9 \end{array}\right] \quad -1\text{ (row 2)} + \text{row 3}$$

This final operation is equivalent to subtracting row 2 from row 3, with the final result being a new row 3.

The new matrix corresponds to the following equivalent system of equations:

① $x + 2y + 2z = 9$
④ $-y - 2z = -8$
⑤ $-3z = -9$

This system can now be solved using back substitution. From equation ⑤, $z = 3$. Substituting into equation ④, $-y - 2(3) = -8$, or $y = 2$. If we substitute $y = 2$ and $z = 3$ into equation ①, we obtain $x + 2(2) + 2(3) = 9$, or $x = -1$.

We conclude that $x = -1$, $y = 2$, and $z = 3$ is the solution to the system of equations. So the original system represents three planes intersecting at the point $(-1, 2, 3)$.

Check:
Substituting into equation ①, $x + 2y + 2z = -1 + 2(2) + 2(3) = 9$.
Substituting into equation ②, $x + y = -1 + 2 = 1$.
Substituting into equation ③, $2x + 3y - z = 2(-1) + 3(2) - 3 = 1$.

When solving a system of equations, either with or without matrices, it is always a good idea to verify the final result.

Important Points about Elementary Row Operations

1. Elementary row operations are used to produce equivalent matrices, but they can be applied in different orders, provided that they are applied properly. There are many ways to get to the final answer.
2. Steps can be combined, and every step does *not* have to be written out in words. In the following examples, we will demonstrate ways of abbreviating steps and reducing the amount of written work.
3. When using elementary row operations, steps should be documented to show how a new matrix was determined. This allows for easy checking in case a mistake is made.

When using elementary row operations to solve systems of equations, the objective is to have the final matrix written in what is described as row-echelon form. In Example 1, we accomplished this but did not define row-echelon form. If a matrix is written in row-echelon form, it must have the following properties:

Properties of a Matrix in Row-Echelon Form

1. All rows that consist entirely of zeros must be written at the bottom of the matrix.
2. In any two successive rows that do not consist entirely of zeros, the first nonzero number in the lower row must occur farther to the right than the first nonzero number in the row directly above.

EXAMPLE 2 Reasoning about the row-echelon form of a matrix

Determine whether the following matrices are in row-echelon form. If they are not in this form, use an elementary row operation to put them in this form.

a. $\left[\begin{array}{ccc|c} 0 & 2 & 3 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right]$

c. $\left[\begin{array}{ccc|c} 2 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$

b. $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$

d. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{array}\right]$

Solution

a. This augmented matrix is not in row-echelon form. It can be changed to the correct form by multiplying row 1 by -1 and adding the product to row 2. The resulting matrix will be in row-echelon form.

$$\left[\begin{array}{ccc|c} 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right] \quad -1\text{ (row 1)} + \text{row 2}$$

b. This matrix is in row-echelon form. The first nonzero number in row 3 is 1, which occurs to the right of the first nonzero number, -1, in row 2. Similarly the -1 in row 2 occurs to the right of the first 1 in row 1.

c. This matrix is not in row-echelon form. To put it in row-echelon form, interchange the second and third rows.

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{Interchanging rows 2 and 3}$$

d. This matrix is in row-echelon form. The first nonzero number in row 3 is 8, which is to the right of -2 in row 2, and -2 occurs to the right of 1 in row 1.

In the next example, by using elementary row operations, we will solve a system in which the three planes intersect along a line.

EXAMPLE 3 Using elementary row operations to solve a system of equations

Solve the following system of equations using matrices:

① $2x + y + 2z = 2$
② $x + y + z = 2$
③ $x + z = 0$

Solution

To solve this system, first write it in augmented matrix form, but interchange the order of the equations so that 1 will be the coefficient of x in the first equation (row 2 becomes row 1, row 3 becomes row 2, and row 1 becomes row 3). There is more than one way to write the augmented matrix, which is a reminder that there is more than one way to solve this system using matrices.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 2 \end{array}\right]$$

This matrix results from interchanging the rows in the original equations.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & 0 & -2 \\ 0 & -1 & 0 & -2 \end{array}\right]$$

-1 (row 1) + row 2
-2 (row 1) + row 3

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

-1 (row 2) + row 3

This system is now in row-echelon form, which means that we can solve it using the method of back substitution. Row 2 corresponds to $-y = -2$, or $y = 2$. Since row 3 consists only of zeros, this corresponds to $0x + 0y + 0z = 0$, which implies a parametric solution. If $z = s$ and $y = 2$, these values can be substituted into the first equation to obtain x in terms of the parameter.

Since the first equation is $x + y + z = 2$, substituting gives $x + 2 + s = 2$. So, $x = -s$ and the solution to the system is $x = -s$, $y = 2$, and $z = s$.

This means that the three planes intersect along a line with parametric equations $x = -s$, $y = 2$, and $z = s$, or, written as a vector equation, $\vec{r} = (0, 2, 0) + s(-1, 0, 1)$, $s \in \mathbf{R}$.

Check:
Substituting into equation ②, $x + y + z = -s + 2 + s = 2$.
Substituting into equation ③, $x + z = -s + s = 0$.
Substituting into equation ①, $2x + y + 2z = 2(-s) + 2 + 2(s) = 2$.

Exercise

PART A

1. Write an augmented matrix for each system of equations.

   a. ① $x + 2y - z = -1$
   ② $-x + 3y - 2z = -1$
   ③ $3y - 2z = -3$

   b. ① $2x - z = 1$
   ② $2y - z = 16$
   ③ $-3x + y = 10$

   c. ① $2x - y - z = -2$
   ② $x - y + 4z = -1$
   ③ $-x - y = 13$

2. Determine two different row-echelon forms for the following augmented matrix:

$$\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 3 & -1 & 1 \end{array}\right]$$

3. Reduce the following augmented matrix to row-echelon form. Make sure that there are no fractions in the final matrix.

$$\left[\begin{array}{ccc|c} 2 & 1 & 6 & 0 \\ 0 & -2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right]$$

4. a. Write the following augmented matrix in row-echelon form. Make sure that every number in this matrix is an integer.

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 2 \\ 0 & -1 & 2 & 0 \\ \frac{1}{2} & -\frac{3}{4} & -2 & \frac{1}{3} \end{array}\right]$$

   b. Solve the system of equations corresponding to the matrix you derived in part a.

5. Write the system of equations that corresponds to each augmented matrix.

   a. $\left[\begin{array}{cc|c} 1 & -2 & -1 \\ 2 & -3 & 1 \\ 2 & -1 & 0 \end{array}\right]$

   b. $\left[\begin{array}{ccc|c} -2 & 0 & -1 & 0 \\ 1 & -2 & 0 & 4 \\ 0 & 1 & 2 & -3 \end{array}\right]$

   c. $\left[\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & 0 \end{array}\right]$

6. The following matrices represent the final row-echelon form matrix in the solution to a system of equations. Write the solution to each system, if it exists.

   a. $\left[\begin{array}{cc|c} -2 & 1 & 6 \\ 0 & -5 & 15 \end{array}\right]$

   b. $\left[\begin{array}{ccc|c} 2 & -1 & 1 & 11 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 6 & -36 \end{array}\right]$

   c. $\left[\begin{array}{ccc|c} -1 & 3 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & -13 \end{array}\right]$

   d. $\left[\begin{array}{ccc|c} 4 & -1 & -1 & 0 \\ 0 & -1 & 0 & 4 \\ 0 & 0 & -1 & 5 \end{array}\right]$

   e. $\left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

   f. $\left[\begin{array}{ccc|c} -1 & 0 & 0 & -4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -1 & 2 \end{array}\right]$

7. A student performs elementary row operations on an augmented matrix and comes up with the following matrix:

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 3 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

a. Explain why this matrix is in row-echelon form.

b. Explain why there is no solution to the corresponding system of equations.

c. Give an example of an augmented matrix consisting of all nonzero numbers that might have produced this matrix.

PART B

8. Determine whether the following matrices are in row-echelon form. If they are not, use elementary row operations to put them in row-echelon form.

a. $\left[\begin{array}{ccc|c} -1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right]$

b. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 3 & 1 & -4 & 2 \\ 0 & 0 & 3 & 6 \end{array}\right]$

c. $\left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -6 \end{array}\right]$

d. $\left[\begin{array}{ccc|c} 1 & -4 & 1 & 0 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

9. a. Write the solution to the system of equations corresponding to each of the augmented matrices in question 8. if a solution exists.

b. Give a geometric interpretation of your result.

10. Solve each system of equations using matrices, and interpret your result geometrically.

a. ① $-x + y + z = 9$
② $x - 2y + z = 15$
③ $2x - y - z = -12$

b. ① $x + y + z = 0$
② $2x + 3y + z = 0$
③ $-3x - 2y - 4z = 0$

c. ① $x - y + 3z = -1$
② $5x + y - 3z = -5$
③ $2x + y - 3z = -2$

d. ① $x + 3y + 4z = 4$
② $-x + 3y + 8z = -4$
③ $x - 3y - 4z = -4$

e. ① $2x + y + z = 1$
② $4x + 2y + 2z = 2$
③ $-2x + y + z = 3$

f. ① $x - y = -500$
② $2y - z = 3500$
③ $x - z = 2000$

11. In the following system of equations, *a*, *b*, and *c* are written as linear combinations of *x*, *y*, and *z*:

$a = x + 2y - z$

$b = x - y + 2z$

$c = 3x + 3y + z$

Express each of *x*, *y*, and *z* as a linear combination of *a*, *b*, and *c*.

12. Determine the equation of the parabola that passes through the points $A(-1, -7)$, $B(2, 20)$, and $C(-3, -5)$, and has an axis of symmetry parallel to the y-axis. (A parabola whose axis of symmetry is parallel to the y-axis has an equation of the form $y = ax^2 + bx + c$.)

PART C

13. Solve for p, q, and r in the following system of equations:

① $pq - 2\sqrt{q} + 3rq = 8$

② $2pq - \sqrt{q} + 2rq = 7$

③ $-pq + \sqrt{q} + 2rq = 4$

14. A system of equations has the following augmented matrix:

$$\left[\begin{array}{ccc|c} a & 1 & 1 & a \\ 1 & a & 1 & a \\ 1 & 1 & a & a \end{array}\right]$$

Determine the values of the parameter a if the corresponding system of equations has

a. no solutions
b. an infinite number of solutions
c. exactly one solution

Gauss-Jordan Method for Solving Systems of Equations

In the previous section, we introduced Gaussian elimination as a method for solving systems of equations. This method uses elementary row operations on an augmented matrix so it can be written in row-echelon form. We will now introduce the concept of Gauss-Jordan elimination as a method for solving systems of equations. This new method uses elementary operations in the same way as before, but the augmented matrix is written in what is called **reduced row-echelon form.** The writing of a matrix in reduced row-echelon form allows us to avoid using the method of back substitution and, instead, to determine the solution(s) to a system of equations directly from the matrix. We begin by first defining what is meant by a reduced row-echelon matrix.

Reduced Row-Echelon Form of a Matrix

A matrix is in reduced row-echelon form if

1. it is in row-echelon form
2. the first nonzero number in every row is a 1 (this is known as a leading 1 for that row)
3. any *column* containing a leading 1 has all of its other column entries equal to zero

For example, a system of three equations and three unknowns might appear as follows in reduced row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

EXAMPLE 1 Reasoning about reduced row-echelon form

Explain why each of the following augmented matrices is in reduced row-echelon form:

a. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{array}\right]$ b. $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & -4 & 0 \end{array}\right]$ c. $\left[\begin{array}{ccc|c} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Solution

We start by noting that each of these augmented matrices is in row-echelon form, so the first condition is met. The leading nonzero entry in each row is a 1, and each of the other column entries for a leading 1 is also 0. This means that each of these three matrices is in reduced row-echelon form.

EXAMPLE 2 Connecting the solution to a system of equations to a matrix in reduced row-echelon form

The following two augmented matrices are in row-echelon form. Determine the solution to the corresponding system of equations by writing each of these matrices in reduced row-echelon form.

a. $\left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -7 \end{array}\right]$ b. $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 5 \end{array}\right]$

Solution

a. This matrix is not in reduced row-echelon form because, in the second column, there is a -2 above the 1. This -2 can be eliminated by multiplying row 2 by 2 and adding it to row 1, which gives the following:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -7 \end{array}\right] \quad 2\text{ (row 2)} + \text{row 1}$$

The solution to the system of equations is $x = 11$, $y = 4$, and $z = -7$, which implies that the original system of equations represents three planes intersecting at the point $(11, 4, -7)$.

b. This matrix is in row-echelon form, but it is not in reduced row-echelon form because, in the third column, there should be a 0 instead of the -2 in row 2, and instead of the 4 in row 1. This matrix can be put in the required form by using elementary row operations.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -18 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & 5 \end{array}\right] \quad \begin{array}{l} -4\,(\text{row } 3) + \text{row } 1 \\ 2\,(\text{row } 3) + \text{row } 2 \\ \end{array}$$

The solution to this system is $x = -18$, $y = 11$, and $z = 5$, which means that the original system of equations represents three planes intersecting at the point $(-18, 11, 5)$.

In the following example, we use the interchanging of rows along with the other elementary operations.

EXAMPLE 3

Solving a system of equations by putting the augmented matrix in reduced row-echelon form

Write the following matrix in reduced row-echelon form. Then determine the solution to the corresponding system of equations.

$$\left[\begin{array}{ccc|c} 0 & 0 & -1 & 4 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right]$$

Solution

We start by writing the given matrix in row-echelon form. This involves interchanging the three rows.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 4 \end{array}\right]$$

In this matrix, the difficulty is now with the third column: $\begin{matrix} 2 \\ 0 \\ -1 \end{matrix}$. There should be a 1 where the -1 is located and a 0 where the 2 is located. Elementary operations must be used to put the matrix in reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 4 \end{array}\right] \quad \begin{array}{l} 2\,(\text{row } 3) + \text{row } 1 \\ \\ \end{array}$$

All that is needed now is to have a leading 1 in each row.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -4 \end{array}\right] \quad \begin{array}{l} \\ \\ -1\,(\text{row } 3) \end{array}$$

The solution is $x = 8$, $y = -1$, and $z = -4$, meaning that we have three planes intersecting at the point $(8, -1, -4)$.

In the next example, we will first put the matrix into row-echelon form and then write it in reduced row-echelon form.

EXAMPLE 4 Using Gauss-Jordan elimination to solve a system of equations

Solve the following system of equations using Gauss-Jordan elimination:

① $x - y + 2z = -9$
② $-x - y + 2z = -7$
③ $x + 2y - z = 6$

Solution

The given system of equations is first written in augmented matrix form.

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & -9 \\ -1 & -1 & 2 & -7 \\ 1 & 2 & -1 & 6 \end{array}\right]$$

Step 1: Write the given augmented matrix in row-echelon form.

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & -9 \\ 0 & -2 & 4 & -16 \\ 0 & 3 & -3 & 15 \end{array}\right] \quad \begin{array}{l} \text{row 1} + \text{row 2} \\ -1\,(\text{row 1}) + \text{row 3} \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & -9 \\ 0 & 1 & -2 & 8 \\ 0 & 3 & -3 & 15 \end{array}\right] \quad -\frac{1}{2}(\text{row 2})$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & -9 \\ 0 & 1 & -2 & 8 \\ 0 & 0 & 3 & -9 \end{array}\right] \quad -3\,(\text{row 2}) + \text{row 3}$$

The original matrix is now in row-echelon form.

Step 2: Write the matrix in reduced row-echelon form.
First change the leading 3 in row 3 to a 1.

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & -9 \\ 0 & 1 & -2 & 8 \\ 0 & 0 & 1 & -3 \end{array}\right] \quad \frac{1}{3}(\text{row 3})$$

Use elementary row operations to obtain 0 in the third column for all entries but the third row.

$$\left[\begin{array}{rrr|r} 1 & -1 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right] \quad \begin{array}{l} -2\,(\text{row 3}) + \text{row 1} \\ 2\,(\text{row 3}) + \text{row 2} \end{array}$$

Finally, convert the entry in the second column of the first row to a 0.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right] \quad \text{row 2} + \text{row 3}$$

The solution to this system of equations is $x = -1$, $y = 2$, and $z = -3$. The three given planes intersect at a point with coordinates $(-1, 2, -3)$.

Check:
As with previous calculations, this result should be checked by substitution in each of the three original equations.

The previous example leads to the following generalization for finding the intersection of three planes at a point using the Gauss-Jordan method of elimination:

Finding the Point of Intersection for Three Planes with the Gauss-Jordan Method

When three planes intersect at a point, the resulting coefficient matrix can always be put in the form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which is known as the **identity matrix**.

Solving equations using the Gauss-Jordan method takes about the same amount of effort as Gaussian elimination when solving small systems of equations. The main advantage of the Gauss-Jordan method is its usefulness in higher-level applications and the understanding it provides regarding the general theory of matrices. To solve smaller systems of equations, such as our examples, either method (Gaussian elimination or Gauss-Jordan elimination) can be used.

Exercises

PART A

1. Using elementary operations, write each of the following matrices in reduced row-echelon form:

   a. $\left[\begin{array}{cc|c} -1 & 3 & 1 \\ 0 & -1 & 2 \end{array}\right]$

   b. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -1 & 0 \end{array}\right]$

   c. $\left[\begin{array}{ccc|c} 1 & 1 & 4 & 2 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

   d. $\left[\begin{array}{ccc|c} 0 & 0 & -1 & 4 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right]$

2. After you have written each of the matrices in question 1 in reduced row-echelon form, determine the solution to the related system of equations.

PART B

3. Solve each system of equations using the Gauss-Jordan method of elimination.

   a. ① $x - y + z = 0$
   ② $x + 2y - z = 8$
   ③ $2x - 2y + z = -11$

   b. ① $3x - 2y + z = 6$
   ② $x - 3y - 2z = -26$
   ③ $-x + y + z = 9$

   c. ① $2x + 2y + 5z = -14$
   ② $-x + z = -5$
   ③ $y - z = 6$

   d. ① $x - y - 3z = 3$
   ② $2x + 2y + z = -1$
   ③ $-x - y + z = -1$

   e. ① $\frac{1}{2}x + 9y - z = 1$
   ② $x - 6y + z = -6$
   ③ $2x + 3y - z = -7$

   f. ① $2x + 3y + 6z = 3$
   ② $x - y - z = 0$
   ③ $4x + 3y - 6z = 2$

4. Using either Gaussian elimination or the Gauss-Jordan method of elimination, solve the following systems of equations:

   a. ① $2x + y - z = -6$
   ② $x - y + 2z = 9$
   ③ $-x + y + z = 9$

   b. ① $x - y + z = -30$
   ② $-2x + y + 6z = 90$
   ③ $2x - y - z = -20$

5. a. Determine the value of k for which the following system will have an infinite number of solutions:

   ① $x + y + z = -1$
   ② $x - y + z = 2$
   ③ $3x - y + 3z = k$

   b. For what value(s) of k will this system have no solutions?

   c. Explain why it is not possible for this system to have a unique solution.

PART C

6. The following system of equations is called a homogeneous system. This term is used to describe a system of equations in which the number to the right of the equal sign in every equation equals 0.

   ① $2x - y + z = 0$
   ② $x + y + z = 0$
   ③ $5x - y + 3z = 0$

   a. Explain why every homogeneous system of equations has at least one solution.

   b. Write the related augmented matrix in reduced row-echelon form, and explain the meaning of this result.

7. Solve the following system of equations using the Gauss-Jordan method of elimination:

   ① $\frac{1}{x} - \frac{2}{y} + \frac{6}{z} = \frac{5}{6}$
   ② $\frac{2}{x} - \frac{3}{y} + \frac{12}{z} = 2$
   ③ $\frac{3}{x} + \frac{6}{y} + \frac{2}{z} = \frac{23}{6}$

Review of Technical Skills Appendix

Part 1 Using the TI-83 Plus and TI-84 Graphing Calculators

Part 2 Using *The Geometer's Sketchpad*

Part 3 Using *Fathom*

PART 1 USING THE TI-83 PLUS AND TI-84 GRAPHING CALCULATORS

1 Preparing the Calculator

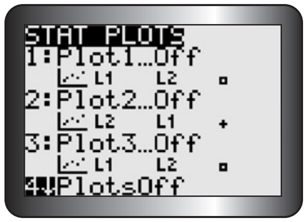
Before you graph a function, be sure to clear any information left on the calculator from the last time it was used. You should always do the following:

1. **Clear all data in the lists.**

   Press 2ND + 4 ENTER.

2. **Turn off all stat plots.**

   Press 2ND Y= 4 ENTER.



3. **Clear all equations in the equation editor.**

   Press Y=, and then press CLEAR for each equation.

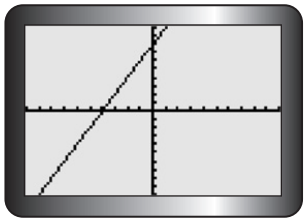
4. **Set the window so that the axes range from −10 to 10.**

   Press ZOOM 6. Press WINDOW to verify.

2 Entering and Graphing a Function

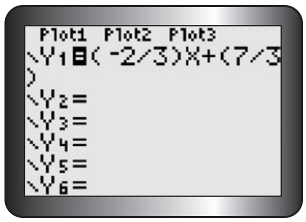
1. **Enter the equation of the function in the equation editor.**

   To graph $y = 2x + 8$, press Y= 2 X, T, Θ, *n* + 8 GRAPH. The graph will be displayed as shown.



2. **Enter all linear equations in the form $y = mx + b$.**

   If m or b are fractions, enter them between brackets. For example, write $2x + 3y = 7$ in the form $y = -\frac{2}{3}x + \frac{7}{3}$, and enter it as shown.

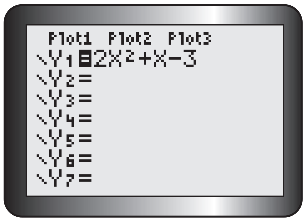


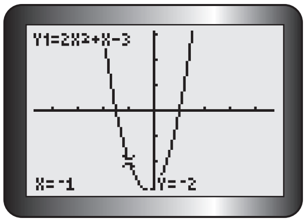
3. **Press GRAPH to view the graph.**

4. **Press TRACE to find the coordinates of any point on the graph.**

   Use the left and right arrow keys to cursor along the graph.
   Press ZOOM 8 ENTER TRACE to trace using integer values. If you are working with several graphs at the same time, use the up and down arrow keys to scroll between graphs.

3 Evaluating a Function





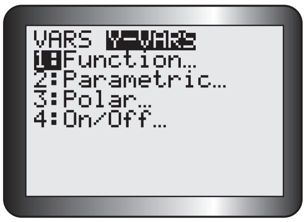
1. **Enter the function in the equation editor.**

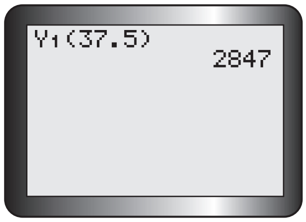
   To enter $y = 2x^2 + x - 3$, press [Y=] [2] [X, T, Θ, *n*] [X^2] [+] [X, T, Θ, *n*] [−] [3].

2. **Use the value operation to evaluate the function.**

   To find the value of the function at $x = -1$, press [2ND] [TRACE] [ENTER], enter [(−)] [1] at the cursor, and then press [ENTER].

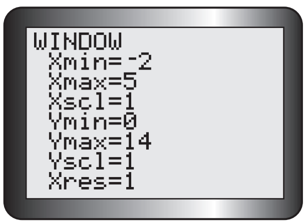
3. **Use function notation and the Y-VARS operation to evaluate the function.**
   This is another way to evaluate the function. To find the value of the function at $x = 37.5$, press [CLEAR] [VARS]. Then cursor right to **Y-VARS**, and press [ENTER]. Press [1] to select **Y1**. Finally, press [(] [3] [7] [.] [5] [)], and then [ENTER].

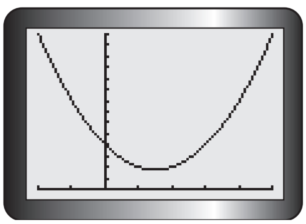




4 Changing Window Settings

The window settings can be changed to show a graph for a given domain and range.





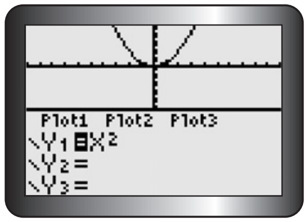
1. **Enter the function $y = x^2 - 3x + 4$ in the equation editor.**
2. **Use the WINDOW function to set the domain and range.**

   To display the function over the domain $\{x \mid -2 \leq x \leq 5\}$ and range $\{y \mid 0 \leq y \leq 14\}$, press [WINDOW] [(−)] [2] [ENTER], then [5] [ENTER], then [1] [ENTER], then [0] [ENTER], then [1] [4] [ENTER], then [1] [ENTER], and finally [1] [ENTER].

3. **Press [GRAPH] to show the function with this domain and range.**

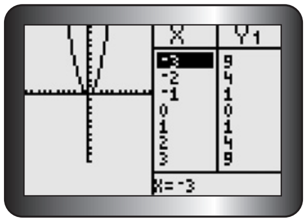
5 Using the Split Screen

1. **The split screen can be used to see a graph and the equation editor at the same time.**

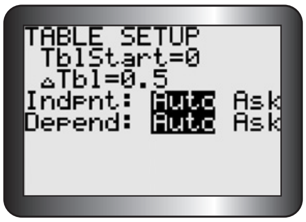
   

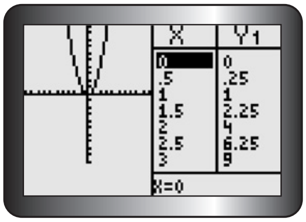
   Press MODE and curser to **Horiz**. Press ENTER to select this, and then press 2ND MODE to return to the home screen. Enter $y = x^2$ in **Y1** of the equation editor, and then press GRAPH.

2. **The split screen can also be used to see a graph and a table at the same time.**

   Press MODE and cursor to **G–T** (Graph-Table). Press ENTER to select this, and then press GRAPH. It is possible to view the table with different increments. For example, to see the table start at $x = 0$ and increase in increments of 0.5, press 2ND WINDOW and adjust the settings as shown. Then press GRAPH.

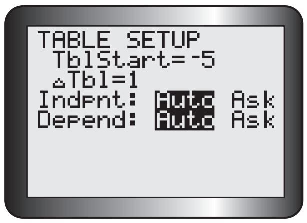
6 Using the TABLE Feature

A function can be displayed in a table of values.

1. **Enter the function in the equation editor.**

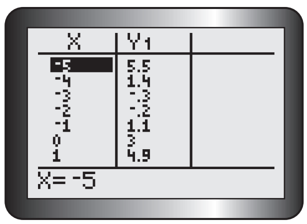
   To enter $y = -0.1x^3 + 2x + 3$, press Y= (−) . 1 X, T, Θ, *n* ^ 3 + 2 X, T, Θ, *n* + 3.

2. **Set the start point and step size for the table.**

   Press 2ND WINDOW. The cursor is beside "TblStart=." To start at $x = -5$, press (−) 5 ENTER. The cursor is now beside **ΔTbl=**. To increase the x-value in steps of 1, press 1 ENTER.

3. **To view the table, press 2ND GRAPH.**

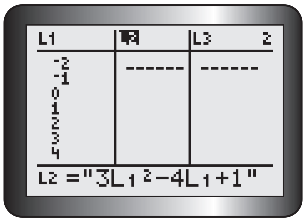
   Use the up and down arrow keys to move up and down the table. Notice that you can look at higher or lower x-values than those in the original range.

7 Making a Table of Differences

To make a table with the first and second differences for a function, use the STAT lists.

1. Press [STAT] [1]**, and enter the *x*-values in L1.**

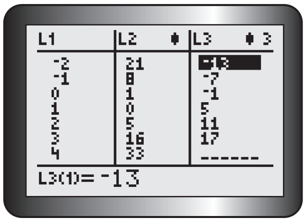
For the function $f(x) = 3x^2 - 4x + 1$, use x-values from -2 to 4.



2. Enter the function.

Scroll right and up to select **L2**. Enter the function $f(x)$, using **L1** as the variable x. Press [ALPHA] [+] [3] [2ND] [1] [X^2] [−] [4] [2ND] [1] [+] [1] [ALPHA] [+].

3. Press [ENTER] **to display the values of the function in L2.**



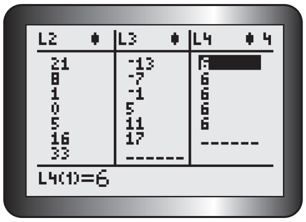
4. Find the first differences.

Scroll right and up to select **L3**. Then press [2ND] [STAT].

Scroll right to **OPS**, and press [7] to choose **ΔList(**.

Enter **L2** by pressing [2ND] [2] [)].

Press [ENTER] to see the first differences displayed in **L3**.



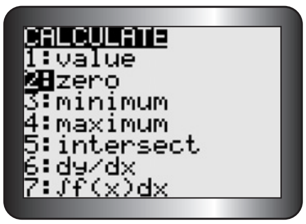
5. Find the second differences.

Scroll right and up to select **L4**. Repeat step 4, using **L3** instead of **L2**.

Press [ENTER] to see the second differences displayed in **L4**.

8 Finding the Zeros of a Function

To find the zeros of a function, use the **zero** operation.



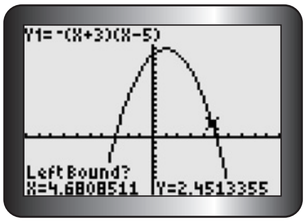
1. Enter $y = -(x + 3)(x - 5)$ in the equation editor.

Press [GRAPH] [ZOOM] [6].

2. Access the zero operation.

Press [2ND] [TRACE] [2].

3. **Use the left and right arrow keys to cursor along the curve to any point that is left of the zero.**



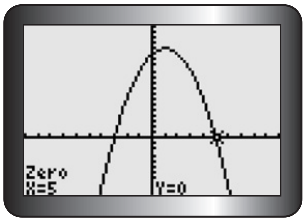
   Press ENTER to set the left bound.

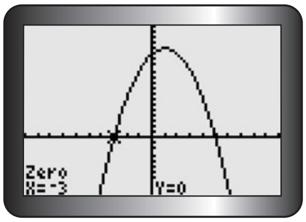
4. **Cursor along the curve to any point that is right of the zero.**

   Press ENTER to set the right bound.

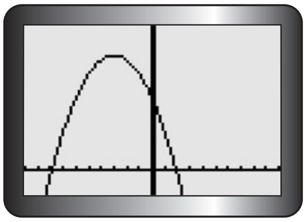
5. **Press ENTER again to display the coordinates of the zero (the *x*-intercept).**

6. **Repeat to find the second zero.**



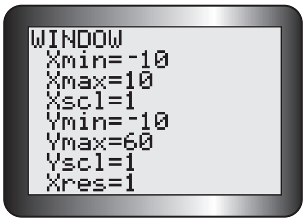


9 Finding the Maximum or Minimum Values of a Function



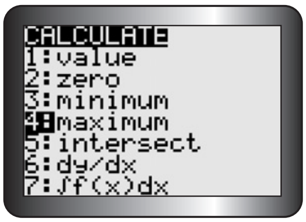
The least or greatest value can be found using the **minimum** operation or the **maximum** operation.

1. **Enter $y = -2x^2 - 12x + 30$.**
   Graph it, and adjust the window as shown. This graph opens downward, so it has a maximum.



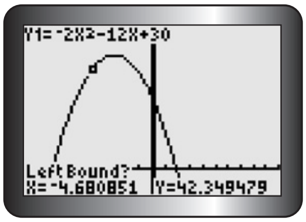
2. **Use the maximum operation.**

   Press 2ND TRACE 4. For parabolas that open upward, press 2ND TRACE 3 to use the **minimum** operation.



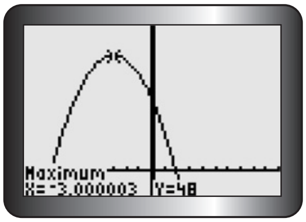
3. **Use the left and right arrow keys to cursor along the curve to any point that is left of the maximum value.**

   Press ENTER to set the left bound.



4. **Cursor along the curve to any point that is right of the maximum value.**

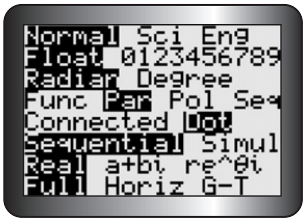
   Press ENTER to set the right bound.



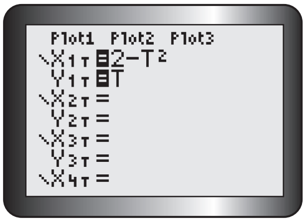
5. **Press ENTER again to display the coordinates of the optimal value.**

10 Graphing the Inverse of a Function

Parametric equations allow you to graph any function and its inverse. For example, the function $y = 2 - x^2$, with domain $x \geq 0$, can be graphed using parametric mode. For a parametric equation, both x and y must be expressed in terms of a parameter, t. Replace x with t. Then $x = t$ and $y = 2 - t^2$. The inverse of this function can now be graphed.



1. **Clear the calculator, and press MODE.**
   Change the setting to the parametric mode by scrolling down to the fourth line and to the right to **Par**, as shown on the screen. Press ENTER.

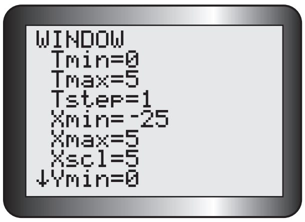


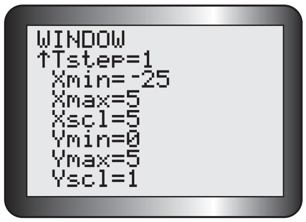
2. **Enter the inverse function by changing the parametric equations $x = t, y = 2 - t^2$ to $x = 2 - t^2, y = t$.**

   Press Y=. At X1T=, enter 2 − X, T, Θ, *n* X^2 ENTER. At Y1T=, enter X, T, Θ, *n*.

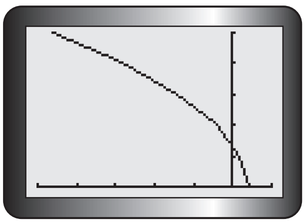
3. **Press WINDOW.**

   The original domain, $x \geq 0$, is also the domain of t. Use window settings such as those shown below to display the graph.





4. **Press GRAPH to display the inverse function.**

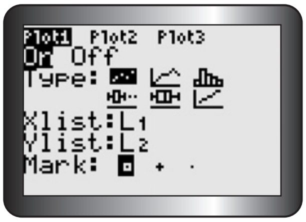


11 Creating a Scatter Plot and Determining a Line or Curve of Best Fit Using Regression

This table gives the height of a baseball above ground, from the time it was hit to the time it touched the ground.

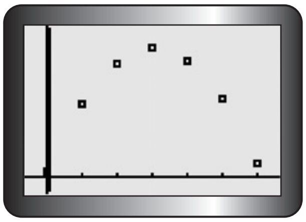
Time (s)	0	1	2	3	4	5	6
Height (m)	2	27	42	48	43	29	5

To create a scatter plot of the data, follow the steps below.



1. **Enter the data into lists.**

   To start, press STAT ENTER. Move the cursor over to the first position in **L1**, and enter the values for time. Press ENTER after each value. Repeat this for height in **L2**.

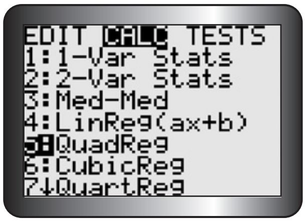


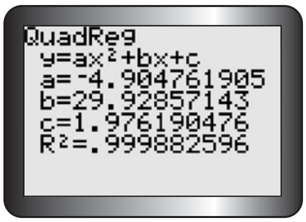
2. **Create a scatter plot.**

   Press 2ND Y= and 1 ENTER. Turn on Plot 1 by making sure that the cursor is over **On**, the **Type** is set to the graph type you prefer, and **L1** and **L2** appear after **Xlist** and **Ylist**.

3. **Display the graph.**

   Press ZOOM 9 to activate **ZoomStat**.





4. **Apply the appropriate regression analysis.**

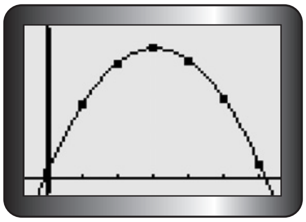
   To determine the equation of the line or curve of best fit, press STAT and scroll over to **CALC**. Press

   - 4 to enable **LinReg(ax+b)**
   - 5 to enable **QuadReg**
   - 6 to enable **CubicReg**
   - 7 to enable **QuartReg**
   - 0 to enable **ExpReg**
   - ALPHA C to enable **SinReg**

   Then press 2ND 1 , 2ND 2 , VARS. Scroll over to **Y-VARS**. Press 1 twice. This action stores the equation of the line or curve of best fit in **Y1** of the equation editor.

5. **Display and analyze the results.**

Press ENTER. In this example, the letters a, b, and c are the coefficients of the general quadratic equation $y = ax^2 + bx + c$ for the curve of best fit. R^2 is the percent of data variation represented by the model. The equation is about $y = -4.90x^2 + 29.93x + 1.98$.
Note: For linear regression, if r is not displayed, turn on the diagnostics function. Press 2ND 0 and scroll down to **DiagnosticOn**. Press ENTER twice. Repeat steps 4 to 6.



6. **Plot the curve.**

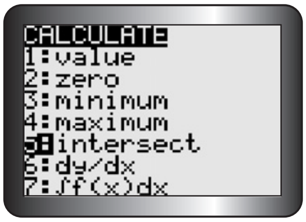
Press GRAPH

12 Finding the Points of Intersection of Two Functions

1. **Enter both functions in the equation editor.**
For example, enter $y = 5x + 4$ and $y = -2x + 18$.

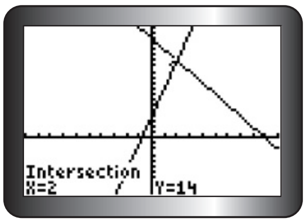
2. **Graph both functions**.

Press GRAPH. Adjust the window settings until one or more points of intersection are displayed.



3. **Use the intersect operation.**

Press 2ND TRACE 5.



4. **Determine a point of intersection**.
You will be asked to verify the two curves and enter a guess (optional) for the point of intersection. Press ENTER after each screen appears.

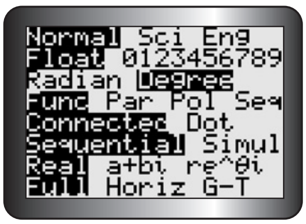
The point of intersection is exactly (2, 14).

5. **Determine any additional points of intersection.**

Press TRACE, and move the cursor close to the other point you wish to identify. Repeat step 4.

13 Evaluating Trigonometric Ratios and Finding Angles

Working with Degrees



1. **Put the calculator in degree mode.**

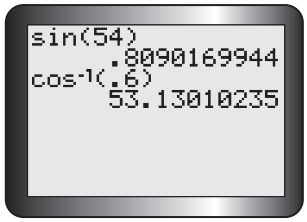
Press MODE. Scroll down and across to **Degree**. Press ENTER.

2. Use the SIN, COS, or TAN key to calculate a trigonometric ratio.

To find the value of sin 54°, press SIN 5 4) ENTER.

3. Use SIN^{-1}, COS^{-1}, or TAN^{-1} to calculate an angle.

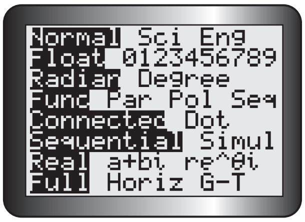
To find the angle whose cosine is 0.6, press 2ND COS . 6) ENTER.



Working with Radians

1. Put the calculator in radian mode.

Press MODE. Scroll down and across to **Radian**. Press ENTER.

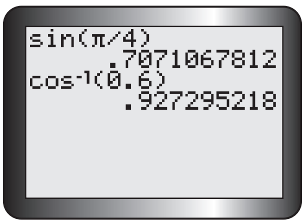


2. Use the SIN, COS, or TAN key to calculate a trigonometric ratio.

To find the value of sin $\frac{\pi}{4}$, press SIN 2ND ^ ÷ 4) ENTER.

3. Use SIN^{-1}, COS^{-1}, or TAN^{-1} to calculate an angle.

To find the angle whose cosine is 0.6, press 2ND COS . 6) ENTER.



14 Graphing a Trigonometric Function

Working with Degrees

You can graph a trigonometric function in degree measure using the TI-83 Plus or TI-84 calculator.

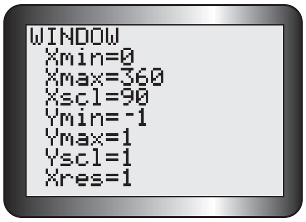
1. **Put the calculator in degree mode.**

   Press MODE . Scroll down and across to **Degree**. Press ENTER .

2. **Enter the function in the equation editor.**

   For example, to graph the function $y = \sin x$, for $0° \le x \le 360°$, press

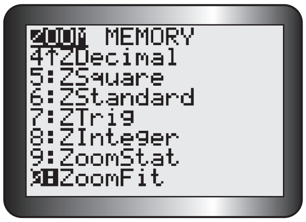
   Y= SIN X, T, Θ, *n*) .

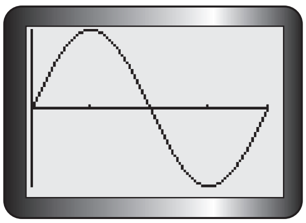


step 3

3. **Adjust the window to correspond to the given domain.**

   Press WINDOW . Set **Xmin = 0**, **Xmax = 360**, and **Xscl = 90**. These settings display the graph from 0° to 360°, using an interval of 90° on the *x*-axis. Then set **Ymin = −1** and **Ymax = 1**, since the sine function being graphed lies between these values. If the domain is not known, this step can be omitted.





step 4

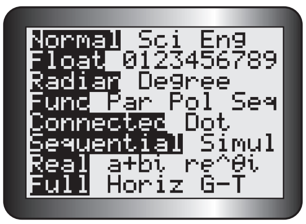
4. **Graph the function using ZoomFit.**

   Press ZOOM 0 . The graph is displayed over the domain, and the calculator determines the best values to use for **Ymax** and **Ymin** in the display window.

   Note: You can use **ZoomTrig** (press ZOOM 7) to graph the function in step 4. **ZoomTrig** will always display the graph in a window where **Xmin = −360°**, **Xmax = 360°**, **Ymin = −4**, and **Ymax = 4**.

Working with Radians

You can also graph a trigonometric function in radians using the TI-83 Plus or TI-84 calculator.



1. **Put the calculator in radian mode.**

   Press MODE . Scroll down and across to **Radian**. Press ENTER .

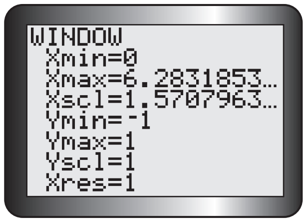
2. **Enter the function in the equation editor.**

   For example, to graph the function $y = \sin x$, for $0 \le x \le 2\pi$, press

   Y= SIN X, T, Θ, *n*) .

3. **Adjust the window to correspond to the given domain.**

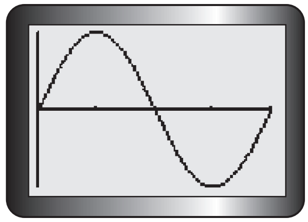
Press WINDOW. Set **Xmin** $= 0$, **Xmax** $= 2\pi$, and **Xscl** $= \frac{\pi}{2}$. These settings display the graph from 0 to 2π, using an interval of $\frac{\pi}{2}$ on the x-axis. Then set **Ymin** $= -1$ and **Ymax** $= 1$, since the sine function being graphed lies between these values. If the domain is not known, this step can be omitted.



4. **Graph the function using ZoomFit.**

Press ZOOM 0. The graph is displayed over the domain, and the calculator determines the best values to use for **Ymax** and **Ymin** in the display window.

Note: You can use **ZoomTrig** (press ZOOM 7) to graph the function in step 4. **ZoomTrig** will always display the graph in a window where **Xmin** $= -2\pi$, **Xmax** $= 2\pi$, **Ymin** $= -4$, and **Ymax** $= 4$.



15 Evaluating Powers and Roots

1. **Evaluate the power $(5.3)^2$.**

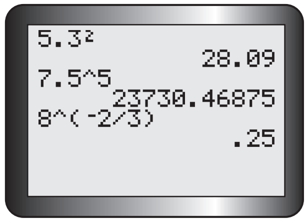
Press 5 . 3 X^2 ENTER.

2. **Evaluate the power 7.5^5.**

Press 7 . 5 ^ 5 ENTER.

3. **Evaluate the power $8^{-\frac{2}{3}}$.**

Press 8 ^ (− 2 ÷ 3) ENTER.

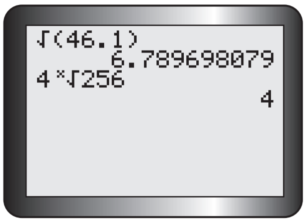


4. **Evaluate the square root of 46.1.**

Press 2ND X^2 4 6 . 1) ENTER.

5. **Evaluate $\sqrt[4]{256}$.**

Press 4 MATH 5 2 5 6 ENTER.



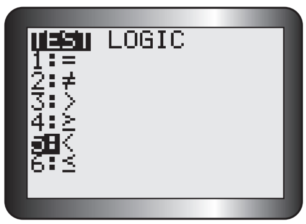
16 Graphing a Piecewise Function

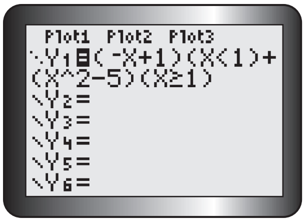
Follow these steps to graph the piecewise function defined by

$$f(x) = \begin{cases} -x + 1, \text{ if } x < 1 \\ x^2 - 5, \text{ if } x \geq 1 \end{cases}$$

1. **Enter the first equation.**

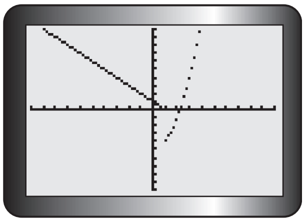
In the equation editor for **Y1**, enter the first equation in brackets. Then enter its corresponding interval in brackets. The inequality signs can be accessed in the **Test** menu by pressing 2ND MATH.





2. **Enter the second equation.**

   Press +, and repeat step 1 for the second equation and its interval. Scroll to the left of **Y1**, and press ENTER until the dotted graphing mode appears.

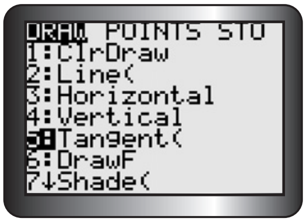


3. **Display the graph.**

   Press GRAPH to display the graph.
   Each equation produces a different graph on each interval. This function is discontinuous at $x = 1$.

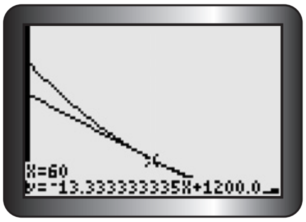
17 Performing Operations Specific to Calculus

Drawing Tangent Lines



1. **Enter a function to be graphed.**
   Enter $V(t) = \frac{1}{9}(120 - t)^2$ in **Y1** of the equation editor. Adjust the window, and display the graph.

2. **Draw a tangent line at the desired point.**
   Use the **Tangent** command in the **Draw** menu to draw a tangent line at a point and estimate its slope. Press 2ND PRGM. Choose **5:Tangent(** . Scroll to 60, or enter 60, for the x-coordinate. Press ENTER . The tangent line is drawn, and its equation is displayed. Press 2ND PRGM 1 to clear the drawn tangent lines. The function will be regraphed without the tangent lines.

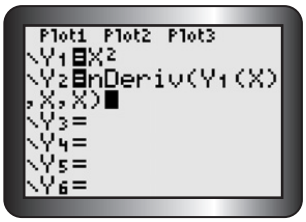


Graphing the First and Second Derivatives of a Function

1. **Enter a function to be graphed.**
   Enter a function, such as $y = x^2$, in **Y1** of the equation editor. Press ENTER.

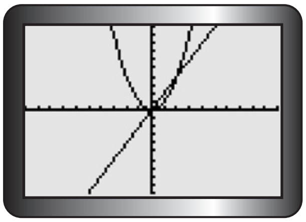
2. **Graph the first derivative.**
   To graph the derivative, use the **nDeriv** operation. Press MATH 8.

   To enter the expression **Y1**, press VARS. Scroll over to **Y-VARS**. Press 1 twice. Press (X, T, Θ, *n*) , X, T, Θ, *n* , X, T, Θ, *n*) to enter the expression, variable name, and general value of x. Press GRAPH. The original function is graphed first, and the derivative is graphed next. **nDeriv(** approximates the derivative.



3. **Graph the second derivative.**

To graph the second derivative, enter **nDeriv(Y2(X), X, X)** in **Y3**. (See step 2.) Remember to select **Y2** from the **Function** menu. You can deselect a function to be graphed. Position the cursor over the equal sign of the desired function in the equation editor. Press ENTER. Only the functions whose equal signs are shaded will be graphed when GRAPH is pressed.

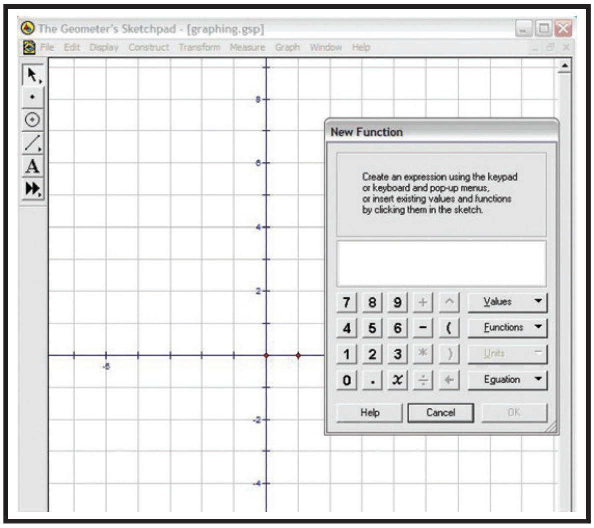


PART 2 USING THE GEOMETER'S SKETCHPAD

18 Graphing a Function

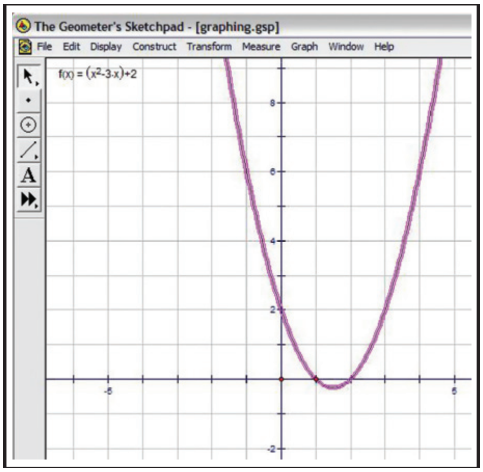
1. **Turn on the grid.**
From the **Graph** menu, choose **Show Grid**.

2. **Enter the function.**
From the **Graph** menu, choose **Plot New Function**. The function calculator should appear.

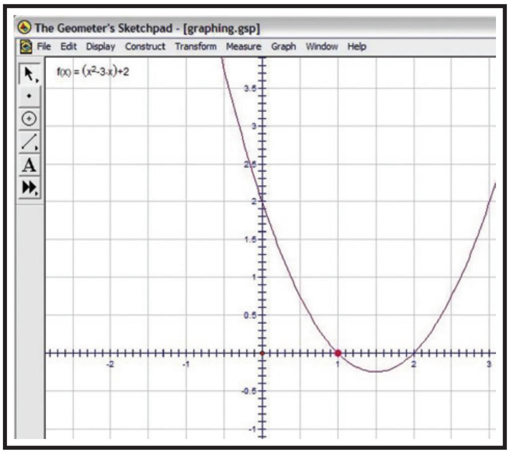


3. **Graph the function $y = x^2 - 3x + 2$.**

Use either the calculator keypad or the keyboard to enter **x ^ 2 - 3 * x + 2.** Then press OK on the calculator keypad. The graph of $y = x^2 - 3x + 2$ should appear on the grid.



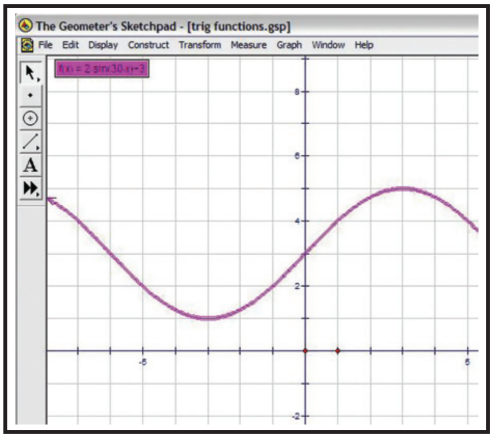
4. Adjust the origin and/or scale.
To adjust the origin, left-click on the point at the origin to select it. Then left-click and drag the origin as desired.
To adjust the scale, left-click in blank space to deselect, and then left-click on the point at (1, 0) to select it. Left-click and drag this point to change the scale.



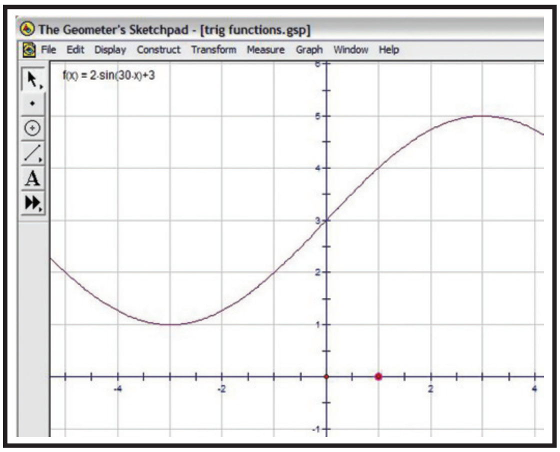
19 Graphing a Trigonometric Function

1. **Turn on the grid.**
   From the **Graph** menu, choose **Show Grid**.

2. **Graph the function $y = 2 \sin (30x) + 3$.**
   From the **Graph** menu, choose **Plot New Function**. The function calculator should appear.
   Use either the calculator keypad or the keyboard to enter **2 * sin (30 * x) + 3**. To enter sin, use the pull-down **Functions** menu on the calculator keypad. Click OK on the calculator keypad.
   Click **No** in the pop-up panel to keep degrees as the angle unit. The graph of $y = 2 \sin (30x) + 3$ should appear on the grid.

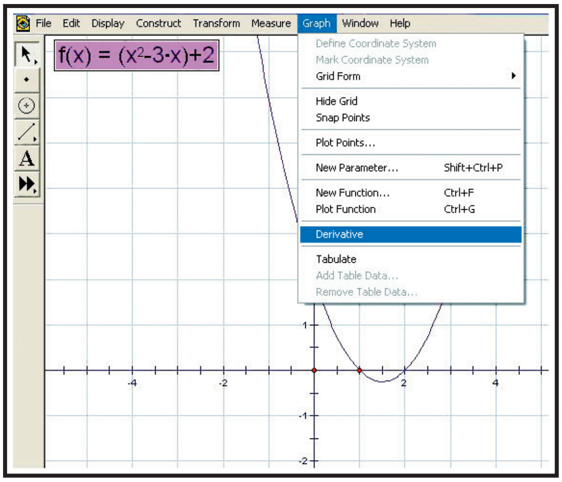


3. **Adjust the origin and/or scale.**
   Left-click on and drag either the origin or the point (1, 0).

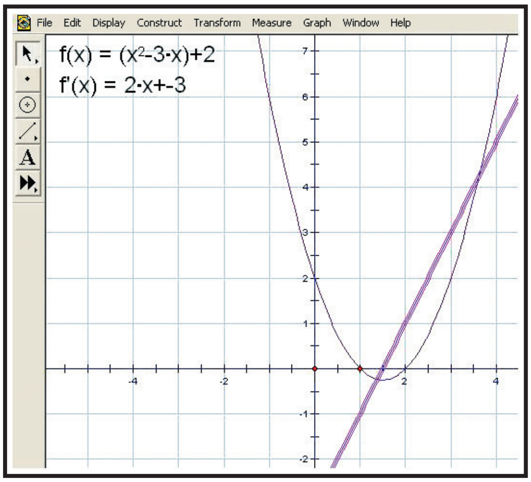


20 Graphing the Derivative of a Function

1. **Graph the function $y = x^2 - 3x + 2$.**
   Follow the instructions as outlined in Technical Appendix 18, Graphing Functions, to graph the given function.
2. **Select the equation of the function whose derivative is to be determined.**
   With the equation of the function selected, choose **Derivative** from the **Graph** menu. The equation of $f'(x)$ will be displayed.

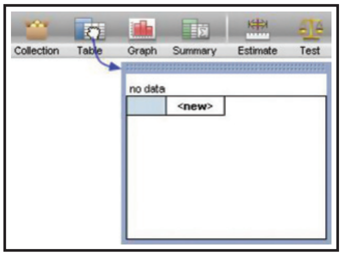


3. **Graph the derivative function.**
   With the equation of the derivative function selected, chose **Plot Function** from the **Graph** menu. The graph of $f'(x)$ will be displayed.



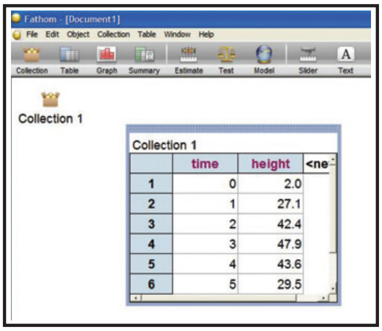
PART 3 USING FATHOM

21 Creating a Scatter Plot and Determining the Equation of a Line or Curve of Good Fit

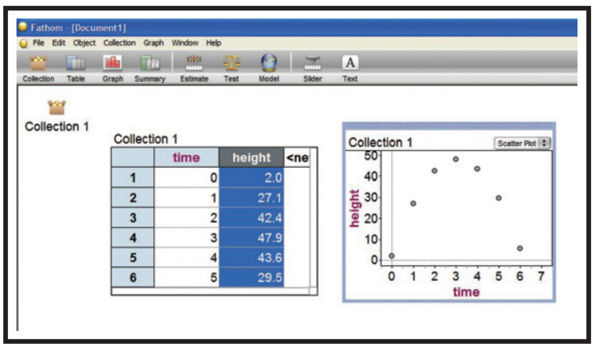


1. **Create a case table.**
   Drag a case table from the object shelf, and drop it in the document.

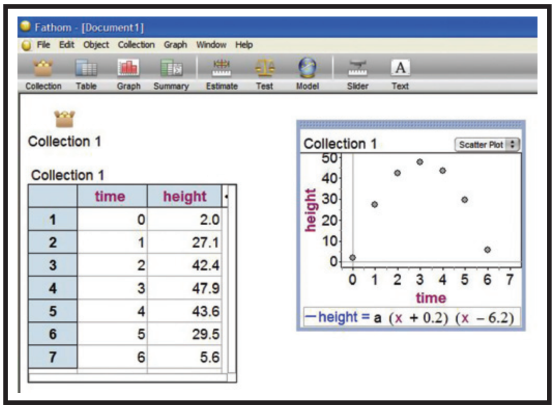
2. **Enter the Variables and Data.**
   Click **<new>**, type a name for the new variable or attribute, and press ENTER. (If necessary, repeat this step to add more attributes; Pressing TAB instead of ENTER moves you to the next column.)
   When you name your first attribute, Fathom creates an empty collection to hold your data (a little, empty box). This is where your data are actually stored. Deleting the collection deletes your data. When you add cases by typing values, the collection icon fills with gold balls. To enter the data, click in the blank cell under the attribute name and begin typing values. (Press TAB to move from cell to cell.)



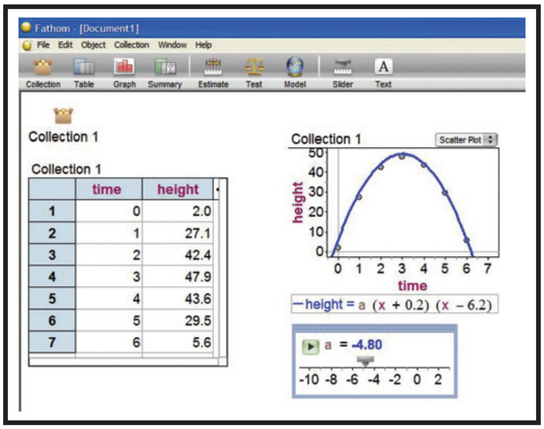
3. **Graph the data.**
   Drag a new graph from the object shelf at the top of the Fathom window, and drop it in a blank space in your document. Drag an attribute from the case table, and drop it on the prompt below and/or to the left of the appropriate axis in the graph.



4. **Create a function.**
Right-click the graph, and select **Plot Function**. Enter your function using a parameter that can be adjusted to fit the curve to the scatter plot (**a** was used below).



5. **Create a slider for the parameter(s) in your equation.**
Drag a new slider from the object shelf at the top of the Fathom window, and drop it in a blank space below your graph. Over V1, type in the letter of the parameter used in your function in step 4. Click on the number, and then adjust the value of the slider until you are satisfied with the fit.



The equation of a curve of good fit is $y = -4.8(x + 0.2)(x - 6.2)$.

Glossary

A

absolute extrema: the largest or smallest value of a function over its entire domain.

acceleration: the rate of change of velocity with respect to time $\left(\frac{dv}{dt}\right)$ or the second derivative of displacement with respect to time $\left(\frac{d^2s}{dt^2}\right)$.

algebraic vectors: vectors that are considered with respect to coordinate axes.

asymptote: a line having the property that the distance from a point P on a curve to the line approaches zero as the distance from P to the origin increases indefinitely. The line and the curve get closer and closer but never touch. See **horizontal**, **vertical**, and **slant asymptote**.

augmented matrix: a matrix made up of the coefficient matrix and one additional column containing the constant terms of the equations to be solved.

average rate of change: given by the difference quotient $\frac{\Delta y}{\Delta x} = \frac{f(a + h) - f(a)}{h}$. The average rate of change of the function $f(x)$ over the interval $x = a$ to $x = a + h$.

B

bearing: a way of specifying the direction from one object to another, often stated in terms of compass directions.

C

calculus: a branch of mathematics, discovered independently by Sir Isaac Newton and Gottfried Wilhelm von Leibniz, that deals with the instantaneous rate of change of a function (differential calculus) and the area under a function (integral calculus).

Cartesian coordinate system: a reference system in two-space, consisting of two axes at right angles, or three-space (three axes) in which any point in the plane is located by its displacements from these fixed lines (axes). The origin is the common point from which each displacement is measured. In two-space, a set of two numbers or coordinates is required to uniquely define a position; in three-space, three coordinates are required.

Cartesian equation of a plane: Cartesian (or scalar) equation of a plane in R^3 is of the form $Ax + By + Cz + D = 0$ with a normal $\vec{n} = (A, B, C)$. The normal n is a nonzero vector perpendicular to all of vectors in the plane.

chain rule: if $f(x)$ and $g(x)$ are continuous and differentiable functions, then the composite function $h(x) = f[g(x)]$ has a derivative given by $h'(x) = f'[g(x)]g'(x)$. In Leibniz notation, if $y = f(u)$ where $u = g(x)$, then y is a composite function and $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

chord: in a circle, the portion of the secant inside the circle.

coefficient matrix: a matrix whose elements are the coefficients of the unknown terms in the equations to be solved by matrix methods.

coincident: Two or more congruent geometric figures (vectors, lines, or planes) which can be translated to lie on top of each other.

collinear vectors: vectors that are parallel and that lie on the same straight line.

composition of forces: the process of finding the resultant of all the forces acting on an object.

composite function: given two functions, $f(x)$ and $g(x)$, the composite function $f \circ g = f[g(x)]$. $g(x)$ is called the inner function and $f(x)$ is the outer function.

composition: the process of combining functions.

concave up/down: $f(x)$ is concave up at x_0 if and only if $f'(x)$ is increasing at x_0. $f(x)$ is concave down at x_0 if and only if $f'(x)$ is decreasing at x_0. If $f''(x)$ exists at x_0 and is positive, then $f(x)$ is concave up at x_0. If $f''(x)$ exists and is negative, then $f(x)$ is concave down at x_0. If $f''(x)$ does not exist or is zero, then the test fails.

conjugate radical: for an expression of the form $\sqrt{a} + \sqrt{b}$, the conjugate radical is $\sqrt{a} - \sqrt{b}$.

constant function rule: if $f(x) = k$, where k is a constant, then $f'(x) = 0$. In Leibniz notation, $\frac{d}{dx}(k) = 0$.

constant multiple rule: if $f(x) = kg(x)$ where k is a constant, then $f'(x) = kg'(x)$. In Leibniz notation: $\frac{df}{dx} = k\frac{dg}{dx}$.

consistent system of equations: a system of equations that has either one solution or an infinite number of solutions.

continuity: the condition of being uninterrupted, without break or irregularity.

continuous function: a function $f(x)$ is continuous at a particular point $x = a$, if $f(a)$ is defined and if $\lim_{x\to a} f(x) = f(a)$. If this property is true for all points in the domain of the function, then the function is said to be continuous over the domain.

coplanar: the description given to two or more geometric objects that lie in the same plane.

critical points (of a function): a critical point on $f(x)$ occurs at x_0 if and only if either $f'(x_0) = 0$ or the derivative doesn't exist.

critical numbers: numbers in the domain of a function that cause its derivative to be zero or undefined.

cross product: the cross product of two vectors $\vec{a}$ and $\vec{b}$ in R^3 (three-space) is the vector that is perpendicular to these factors and has a magnitude of $|\vec{a}||\vec{b}|\sin\theta$ such that the vectors $\vec{a}$, $\vec{b}$, and $\vec{a} \times \vec{b}$ form a right-handed system.

cusp: a type of double point. A cusp is a point on a continuous curve where the tangent line reverses sign.

D

decreasing function: a function $f(x)$ is decreasing at a point x_0 if and only if there exists some interval I containing x_0, such that $f(x_0) < f(x)$ for all x in I to the left of x_0 and $f(x_0) > f(x)$ for all x in I to the right of x_0.

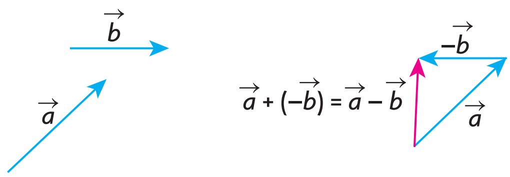
delta: the fourth letter of the Greek alphabet: lower case [δ]; upper case [Δ].

dependent variable: in a relation, the variable whose value depends upon the value of the independent variable. On a coordinate grid, the values of the independent variable are usually plotted on the horizontal axis, and the values of the dependent variable on the vertical axis.

derivative: the instantaneous rate of change of a function with respect to the variable. The derivative at a particular point is equal to the slope of the tangent line drawn to the curve at that point. The derivative of $f(x)$ at the point $x = a$: $f'(a) = \lim_{x\to a}\frac{f(x) - f(a)}{x - a}$ provided the limit exists.

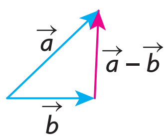
derivative function: for a function $f(x)$, the derivative function is $f'(x) = \lim_{h\to 0}\frac{f(x + h) - f(x)}{h}$ for all x for which the limit exists.

difference of two vectors:



In the diagram above, the difference between vectors $\vec{a}$ and $\vec{b}$ is found by adding the opposite vector $\vec{b}$ to $\vec{a}$ using the triangle law of addition.

Another way to think about $\vec{a} - \vec{b}$ is to arrange vectors tail to tail. In this case, $\vec{a} - \vec{b}$ is the vector that must be added to $\vec{b}$ to get $\vec{a}$. This is illustrated in the following diagram. Using the vectors above, the difference vector is the same as the one produced by adding the opposite.



difference quotient: the slope of the secant drawn to a curve $f(x)$ between the points on the curve $(a, f(a))$ and $(a + h, f(a + h))$: $\frac{\Delta y}{\Delta x} = \frac{f(a + h) - f(a)}{h}$.

difference rule: if functions $p(x)$ and $q(x)$ are differentiable and $f(x) = p(x) - q(x)$, then $f'(x) = p'(x) - q'(x)$. In Leibniz notation: $\frac{df}{dx} = \frac{dp}{dx} - \frac{dq}{dx}$.

differentiability: a real function is said to be differentiable at a point if its derivative exists at that point.

differential calculus: that portion of calculus dealing with derivatives.

direction angles: the angles that a vector makes with each of the coordinate axes.

direction cosines: the cosines of the direction angles.

direction numbers: the components of a direction vector; for the vector (a, b), the direction numbers are a and b.

direction vector: a vector that determines the direction of a particular line.

discontinuity: an interrupted or broken connection. A value for x, on an $x - y$ grid, for which a value for y is not defined. A formal mathematical definition: a function $f(x)$ is discontinuous at a particular point $x = a$ if $f(a)$ is not defined and/or if $\lim_{x \to a} f(x) \neq f(a)$.

displacement: a translation from one position to another, without consideration of any intervening positions. The minimal distance between two points.

dot product: for two vectors, the dot product is the product of the magnitudes of the vectors and the cosine of the angle between the two vectors.

E

***e*:** the base of the natural logarithm, whose symbol "e" honours Euler. It can be defined as the $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$ and is equal to 2.718 281 828 6… (a non-repeating, infinite decimal).

elementary operations: operations that produce equivalent systems:

1. multiplication of an equation by a nonzero constant
2. interchanging any pair of equations
3. adding a nonzero multiple of one equation to a second equation to replace the second equation

equivalent systems: two systems of equations are equivalent if every solution of one system is also a solution to the second system.

exponent laws:

Action	Result
$a^m \times a^n$	a^{m+n}
$\frac{a^m}{a^n}$	a^{m-n}
$(a^m)^n$	$a^{m \times n}$
$(ab)^m$	$a^m b^m$
$\left(\frac{a}{b}\right)^m, b \neq 0$	$\frac{a^m}{b^m}$

Action	Result
a^0	1
a^{-n}	$\frac{1}{a^n}$
$\left(\frac{a}{b}\right)^{-n}$	$\frac{b^n}{a^n}$
$a^{\frac{p}{q}}$	$(\sqrt[q]{a})^p$ or $\sqrt[q]{a^p}$

equilibrant: equal in magnitude but acting in the opposite direction to the resultant force, resulting in a state of equilibrium.

extended power rule: a symmetric expression that extends the power rule for the product of two functions to three functions and beyond. For example, if $f(x) = g(x)h(x)k(x)$, then $f'(x) = g'(x)h(x)k(x) + g(x)h'(x)k(x) + g(x)h(x)k'(x)$ Note the symmetry.

extreme values (of a function): the maximum and minimum values of a function over a particular interval of values (domain).

F

first derivative test: if $f'(x)$ changes sign from negative to positive at $x = c$, then $f(x)$ has a local minimum at this point; if $f'(x)$ changes sign from positive to negative at $x = c$, then $f(x)$ has a local maximum at this point.

force: something that either pushes or pulls an object.

G

geometric vectors: vectors that are considered without reference to coordinate axes.

H

horizontal asymptote: the line $y = y_0$ is a horizontal asymptote of $f(x)$ if and only if $f(x)$ approaches y_0 as $x \to \pm\infty$.

I

identity matrix: the matrix that consists entirely of a diagonal of 1's with all other numbers in the matrix 0.

implicit differentiation: a method for differentiating an implicit function, utilizing the chain rule and ultimately solving for the derivative desired $\left(\frac{dy}{dx}\right)$.

implicit function: a function in which the dependent variable is not directly stated as a function of the independent variable.

inconsistent system of equations: a system of equations that has no solutions.

increasing function: a function $f(x)$ is increasing at a point x_0 if and only if there exists some interval I containing x_0, such that $f(x_0) > f(x)$ for all x in I to the left of x_0, and $f(x_0) < f(x)$ for all x in I to the right of x_0.

independent variable: in a relation, the variable whose value determines the value of the dependent variable. See **dependent variable.**

indeterminate form: a quotient $\lim_{x \to a} \frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ both approach 0 or $\pm\infty$ as x approaches a is an indeterminate form: $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$.

inflection point: an inflection point occurs on $f(x)$ at x_0 if and only if $f(x)$ has a tangent line at x_0 and there exists an interval I containing x_0 such that $f(x)$ is concave up on one side of x_0 and concave down on the other side.

instantaneous rate of change: the rate of change of $y = f(x)$ at a particular point $x = a$ is given by $\lim_{x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x \to 0} \frac{f(a + h) - f(a)}{h}$ provided the limit exists.

L

Leibniz notation: for example, $\frac{dy}{dx}$ is Leibniz notation for the derivative of y with respect to x. The notation we use in everyday calculus is attributable primarily to Leibniz.

limit (of a function): the notation $\lim_{x \to a} f(x) = L$ implies that, as x approaches closer and closer to the value a, the value of the function approaches a limiting value of L.

linear combination: the sum of nonzero multiples of two or more vectors or equations.

local maximum: a function $f(x)$ has a local maximum at x_0 if and only if there exists some interval I containing x_0 such that $f(x_0) \geq f(x)$ for all x in I.

local minimum: a function $f(x)$ has a local minimum at x_0 if and only if there exists some interval I containing x_0 such that $f(x_0) \leq f(x)$ for all x in I.

logarithm (natural): logarithms of numbers using a base of e. Usually written as $\ln x$.

logarithmic differentiation: a process using logarithms to differentiate algebraically complicated functions or functions for which the ordinary rules of differentiation do not apply.

logarithmic function: the inverse of the exponential function. If $y = b^x$ represents the exponential function, then $y = \log_b y$ is the logarithmic function. Usually written as $y = \log_b x$.

logistic model: a mathematical model that describes a population that grows exponentially at the beginning and then levels off to a limiting value. There are several different forms of equations representing this model.

M

magnitude: the absolute value of a quantity.

maximum: the largest value of a function on a given interval of values.

minimum: the smallest value of a function on a given interval of values.

N

natural logarithm function: the logarithm function with base e, written $y = \log_e x$ or $y = \ln x$.

normal axis: for a given line, the normal axis is the only line that can be drawn from the origin perpendicular to the given line.

normal line: the line drawn at a point on a graph of $f(x)$, perpendicular to the tangent line drawn at that point.

O

oblique asymptote: See **slant asymptote**.

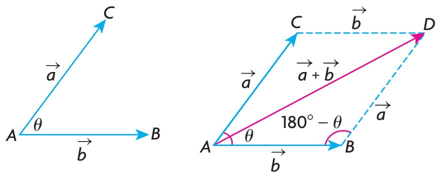
opposite vectors: vectors that have the same magnitude but point in opposite directions.

optimization: a procedure to determine the best possible outcome of a situation. If the situation can be modelled as a function, it may involve finding either the maximum or minimum value of the function over a set of values (domain).

optimize: to realize the best possible outcome for a situation, subject to a set of restrictions.

P

parallelogram law for adding two vectors: To determine the sum of the two vectors $\vec{a}$ and $\vec{b}$, complete the parallelogram formed by these two vectors when placed tail to tail. Their sum is the vector $\overrightarrow{AD}$, the diagonal of the constructed parallelogram, $\vec{a} + \vec{b} = \overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$.



parameter: a variable that permits the description of a relation among other variables (two or more) to be expressed in an indirect manner using that variable.

parametric equations of a line: derived from vector equation; in $R^2, x = x_0 + ta, y = y_0 + tb, t \in \mathbf{R}$.

parametric equations for a plane: in R^3, $\vec{r}_0 = (x_0, y_0, z_0)$ is determined by a point on a plane and $a = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ are vectors that lie on the same plane as the point.
$x = x_0 + sa_1 + tb_1,$
$y = y_0 + sa_2 + tb_2,$
$z = z_0 + sa_3 + tb_3, s, t \in \mathbf{R}.$

point of inflection: See **inflection point**.

position vector: the position vector $\overrightarrow{OP}$ has its head at the point $P(a, 0)$ and its tail at the origin $O(0, 0)$.

power function: a function of the form $f(x) = x^n$, where n is a real number.

power of a function rule: if u is a function of x and n is a positive integer, then in Leibniz notation $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$. In function notation, if $f(x) = [g(x)]^n$, then $f'(x) = n[g(x)]^{n-1}g(x)$.

power rule: if $f(x) = x^n$, where n is a real number, then $f'(x) = nx^{n-1}$.

product rule: if $h(x) = f(x)g(x)$, then $h'(x) = f'(x)g(x) + f(x)g'(x)$. See **extended power rule**.

projection: a mapping of a geometric figure formed by dropping a perpendicular from each of the points onto a line or plane.

Q

quotient rule: if $h(x) = \frac{f(x)}{g(x)}$, then
$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, g(x) \neq 0.$

R

rate of change: a measure of how rapidly the dependent variable changes when there is a change in the independent variable.

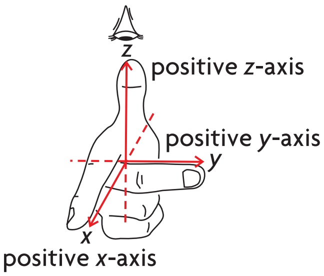
rational function: a function that can be expressed as $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$.

rationalizing the denominator: the process of multiplying the numerator and denominator of a rational expression by the conjugate radical of the denominator.

resolution: the opposite of composition; taking a single force and decomposing it into two components, often parallel to the vertical and horizontal axes.

resultant: the sum of two or more vectors; represents the combined effect of the vectors.

right-handed system of coordinates: one method of specifying the relative position of the coordinate axes in three dimensions; illustrated in the figure below.



row-echelon form: any matrix that has the following characteristics:

1. All zero rows are at the bottom of the matrix
2. The leading entry of each nonzero row after the first occurs to the right of the leading entry of the previous row.
3. The leading entry in any nonzero row is 1.
4. All entries in the column above and below a leading 1 are zero.

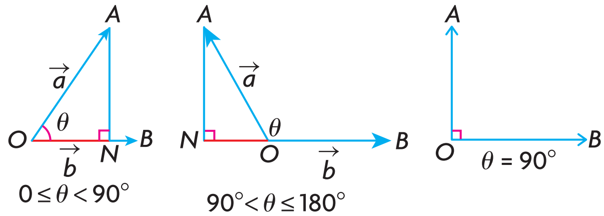
reduced row-echelon form: a matrix derived by the method of Gauss-Jordan elimination that permits the solution of a system of linear equations.

S

scalar: a quantity whose magnitude can be completely specified by just one number.

scalar product: See **dot product**.

scalar projection: the scalar projection of vector $\vec{a}$ onto $\vec{b}$ is ON where $ON = |\vec{a}|\cos\theta$.



second derivative: for a function $f(x)$, the second derivative of $y = f(x)$ is the derivative of $y = f'(x)$.

second derivative test: if $f(x)$ is a function for which $f''(c) = 0$, and the second derivative of $f(x)$ exists on an interval containing c, then

- $f(c)$ is a local minimum value if $f''(x) > 0$
- $f(c)$ is a local maximum value if $f''(x) < 0$
- the test is indeterminate if $f'(x) = 0$, and the first derivative test must be used

secant: a line through two points on a curve.

slant asymptote: the line $y = ax + b$ is a slant or oblique asymptote of $f(x)$ if and only if $\lim_{x \to \pm\infty} f(x) = ax + b$.

slope of tangent: the slope of the tangent to the function $y = f(x)$ at the point $(a, f(a))$ on the curve is given by $\lim_{h \to 0} \frac{\Delta y}{\Delta x} = \lim_{x \to 0} \frac{f(a+h) - f(a)}{h}$.

speed: distance travelled per unit of time. The absolute value of velocity.

spanning set: a set of two vectors forms a spanning set for R^2 if every vector in R^2 can be written as a linear combination of these two vectors; a spanning set for R^3contains three vectors.

standard basis vectors: unit vectors that lie along the axes; $\vec{i}$ and $\vec{j}$ for R^2 and $\vec{i}$, $\vec{j}$, and $\vec{k}$ for R^3.

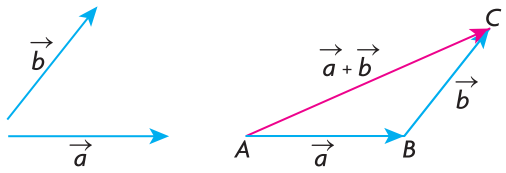
sum rule: if functions $p(x)$ and $q(x)$ are differentiable and $f(x) = p(x) + q(x)$, then $f'(x) = p'(x) + q'(x)$. In Leibniz notation: $\frac{df}{dx} = \frac{dp}{dx} + \frac{dq}{dx}$.

symmetric equation: for a line in R^3,
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} =, a \neq 0, b \neq 0, c \neq 0.$$
(x_0, y_0, z_0) is the vector from the origin to a point on the line and is a direction vector of the line.

T

tangent: the straight line that most resembles the graph near that point.

triangle law of addition: in the diagram, the sum of the vectors $\vec{a}$ and $\vec{b}$, $\vec{a} + \vec{b}$, is found by translating the tail of vector $\vec{b}$ to the head of vector $\vec{a}$. This could also have been done by translating $\vec{a}$ so that its tail was at the head of $\vec{b}$. In either case, the sum of the vectors $\vec{a}$ and $\vec{b}$ is $\overrightarrow{AC}$.



U

unit vector: a vector with magnitude 1.

V

vector: a quantity that requires both a magnitude and a direction for a complete description.

vector equation of a line: equation of a line written in terms of a position vector for a point on the line and a vector specifying the direction of the vector; in R^2, $\vec{r} = \vec{r_0} + t\vec{m}$, $t \in \mathbf{R}$.

vector equation of a plane: in R_3, $\vec{r_0} = (x_0, y_0, z_0)$ is determined by a point on a plane and $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ are vectors that lie on the same plane as the point: $\vec{r} = \vec{r_0} + s\vec{a} + t\vec{b}$, $s, t \in \mathbf{R}$.

velocity: the rate of change of displacement with respect to time: $\left(\frac{ds}{dt}\right)$.

vertical asymptote: the line $x = x_0$ is a vertical asymptote of $f(x)$ if and only if $f(x) \to \pm\infty$ as $x \to x_0$ from the left or from the right.

vector product: See **cross product**.

vector projection: the vector projection of $\vec{a}$ on $\vec{b}$ is the product of the dot product for the two vectors and a unit vector in the direction of $\vec{b}$.

Z

zero vector: the vector with a magnitude of 0 and no defined direction.

Answers

Chapter 1

Review of Prerequisite Skills, pp. 2–3

1. **a.** -3 **c.** 4 **e.** -4.1
b. -2 **d.** -4 **f.** $-\frac{1}{2}$

2. **a.** $y = 4x - 2$
b. $y = -2x + 5$
c. $y = \frac{6}{5}(x + 1) + 6$
d. $x + y - 2 = 0$
e. $x = -3$
f. $y = 5$

3. **a.** -1 **c.** -9
b. 0 **d.** 144

4. **a.** $-\frac{5}{52}$ **c.** 0
b. $-\frac{3}{13}$ **d.** $\frac{5}{52}$

5. **a.** 6 **c.** 9
b. $\sqrt{3}$ **d.** $\sqrt{6}$

6. **a.** $-\frac{1}{2}$ **c.** 5 **e.** 10^6
b. -1 **d.** 1

7. **a.** $x^2 - 4x - 12$
b. $15 + 17x - 4x^2$
c. $-x^2 - 7x$
d. $-x^2 + x + 7$
e. $a^3 + 6a^2 + 12a + 8$
f. $729a^3 - 1215a^2 + 675a - 125$

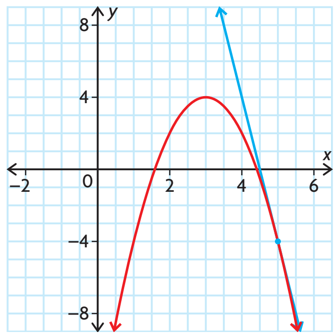
8. **a.** $x(x + 1)(x - 1)$
b. $(x + 3)(x - 2)$
c. $(2x - 3)(x - 2)$
d. $x(x + 1)(x + 1)$
e. $(3x - 4)(9x^2 + 12x + 16)$
f. $(x - 1)(2x - 3)(x + 2)$

9. **a.** $\{x \in \mathbf{R} \mid x \geq -5\}$
b. $\{x \in \mathbf{R}\}$
c. $\{x \in \mathbf{R} \mid x \neq 1\}$
d. $\{x \in \mathbf{R} \mid x \neq 0\}$
e. $\left\{x \in \mathbf{R} \mid x \neq -\frac{1}{2}, 3\right\}$
f. $\{x \in \mathbf{R} \mid x \neq -5, -2, 1\}$

10. **a.** 20.1 m/s **b.** 10.3 m/s

11. **a.** -20 L/min
b. about -13.33 L/min
c. The volume of water in the hot tub is always decreasing during that time period, a negative change.

12. **a.** **b.**



$m = -8$
c. -8

Section 1.1, p. 9

1. **a.** $2\sqrt{3} + 4$ **d.** $3\sqrt{3} - \sqrt{2}$
b. $\sqrt{3} - \sqrt{2}$ **e.** $\sqrt{2} + \sqrt{5}$
c. $2\sqrt{3} + \sqrt{2}$ **f.** $-\sqrt{5} - 2\sqrt{2}$

2. **a.** $\frac{\sqrt{6} + \sqrt{10}}{2}$ **c.** $\frac{4 + \sqrt{6}}{2}$
b. $\sqrt{6} - 3$ **d.** $\frac{3\sqrt{10} - 2}{4}$

3. **a.** $\sqrt{5} + \sqrt{2}$ **d.** $4 - 2\sqrt{5}$
b. $10 - 3\sqrt{10}$ **e.** $\frac{11\sqrt{6} - 16}{47}$
c. $5 + 2\sqrt{6}$ **f.** $\frac{35 - 12\sqrt{6}}{19}$

4. **a.** $\frac{1}{\sqrt{5} + 1}$
b. $\frac{-7}{2 + 3\sqrt{2}}$
c. $\frac{1}{12 - 5\sqrt{5}}$

5. **a.** $8\sqrt{10} + 24$
b. $8\sqrt{10} + 24$
c. The expressions are equivalent. The radicals in the denominator of part a. have been simplified in part b.

6. **a.** $-2\sqrt{3} - 4$
b. $\frac{18\sqrt{2} + 4\sqrt{3}}{23}$
c. $2\sqrt{2} + \sqrt{6}$
d. $\frac{24 + 15\sqrt{3}}{4}$
e. $-\frac{12\sqrt{15} + 15\sqrt{10}}{2}$
f. $5 + 2\sqrt{6}$

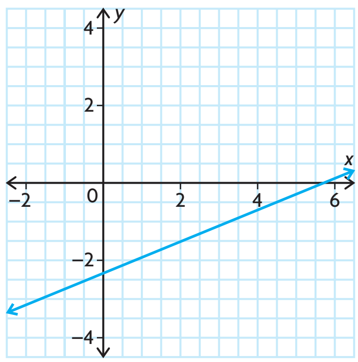
7. **a.** $\frac{1}{\sqrt{a} - 2}$
b. $\frac{1}{\sqrt{x + 4} - 2}$
c. $\frac{1}{\sqrt{x + h} + \sqrt{x}}$

Section 1.2, pp. 18–21

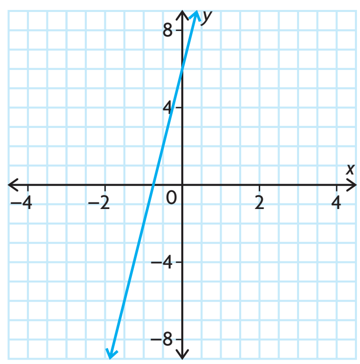
1. **a.** 3 **b.** $-\frac{5}{3}$ **c.** $-\frac{1}{3}$

2. **a.** $-\frac{1}{3}$ **b.** $-\frac{7}{13}$

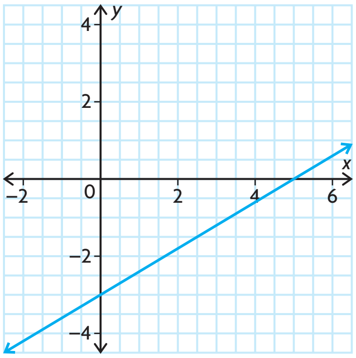
3. **a.** $7x - 17y - 40 = 0$



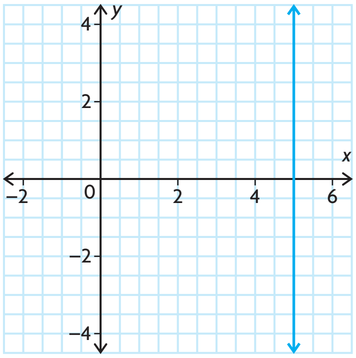
b. $y = 8x + 6$



c. $3x - 5y - 15 = 0$



d. $x = 5$



4. **a.** $75 + 15h + h^2$
b. $108 + 54h + 12h^2 + h^3$
c. $-\dfrac{1}{1 + h}$
d. $6 + 3h$
e. $\dfrac{-3}{4(4 + h)}$
f. $\dfrac{1}{4 + 2h}$

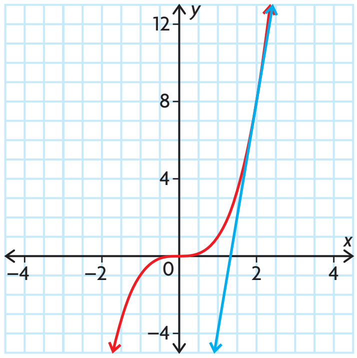
5. **a.** $\dfrac{1}{\sqrt{16 + h} + 4}$
b. $\dfrac{h + 5}{\sqrt{h^2 + 5h + 4} + 2}$
c. $\dfrac{1}{\sqrt{5 + h} + \sqrt{5}}$

6. **a.** $6 + 3h$
b. $3 + 3h + h^2$
c. $\dfrac{1}{\sqrt{9 + h} + 3}$

7. **a.**

P	Q	Slope of Line PQ
(2, 8)	(3, 27)	19
(2, 8)	(2.5, 16.625)	15.25
(2, 8)	(2.1, 9.261)	12.61
(2, 8)	(2.01, 8.120 601)	12.0601
(2, 8)	(1, 1)	7
(2, 8)	(1.5, 3.375)	9.25
(2, 8)	(1.9, 6.859)	11.41
(2, 8)	(1.99, 7.880 599)	11.9401

b. 12
c. $12 + 6h + h^2$
d. 12
e. They are the same.
f.



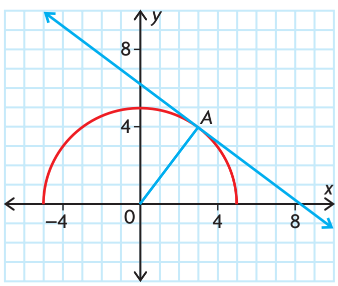
8. **a.** -12 **b.** 5 **c.** 12

9. **a.** $\dfrac{1}{2}$ **b.** $\dfrac{1}{4}$ **c.** $\dfrac{5}{6}$

10. **a.** -2 **b.** $-\dfrac{1}{2}$ **c.** $-\dfrac{1}{10}$

11. **a.** 1 **d.** $\dfrac{1}{6}$
b. -1 **e.** $-\dfrac{3}{4}$
c. 9 **f.** $-\dfrac{1}{6}$

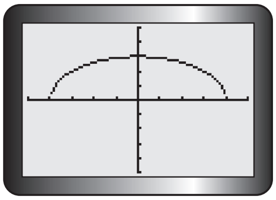
12.



$y = \sqrt{25 - x^2} \rightarrow$ Semi-circle
centre (0, 0), rad 5, $y \geq 0$
OA is a radius. The slope of OA is $\frac{4}{3}$.
The slope of tangent is $-\frac{3}{4}$.

13. Take values of x close to the point, then determine $\frac{\Delta y}{\Delta x}$.

14.



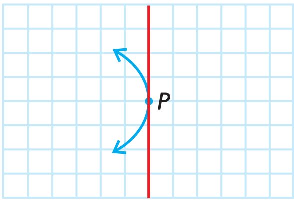
Since the tangent is horizontal, the slope is 0.

15. $3x - y - 8 = 0$

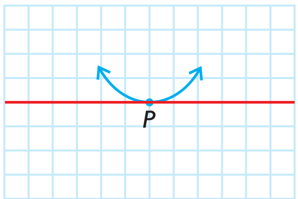
16. $3x + y - 8 = 0$

17. **a.** $(3, -2)$
b. $(5, 6)$
c. $y = 4x - 14$
d. $y = 2x - 8$
e. $y = 6x - 24$

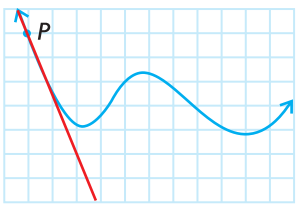
18. **a.** undefined



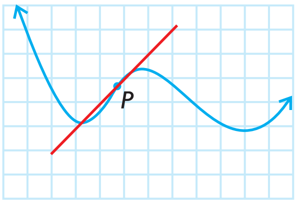
b. 0



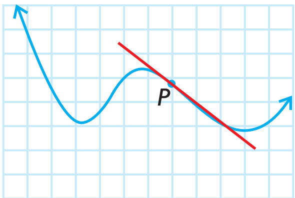
c. about -2.5



d. about 1



e. about $-\frac{7}{8}$



f. no tangent at point P

19. $-\frac{5}{4}$

20. 1600 papers

21. $(2, 4)$

22. $\left(-2, \frac{23}{3}\right), \left(-1, \frac{26}{3}\right), \left(1, -\frac{26}{3}\right), \left(2, -\frac{28}{3}\right)$

23. $y = x^2$ and $y = \frac{1}{2} - x^2$

$$x^2 = \frac{1}{2} - x^2$$

$$x^2 = \frac{1}{4}$$

$$x = \frac{1}{2} \text{ or } x = -\frac{1}{2}$$

The points of intersection are $P\left(\frac{1}{2}, \frac{1}{4}\right)$ and $Q\left(-\frac{1}{2}, \frac{1}{4}\right)$.

Tangent to $y = x$:

$$m = \lim_{h \to 0} \frac{(a + h)^2 - a^2}{h}$$

$$= \lim_{h \to 0} \frac{2ah + h^2}{h}$$

$$= 2a$$

The slope of the tangent at $a = \frac{1}{2}$ is $1 = m_p$ and at $a = -\frac{1}{2}$ is $-1 = m_q$.

Tangents to $y = \frac{1}{2} - x^2$:

$$m = \lim_{h \to 0} \frac{\left[\frac{1}{2} - (a + h)^2\right] - \left[\frac{1}{2} - a^2\right]}{h}$$

$$= \lim_{h \to 0} \frac{-2ah - h^2}{h}$$

$$= -2a$$

The slope of the tangents at $a = \frac{1}{2}$ is $-1 = M_P$ and at $a = -\frac{1}{2}$ is $1 = M_q$; $m_p M_P = -1$ and $m_q M_q = -1$.

Therefore, the tangents are perpendicular at the points of intersection.

24. $y = -11x + 24$

25. **a.** $8a + 5$
b. $(0, -2)$
c. $(-5, 73)$

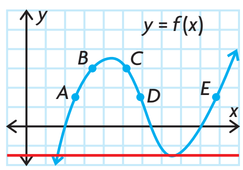
Section 1.3, pp. 29–31

1. 0 s or 4 s

2. **a.** Slope of the secant between the points $(2, s(2))$ and $(9, s(9))$
b. Slope of the tangent at the point $(6, s(6))$

3. Slope of the tangent to the function with equation $y = \sqrt{x}$ at the point $(4, 2)$

4. **a.** A and B
b. greater; the secant line through these two points is steeper than the tangent line at B.
c.



5. Speed is represented only by a number, not a direction.

6. Yes, velocity needs to be described by a number and a direction. Only the speed of the school bus was given, not the direction, so it is not correct to use the word "velocity."

7. **a.** first second = 5 m/s, third second = 25 m/s, eighth second = 75 m/s
b. 55 m/s
c. −20 m/s

8. **a.** **i.** 72 km/h
ii. 64.8 km/h
iii. 64.08 km/h
b. 64 km/h
c. 64 km/h

9. **a.** 15 terms
b. 16 terms/h

10. **a.** $-\frac{1}{3}$ mg/h
b. Amount of medicine in 1 mL of blood being dissipated throughout the system

11. $\frac{1}{50}$ s/m

12. $-\frac{12}{5}$ °C/km

13. 2 s; 0 m/s

14. **a.** \$4800
b. \$80 per ball
c. $x < 80$

15. **a.** 6
b. −1
c. $\frac{1}{10}$

16. \$1 162 250 years since 1982

17. **a.** 75 m
b. 30 m/s
c. 60 m/s
d. 14 s

18. The coordinates of the point are $\left(a, \frac{1}{a}\right)$.

The slope of the tangent is $-\frac{1}{a^2}$.

The equation of the tangent is $y - \frac{1}{a} = -\frac{1}{a^2}(x - a)$, or $y = -\frac{1}{a^2}x + \frac{2}{a}$. The intercepts are $\left(0, \frac{2}{a}\right)$ and $(-2a, 0)$. The tangent line and the axes form a right triangle with legs of length $\frac{2}{a}$ and $2a$. The area of the triangle is $\frac{1}{2}\left(\frac{2}{a}\right)(2a) = 2$.

19. $C(x) = F + V(x)$

$C(x + h) = F + V(x + h)$

Rate of change of cost is

$$\lim_{x \to R} \frac{C(x + h) - C(x)}{h}$$

$$= \lim_{x \to h} \frac{V(x + h) - V(x)}{h} h,$$

which is independent of F − (fixed costs)

20. 200π m²/m

21. Cube of dimensions x by x by x has volume $V = x^3$. Surface area is $6x^2$. $V'(x) = 3x^2 = \frac{1}{2}$ surface area.

22. **a.** 80π cm²/unit of time
b. -100π cm³/unit of time

Mid-Chapter Review, pp. 32–33

1. **a.** 3
b. 37
c. 61
d. 5

2. **a.** $\frac{6\sqrt{3} + \sqrt{6}}{3}$
b. $\frac{6 + 4\sqrt{3}}{3}$
c. $-\frac{5(\sqrt{7} + 4)}{9}$
d. $-2(3 + 2\sqrt{3})$

e. $\dfrac{10\sqrt{3} - 15}{2}$

f. $-\dfrac{3\sqrt{2}(2\sqrt{3} + 5)}{13}$

3. a. $\dfrac{2}{5\sqrt{2}}$

b. $\dfrac{3}{\sqrt{3}(6 + \sqrt{2})}$

c. $-\dfrac{9}{5(\sqrt{7} + 4)}$

d. $-\dfrac{13}{3\sqrt{2}(2\sqrt{3} + 5)}$

e. $-\dfrac{1}{(\sqrt{3} + \sqrt{7})}$

f. $\dfrac{1}{(2\sqrt{3} - \sqrt{7})}$

4. a. $\frac{2}{3}x + y - 6 = 0$

b. $x - y + 5 = 0$

c. $4x - y - 2 = 0$

d. $x - 5y - 9 = 0$

5. -2

6. a.

P	*Q*	Slope of Line *PQ*
$(-1, 1)$	$(-2, 6)$	-5
$(-1, 1)$	$(-1.5, 3.25)$	-4.5
$(-1, 1)$	$(-1.1, 1.41)$	-4.1
$(-1, 1)$	$(-1.01, 1.0401)$	-4.01
$(-1, 1)$	$(-1.001, 1.004\,001)$	-4.001

P	*Q*	Slope of Line *PQ*
$(-1, 1)$	$(0, -2)$	-3
$(-1, 1)$	$(-0.5, -0.75)$	-3.5
$(-1, 1)$	$(-0.9, 0.61)$	-3.9
$(-1, 1)$	$(-0.99, 0.960\,1)$	-3.99
$(-1, 1)$	$(-0.999, 0.996\,001)$	-3.999

b. -4

c. $h - 4$

d. -4

e. The answers are equal.

7. a. -3 c. $-\dfrac{1}{4}$

b. -9 d. $\dfrac{1}{6}$

8. a. i. 36 km/h

ii. 30.6 km/h

iii. 30.06 km/h

b. velocity of car appears to approach 30 km/h

c. $(6h + 30)$ km/h

d. 30 km/h

9. a. -4

b. -12

10. a. -2000 L/min

b. -1000 L/min

11. a. $-9x + y + 19 = 0$

b. $8x + y + 15 = 0$

c. $4x + y + 8 = 0$

d. $-2x + y + 2 = 0$

12. a. $-3x + 4y - 25 = 0$

b. $3x + 4y + 5 = 0$

Section 1.4, pp. 37–39

1. a. $\dfrac{27}{99}$ b. π

2. Evaluate the function for values of the independent variable that get progressively closer to the given value of the independent variable.

3. a. A right-sided limit is the value that a function gets close to as the values of the independent variable decrease and get close to a given value.

b. A left-sided limit is the value that a function gets close to as the values of the independent variable increase and get close to a given value.

c. A (two-sided) limit is the value that a function gets close to as the values of the independent variable get close to a given value, regardless of whether the values increase or decrease toward the given value.

4. a. -5 d. -8

b. 10 e. 4

c. 100 f. 8

5. 1

6. a. 0 c. -1

b. 2 d. 2

7. a. 2

b. 1

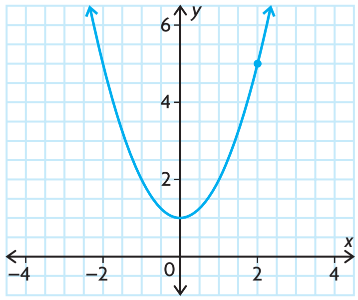
c. does not exist

8. a. 8

b. 2

c. 2

9. 5



10. a. 0 d. $-\dfrac{1}{2}$

b. 0 e. $\dfrac{1}{5}$

c. 5

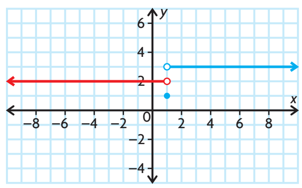
f. does not exist; substitution causes division by zero, and there is no way to remove the factor from the denominator.

11. a. does not exist c. 2

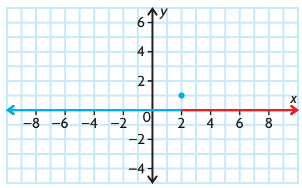
b. 2 d. does not exist

12. Answers may vary. For example:

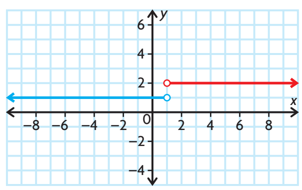
a.



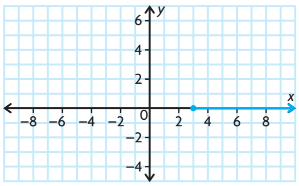
b.



c.



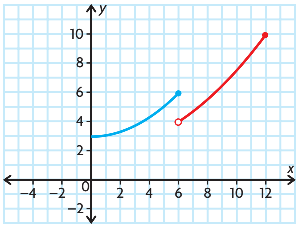
d.



13. $m = -3;\ b = 1$

14. $a = 3, b = 2, c = 0$

15. a.



b. 6; 4

c. 2000

d. about 8.49 years

Section 1.5, pp. 45–47

1. $\lim_{x\to 2}(3 + x)$ and $\lim_{x\to 2}(x + 3)$ have the same value, but $\lim_{x\to 2} 3 + x$ does not. Since there are no brackets around the expression, the limit only applies to 3, and there is no value for the last term, x.

2. Factor the numerator and denominator. Cancel any common factors. Substitute the given value of x.

3. If the two one-sided limits have the same value, then the value of the limit is equal to the value of the one-sided limits. If the one-sided limits do not have the same value, then the limit does not exist.

4. **a.** 1 **d.** $5\pi^3$
b. 1 **e.** 2
c. $\frac{100}{9}$ **f.** $\sqrt{3}$

5. **a.** -2
b. $\sqrt{2}$

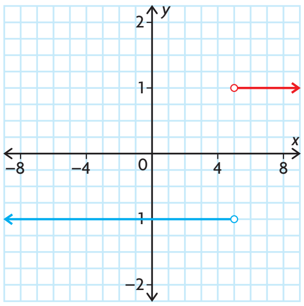
6. Since substituting $t = 1$ does not make the denominator 0, direct substitution works. $\frac{1 - 1 - 5}{6 - 1} = \frac{-5}{5}$
$= -1$

7. **a.** 4 **d.** $-\frac{1}{4}$
b. 5 **e.** $\frac{1}{4}$
c. 27 **f.** $-\frac{1}{\sqrt{7}}$

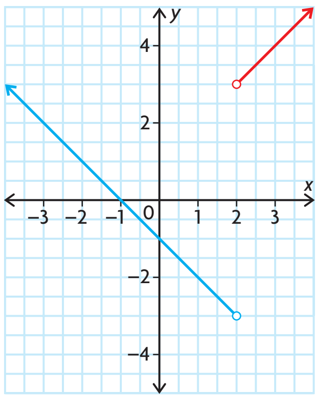
8. **a.** $\frac{1}{12}$ **d.** $\frac{1}{2}$
b. -27 **e.** $\frac{1}{12}$
c. $\frac{1}{6}$ **f.** $\frac{1}{12}$

9. **a.** 0 **d.** $\frac{1}{2}$
b. 0 **e.** $2x$
c. -1 **f.** $\frac{1}{32}$

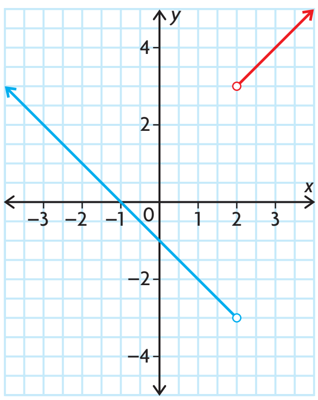
10. **a.** does not exist



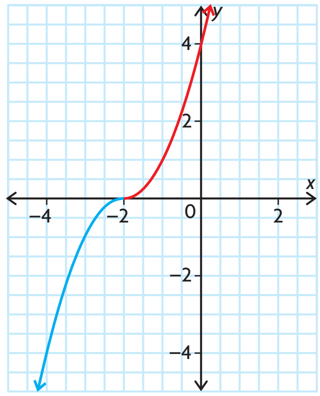
b. does not exist



c. exists



d. exists



11. **a.**

ΔT	T	V	ΔV
	−40	19.1482	
20			1.6426
	−20	20.7908	
20			1.6426
	0	22.4334	
20			1.6426
	20	24.0760	
20			1.6426
	40	25.7186	
20			1.6426
	60	27.3612	
20			1.6426
	80	29.0038	

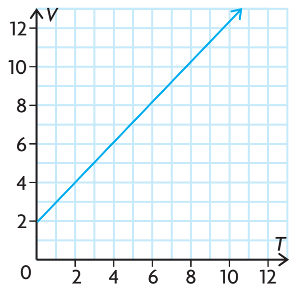
ΔV is constant; therefore, T and V form a linear relationship.

b. $V = 0.082\,13T + 22.4334$

c. $T = \frac{V - 22.4334}{0.082\,13}$

d. $\lim_{v\to 0} T = \frac{0 - 22.4334}{0.082\,13}$
$= -273.145$

e.



12. $\lim_{x\to 5} \frac{x^2 - 4}{f(x)}$
$= \frac{\lim_{x\to 5}(x^2 - 4)}{\lim_{x\to 5} f(x)}$
$= \frac{21}{3}$
$= 7$

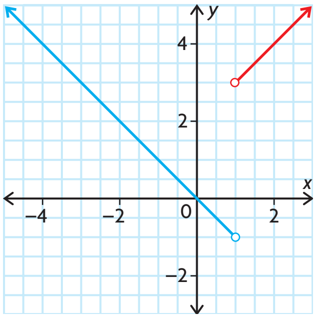
13. **a.** 27 **b.** -1 **c.** 1

14. **a.** 0 **b.** 0

15. **a.** 0 **b.** $\frac{1}{2}$

16. -2

17. does not exist

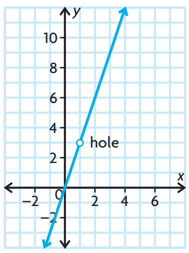


Section 1.6, pp. 51–53

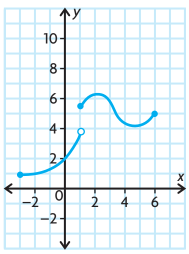
1. Anywhere that you can see breaks or jumps is a place where the function is not continuous.

2. On a given domain, you can trace the graph of the function without lifting your pencil.

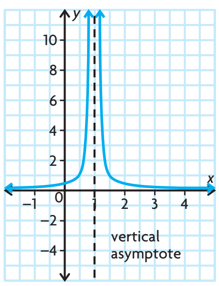
3. point discontinuity



jump discontinuity



infinite discontinuity

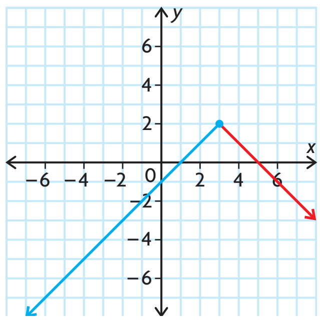


4. **a.** $x = 3$
b. $x = 0$
c. $x = 0$
d. $x = 3$ and $x = -3$
e. $x = -3$ and $x = 2$
f. $x = 3$

5. **a.** continuous for all real numbers
b. continuous for all real numbers
c. continuous for all real numbers, except 0 and 5
d. continuous for all real numbers greater than or equal to -2
e. continuous for all real numbers
f. continuous for all real numbers

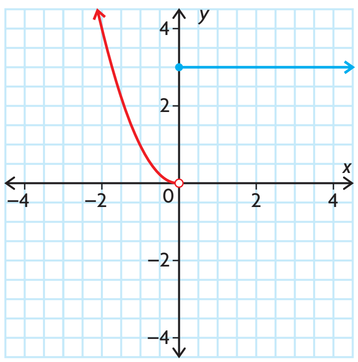
6. $g(x)$ is a linear function (a polynomial), and so is continuous everywhere, including $x = 2$.

7.



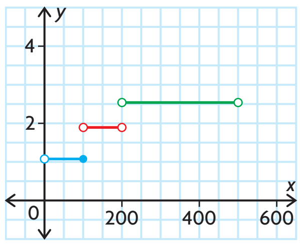
Yes, the function is continuous everywhere.

8.



The function is discontinuous at $x = 0$.

9.



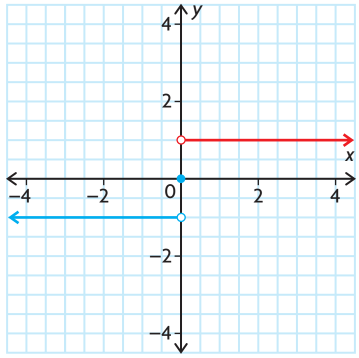
Discontinuities at 0, 100, 200, and 500

10. no

11. Discontinuous at $x = 2$

12. $k = 16$

13. **a.**



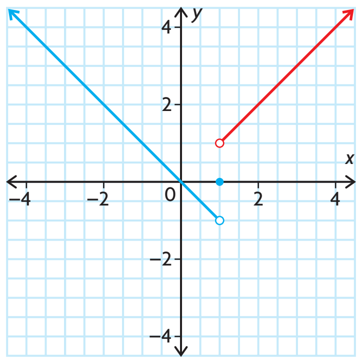
b. **i.** -1
ii. 1
iii. does not exist
c. f is not continuous since $\lim_{x \to 0} f(x)$ does not exist.

14. **a.** 2
b. 4
c. $\lim_{x \to 3^-} f(x) = 4 = \lim_{x \to 3^+} f(x)$
Thus, $\lim_{x \to 3} f(x) = 4$. But, $f(3) = 2$.
Hence, f is not continuous at $x = 2$ and also not continuous on $-3 < x < 8$.

15. (1) $A = B - 3$
(2) $4B - A \neq 6$ (if $B > 1$, then $A > -2$; if $B < 1$, then $A < -2$)

16. $a = -1, b = 6$

17. **a.** $\left.\begin{array}{l} \lim_{x \to 1^-} g(x) = -1 \\ \lim_{x \to 1^+} g(x) = 1 \end{array}\right\} \lim_{x \to 1} g(x)$
$\lim_{x \to 1} g(x)$ does not exist.
b.



$g(x)$ is discontinuous at $x = 1$.

Review Exercise, pp. 56–59

1. **a.** -3 **b.** 7
c. $2x - y - 5 = 0$

2. **a.** $\frac{-1}{3}$ **c.** $-\frac{1}{27}$
b. $\frac{1}{2}$ **d.** $-\frac{5}{4}$

3. **a.** 2 **b.** 2

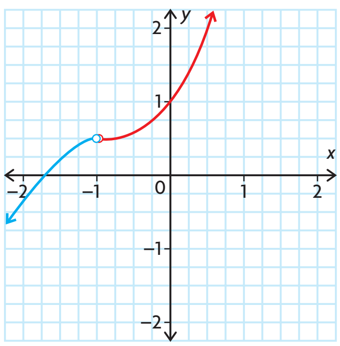
4. **a.** 1st second $= -5$ m/s,
2nd second $= -15$ m/s
b. -40 m/s
c. -60 m/s

5. **a.** 0.0601 g
b. 6.01 g/min
c. 6 g/min

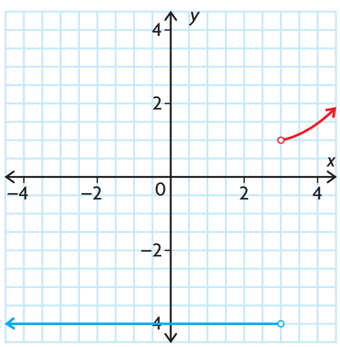
6. **a.** 700 000 t
b. 18×10^4 t per year
c. 15×10^4 t per year
d. 7.5 years

7. **a.** 10
b. 7; 0
c. $t = 3$ and $t = 4$

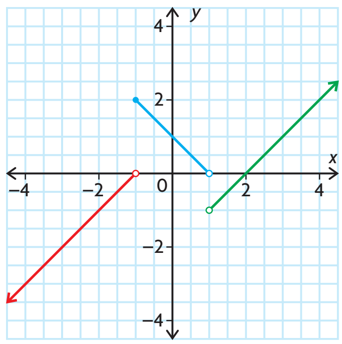
8. **a.** Answers may vary. For example:



b. Answers may vary. For example:



9. **a.**



b. $x = -1$ and $x = 1$
c. They do not exist.

10. not continuous at $x = -4$

11. **a.** $x = 1$ and $x = -2$
b. $\lim_{x\to 1} f(x) = \frac{2}{3}$,
$\lim_{x\to -2} f(x)$ does not exist.

12. **a.** $\lim_{x\to 0} f(x)$ does not exist.
b. $\lim_{x\to 0} g(x) = 0$
c. $\lim_{x\to -3} h(x)$ does not exist.

13. **a.**

x	1.9	1.99	1.999	2.001	2.01	2.1
$\frac{x-2}{x^2-x-2}$	0.344 83	0.334 45	0.333 44	0.333 22	0.332 23	0.322 58

$\frac{1}{3}$

b.

x	0.9	0.99	0.999	1.001	1.01	1.1
$\frac{x-1}{x^2-1}$	0.526 32	0.502 51	0.500 25	0.499 75	0.497 51	0.476 19

$\frac{1}{2}$

14.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$\frac{\sqrt{x+3}-\sqrt{3}}{x}$	0.291 12	0.288 92	0.2887	0.288 65	0.288 43	0.286 31

$\frac{1}{2\sqrt{3}}$; This agrees well with the values in the table.

15. **a.**

x	2.1	2.01	2.001	2.0001
$f(x)$	0.248 46	0.249 84	0.249 98	0.25

$\lim_{x\to 2} f(x) \doteq 0.25$
b. $\lim_{x\to 2} f(x) = 0.25$
c. 0.25

16. **a.** 10 **b.** $\frac{1}{4}$ **c.** $-\frac{1}{16}$

17. **a.** 4 **c.** $\frac{1}{\sqrt{5}}$ **e.** $-\frac{1}{8}$
b. $10a$ **d.** $\frac{1}{3}$ **f.** $-\frac{1}{4}$

18. **a.** The function is not defined for $x < 3$, so there is no left-side limit.
b. Even after dividing out common factors from numerator and denominator, there is a factor of $x - 2$ in the denominator; the graph has a vertical asymptote at $x = 2$.
c. $\lim_{x\to 1^-} f(x) = -5 \neq \lim_{x\to 1^+} f(x) = 2$
d. The function has a vertical asymptote at $x = 2$.
e. $x \to 0^-\ |x| = -x$

$$\lim_{x\to 0^-} \frac{|x|}{x} = -1$$
$$\lim_{x\to 0^+} \frac{|x|}{x} = 1$$
$$\lim_{x\to 0^+} \frac{|x|}{x} \neq \lim_{x\to 0^-} \frac{|x|}{x}$$

f. $\lim_{x\to -1^+} f(x) = -1$
$\lim_{x\to -1^-} f(x) = 5$
$\lim_{x\to -1^+} f(x) \neq \lim_{x\to -1^-} f(x)$
Therefore, $\lim_{x\to -1} f(x)$ does not exist.

19. **a.** $y = 7$
b. $y = -5x - 5$
c. $y = 18x + 9$
d. $y = -216x + 486$

20. **a.** 700 000
b. 109 000/h

Chapter 1 Test, p. 60

1. $\lim\limits_{x \to 1^+} \dfrac{1}{x-1} = +\infty \neq \lim\limits_{x \to 1^-} \dfrac{1}{x-1} = -\infty$

2. -13

3. **a.** $\lim\limits_{x \to 1} f(x)$ does not exist.
b. 1
c. 1
d. $x = 1$ and $x = 2$

4. **a.** 1 km/h
b. 2 km/h

5. $\dfrac{\sqrt{16+h} - \sqrt{16}}{h}$

6. -31

7. **a.** 12 **d.** $-\dfrac{3}{4}$
b. $\dfrac{7}{5}$ **e.** $\dfrac{1}{6}$
c. 4 **f.** $\dfrac{1}{12}$

8. **a.** $a = 1, b = -\dfrac{18}{5}$

Chapter 2

Review of Prerequisite Skills, pp. 62–63

1. **a.** a^8 **d.** $\dfrac{1}{a^2b^7}$
b. $-8a^6$ **e.** $48e^{18}$
c. $2p$ **f.** $-\dfrac{b}{2a^6}$

2. **a.** $x^{\frac{7}{6}}$ **b.** $4x^4$ **c.** $a^{\frac{1}{3}}$

3. **a.** $-\dfrac{3}{2}$ **c.** $-\dfrac{3}{5}$
b. 2 **d.** 1

4. **a.** $x - 6y - 21 = 0$
b. $3x - 2y - 4 = 0$
c. $4x + 3y - 7 = 0$

5. **a.** $2x^2 - 5xy - 3y^2$
b. $x^3 - 5x^2 + 10x - 8$
c. $12x^2 + 36x - 21$
d. $-13x + 42y$
e. $29x^2 - 2xy + 10y^2$
f. $-13x^3 - 12x^2y + 4xy^2$

6. **a.** $\dfrac{15}{2}x; x \neq 0, -2$
b. $\dfrac{y-5}{4y^2(y+2)}; y \neq -2, 0, 5$
c. $\dfrac{8}{9}; h \neq -k$
d. $\dfrac{2}{(x+y)^2}; x \neq -y, +y$
e. $\dfrac{11x^2 - 8x + 7}{2x(x-1)}; x \neq 0, 1$
f. $\dfrac{4x+7}{(x+3)(x-2)}; x \neq -3, 2$

7. **a.** $(2k+3)(2k-3)$
b. $(x-4)(x+8)$
c. $(a+1)(3a-7)$
d. $(x^2+1)(x+1)(x-1)$
e. $(x-y)(x^2+xy+y^2)$
f. $(r+1)(r-1)(r+2)(r-2)$

8. **a.** $(a-b)(a^2+ab+b^2)$
b. $(a-b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$
c. $(a-b)(a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6)$
d. $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + a^3b^{n-3} + ab^{n-2} + b^{n-1})$

9. **a.** -17 **c.** $\dfrac{53}{8}$
b. 10 **d.** about 7.68

10. **a.** $\dfrac{3\sqrt{2}}{2}$
b. $\dfrac{4\sqrt{3} - \sqrt{6}}{3}$
c. $-\dfrac{30 + 17\sqrt{2}}{23}$
d. $-\dfrac{11 - 4\sqrt{6}}{5}$

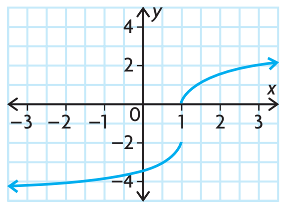
11. **a.** $3h + 10$; expression can be used to determine the slope of the secant line between $(2, 8)$ and $(2 + h, f(2 + h))$
b. For $h = 0.01$: 10.03
c. value represents the slope of the secant line through $(2, 8)$ and $(2.01, 8.1003)$

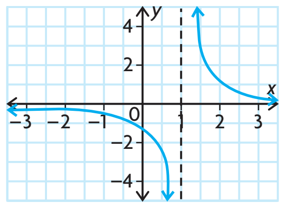
Section 2.1, pp. 73–75

1. **a.** $\{x \in \mathbf{R} \mid x \neq -2\}$
b. $\{x \in \mathbf{R} \mid x \neq 2\}$
c. $\{x \in \mathbf{R}\}$
d. $\{x \in \mathbf{R} \mid x \neq 1\}$
e. $\{x \in \mathbf{R}\}$
f. $\{x \in \mathbf{R} \mid x > 2\}$

2. The derivative of a function represents the slope of the tangent line at a give value of the independent variable or the instantaneous rate of change of the function at a given value of the independent variable.

3. Answers may vary. For example:





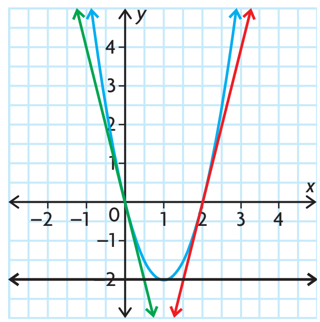
4. **a.** $5a + 5h - 2; 5h$
b. $a^2 + 2ah + h^2 + 3a + 3h - 1;$ $2ah + h^2 + 3h$
c. $a^3 + 3a^{2h} + 3ah^2 + h^3 - 4a - 4h + 1;$ $3a^2h + 3ah^2 + h^3 - 4h$
d. $a^2 + 2ah + h^2 + a + h - 6;$ $2ah + h^2 + h$
e. $-7a - 7h + 4; -7h$
f. $4 - 2a - 2h - a^2 - 2ah - h^2;$ $-2h - h^2 - 2ah$

5. **a.** 2 **c.** $\dfrac{1}{2}$
b. 9 **d.** -5

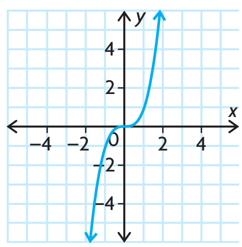
6. **a.** -5 **c.** $18x^2 - 7$
b. $4x + 4$ **d.** $\dfrac{3}{2\sqrt{3x+2}}$

7. **a.** -7
b. $-\dfrac{2}{(x-1)^2}$ **c.** $6x$

8. $f'(0) = -4; f'(1) = 0; f'(2) = 4$



9. **a.**



19. **a.** $y = 7$
b. $y = -5x - 5$
c. $y = 18x + 9$
d. $y = -216x + 486$

20. **a.** 700 000
b. 109 000/h

Chapter 1 Test, p. 60

1. $\lim_{x\to 1^+} \frac{1}{x-1} = +\infty \neq \lim_{x\to 1^-} \frac{1}{x-1} = -\infty$

2. -13

3. **a.** $\lim_{x\to 1} f(x)$ does not exist.
b. 1
c. 1
d. $x = 1$ and $x = 2$

4. **a.** 1 km/h
b. 2 km/h

5. $\frac{\sqrt{16+h} - \sqrt{16}}{h}$

6. -31

7. **a.** 12 **d.** $-\frac{3}{4}$
b. $\frac{7}{5}$ **e.** $\frac{1}{6}$
c. 4 **f.** $\frac{1}{12}$

8. **a.** $a = 1, b = -\frac{18}{5}$

Chapter 2

Review of Prerequisite Skills, pp. 62–63

1. **a.** a^8 **d.** $\frac{1}{a^2b^7}$
b. $-8a^6$ **e.** $48e^{18}$
c. $2p$ **f.** $-\frac{b}{2a^6}$

2. **a.** $x^{\frac{7}{6}}$ **b.** $4x^4$ **c.** $a^{\frac{1}{3}}$

3. **a.** $-\frac{3}{2}$ **c.** $-\frac{3}{5}$
b. 2 **d.** 1

4. **a.** $x - 6y - 21 = 0$
b. $3x - 2y - 4 = 0$
c. $4x + 3y - 7 = 0$

5. **a.** $2x^2 - 5xy - 3y^2$
b. $x^3 - 5x^2 + 10x - 8$
c. $12x^2 + 36x - 21$
d. $-13x + 42y$
e. $29x^2 - 2xy + 10y^2$
f. $-13x^3 - 12x^2y + 4xy^2$

6. **a.** $\frac{15}{2}x; x \neq 0, -2$
b. $\frac{y-5}{4y^2(y+2)}; y \neq -2, 0, 5$
c. $\frac{8}{9}; h \neq -k$
d. $\frac{2}{(x+y)^2}; x \neq -y, +y$
e. $\frac{11x^2 - 8x + 7}{2x(x-1)}; x \neq 0, 1$
f. $\frac{4x+7}{(x+3)(x-2)}; x \neq -3, 2$

7. **a.** $(2k+3)(2k-3)$
b. $(x-4)(x+8)$
c. $(a+1)(3a-7)$
d. $(x^2+1)(x+1)(x-1)$
e. $(x-y)(x^2+xy+y^2)$
f. $(r+1)(r-1)(r+2)(r-2)$

8. **a.** $(a-b)(a^2+ab+b^2)$
b. $(a-b)(a^4+a^3b+a^2b^2$
$+ ab^3 + b^4)$
c. $(a-b)(a^6+a^5b+a^4b^2$
$+ a^3b^3 + a^2b^4 + ab^5 + b^6)$
d. $a^n - b^n = (a-b)(a^{n-1}$
$+ a^{n-2}b + a^{n-3}b^2 + a^3b^{n-3}$
$+ ab^{n-2} + b^{n-1})$

9. **a.** -17 **c.** $\frac{53}{8}$
b. 10 **d.** about 7.68

10. **a.** $\frac{3\sqrt{2}}{2}$
b. $\frac{4\sqrt{3} - \sqrt{6}}{3}$
c. $-\frac{30 + 17\sqrt{2}}{23}$
d. $-\frac{11 - 4\sqrt{6}}{5}$

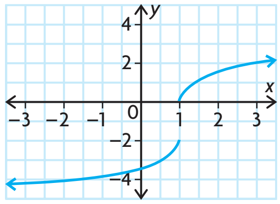
11. **a.** $3h + 10$; expression can be used to determine the slope of the secant line between $(2, 8)$ and $(2 + h, f(2+h))$
b. For $h = 0.01$: 10.03
c. value represents the slope of the secant line through $(2, 8)$ and $(2.01, 8.1003)$

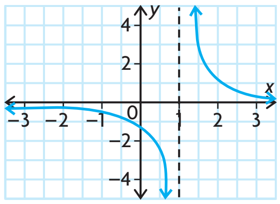
Section 2.1, pp. 73–75

1. **a.** $\{x \in \mathbf{R} \mid x \neq -2\}$
b. $\{x \in \mathbf{R} \mid x \neq 2\}$
c. $\{x \in \mathbf{R}\}$
d. $\{x \in \mathbf{R} \mid x \neq 1\}$
e. $\{x \in \mathbf{R}\}$
f. $\{x \in \mathbf{R} \mid x > 2\}$

2. The derivative of a function represents the slope of the tangent line at a give value of the independent variable or the instantaneous rate of change of the function at a given value of the independent variable.

3. Answers may vary. For example:





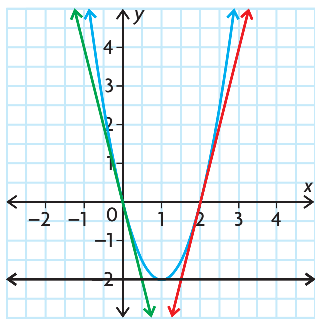
4. **a.** $5a + 5h - 2; 5h$
b. $a^2 + 2ah + h^2 + 3a + 3h - 1;$
$2ah + h^2 + 3h$
c. $a^3 + 3a^{2h} + 3ah^2 + h^3 - 4a$
$- 4h + 1;$
$3a^2h + 3ah^2 + h^3 - 4h$
d. $a^2 + 2ah + h^2 + a + h - 6;$
$2ah + h^2 + h$
e. $-7a - 7h + 4; -7h$
f. $4 - 2a - 2h - a^2 - 2ah - h^2;$
$-2h - h^2 - 2ah$

5. **a.** 2 **c.** $\frac{1}{2}$
b. 9 **d.** -5

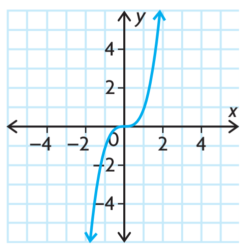
6. **a.** -5 **c.** $18x^2 - 7$
b. $4x + 4$ **d.** $\frac{3}{2\sqrt{3x+2}}$

7. **a.** -7
b. $-\frac{2}{(x-1)^2}$ **c.** $6x$

8. $f'(0) = -4; f'(1) = 0; f'(2) = 4$

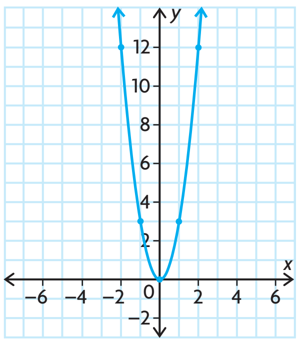


9. **a.**



b. $f'(-2) = 12; f'(-1) = 3;$
$f'(0) = 0; f'(1) = 3; f'(2) = 12$

c.



d. graph of $f(x)$ is cubic; graph of $f'(x)$ seems to be a parabola

10. $s'(0) = 8$ m/s; $s'(4) = 0$ m/s; $s'(6) = -4$ m/s

11. $x - 6y + 10 = 0$

12. **a.** 0 **c.** m
b. 1 **d.** $2ax + b$

13. Since $3x^2$ is nonnegative for all x, the original function never has a negative slope.

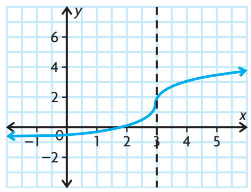
14. **a.** -1.6 m/s
b. $h'(2)$ measures the rate of change in the height of the ball with respect to time when $t = 2$.

15. **a.** e. **b.** f. **c.** d.

16.
$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-(0+h)^2 - (-0^2)}{h} = \lim_{h \to 0^-} \frac{-h^2}{h} = \lim_{h \to 0^-} (-h) = 0$$
$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{(0+h)^2 - (0^2)}{h} = \lim_{h \to 0^+} \frac{h^2}{h} = \lim_{h \to 0^+} (h) = 0$$
Since the limits are equal for both sides, the derivative exists and $f'(0) = 0$.

17. 3

18. Answers may vary. For example:



19. $(3, -8)$

20. $2x + y + 1 = 0$ and $6x - y - 9 = 0$

Section 2.2, pp. 82–84

1. Answers may vary. For example:
constant function rule: $\frac{d}{dx}(5) = 0$
power rule: $\frac{d}{dx}(x^3) = 3x^2$
constant multiple rule: $\frac{d}{dx}(4x^3) = 12x^2$
sum rule: $\frac{d}{dx}(x^2 + x) = 2x + 1$
difference rule: $\frac{d}{dx}(x^3 - x^2 + 3x) = 3x^2 - 2x + 3$

2. **a.** 4 **d.** $\frac{1}{3\sqrt[3]{x^2}}$
b. $3x^2 - 2x$ **e.** $\frac{x^3}{4}$
c. $-2x + 5$ **f.** $-3x^{-4}$

3. **a.** $4x + 11$ **d.** $x^4 + x^2 - x$
b. $6x^2 + 10x - 4$ **e.** $40x^7$
c. $4t^3 - 6t^2$ **f.** $2t^3 - \frac{3}{2}$

4. **a.** $5x^{\frac{2}{3}}$
b. $-2x^{-\frac{3}{2}} + 6x^{-2}$
c. $\frac{-18}{x^4} - \frac{4}{x^3}$
d. $-18x^{-3} + \frac{3}{2}x^{-\frac{1}{2}}$
e. $\frac{1}{2}(x^{-\frac{1}{2}}) + 9x^{\frac{1}{2}}$
f. $-x^{-2} - \frac{1}{2}x^{-\frac{3}{2}}$

5. **a.** $-4t + 7$ **b.** $5 - t^2$ **c.** $2t - 6$

6. **a.** 47.75 **b.** $\frac{11}{24}$

7. **a.** 12 **c.** $-\frac{1}{2}$
b. 5 **d.** 12

8. **a.** 9 **c.** 4
b. $\frac{1}{2}$ **d.** -7

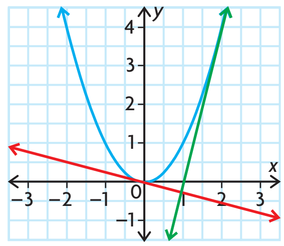
9. **a.** $6x - y - 4 = 0$
b. $18x - y + 25 = 0$
c. $9x - 2y - 9 = 0$
d. $x + y - 3 = 0$
e. $7x - 2y - 28 = 0$
f. $5x - 6y - 11 = 0$

10. A normal to the graph of a function at a point is a line that is perpendicular to the tangent at the given point;
$x + 18y - 125 = 0$

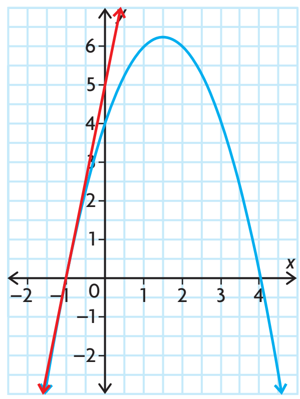
11. 8

12. no

13. $y = x^2, \frac{dy}{dx} = 2x$
The slope of the tangent at $A(2, 4)$ is 4 and at $B\left(-\frac{1}{8}, \frac{1}{64}\right)$ is $-\frac{1}{4}$.
Since the product of the slopes is -1, the tangents at $A(2, 4)$ and $B\left(-\frac{1}{8}, \frac{1}{64}\right)$ will be perpendicular.



14. $(-1, 0)$



15. $(2, 10)$ and $(-2, -6)$

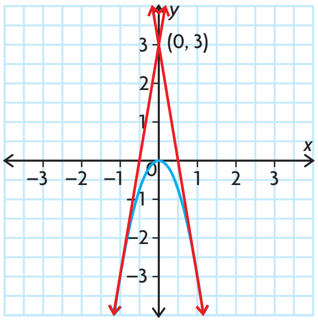
16. $y = \frac{1}{5}x^5 - 10x$, slope is 6
$\frac{dy}{dx} = x^4 - 10 = 6$
$x^4 = 16$
$x^2 = 4$ or $x^2 = -4$
$x = \pm 2$ non-real
Tangents with slope 6 are at the points $\left(2, -\frac{68}{5}\right)$ and $\left(-2, \frac{68}{5}\right)$.

17. **a.** $y - 3 = 0$; $16x - y - 29 = 0$
b. $20x - y - 47 = 0$; $4x + y - 1 = 0$

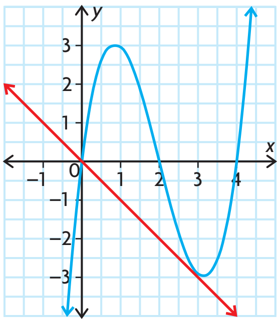
18. 7

19. **a.** 49.9 km
b. 0.12 km/m

20. **a.** 34.3 m/s
b. 39.2 m/s
c. 54.2 m/s
21. 0.29 min and 1.71 min
22. -20 m/s
23. $(1, -3)$ and $(-1, -3)$



24. $B(0, 0)$

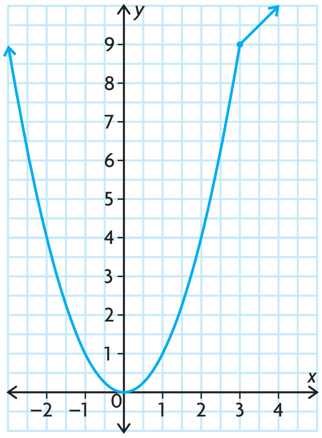


25. **a.** **i.** $\left(\frac{1}{5}, \frac{1}{5}\right)$
ii. $\left(-\frac{1}{4}, -\frac{13}{4}\right)$
iii. $\left(\frac{1}{3}, \frac{103}{27}\right)$ and $(5, -47)$
b. At these points, the slopes of the tangents are zero, meaning that the rate of change of the value of the function with respect to the domain is zero. These points are also local maximum and minimum points.

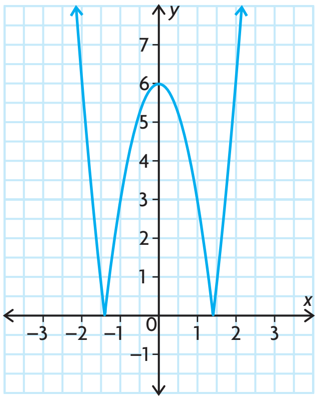
26. $\sqrt{x} + \sqrt{y} = 1$
$P(a, b)$ is on the curve; therefore, $a \geq 0, b \geq 0$.
$\sqrt{y} = 1 - \sqrt{x}$
$y = 1 - 2\sqrt{x} + x$
$\frac{dy}{dx} = -\frac{1}{2}\left(2x^{-\frac{1}{2}} + 1\right)$
At $x = a$. Slope is
$-\frac{1}{\sqrt{a}} + 1 = \frac{-1 + \sqrt{a}}{\sqrt{a}}.$
But, $\sqrt{a} + \sqrt{b} = 1$
$-\sqrt{b} = \sqrt{a} - 1$
Therefore, slope is $-\frac{\sqrt{b}}{\sqrt{a}} = -\sqrt{\frac{b}{a}}.$

27. The x-intercept is $1 - \frac{1}{n}$, as $n \to \infty$, $\frac{1}{n} \to 0$, and the x-intercept approaches 1. As $n \to \infty$, the slope of the tangent at $(1, 1)$ increase without bound, and the tangent approaches a vertical line having equation $x - 1 = 0$.

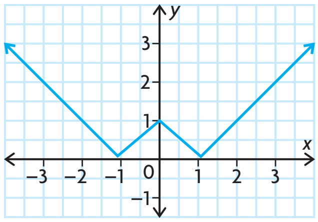
28. **a.** $f'(x) = \begin{cases} 2x, \text{ if } x < 3 \\ 1, \text{ if } x \geq 3 \end{cases}$
$f'(3)$ does not exist.



b.
$$f'(x) = \begin{cases} 6x, \text{ if } x < -\sqrt{2} \text{ or } x > \sqrt{2} \\ -6x, \text{ if } -\sqrt{2} \leq x \leq \sqrt{2} \end{cases}$$
$f'\left(\sqrt{2}\right)$ and $f'\left(-\sqrt{2}\right)$ do not exist.



c. $f'(x) = \begin{cases} 1, \text{ if } x > 1 \\ -1, \text{ if } 0 < x < 1 \\ 1, \text{ if } -1 < x < 0 \\ -1, \text{ if } x < -1 \end{cases}$
$f'(0), f'(-1)$, and $f'(1)$ do not exist.



Section 2.3, pp. 90–91

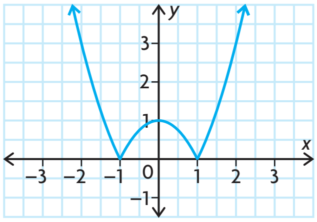
1. **a.** $2x - 4$
b. $6x^2 - 2x$
c. $12x - 17$
d. $45x^8 - 80x^7 + 2x - 2$
e. $-8t^3 + 2t$
f. $\frac{6}{(x + 3)^2}$
2. **a.** $(5x + 1)^3 + 15(5x + 1)^2(x - 4)$
b. $15x^2(3x^2 + 4)(3 + x^3)^4 + 6x(3 + x^3)^5$
c. $-8x(1 - x^2)^3(2x + 6)^3 + 6(1 - x^2)^4(2x + 6)^2$
d. $6(x^2 - 9)^4(2x - 1)^2 + 8x(x^2 - 9)^3(2x - 1)^3$
3. It is not appropriate or necessary to use the product rule when one of the factors is a constant or when it would be easier to first determine the product of the factors and then use other rules to determine the derivative. For example, it would not be best to use the product rule for $f(x) = 3(x^2 + 1)$ or $g(x) = (x + 1)(x - 1)$.
4. $F'(x) = [b(x)][c'(x)] + [b'(x)][c(x)]$
5. **a.** 9
b. -4
c. -9
d. -36
e. 22
f. 671
6. $10x + y - 8 = 0$
7. **a.** $(14, -450)$
b. $(-1, 0)$
8. **a.** $3(x + 1)^2(x + 4)(x - 3)^2 + (x + 1)^3(1)(x - 3)^2 + (x + 1)^3(x + 4)[2(x - 3)]$
b. $2x(3x^2 + 4)^2(3 - x^3)^4 + x^2[2(3x^2 + 4)(6x)](3 - x^3)^4 + x^2(3x^2 + 4)^2 \times [4(3 - x^3)^3(-3x^2)]$
9. -4.84 L/h
10. -30; Determine the point of tangency, and then find the negative reciprocal of the slope of the tangent. Use this information to find the equation of the normal.

11. a. $f'(x) = g_1'(x)g_2(x)g_3(x)\ldots g_{n-1}(x)g_n(x) + g_1(x)g_2'(x)g_3(x)\ldots g_{n-1}(x)g_n(x) + g_1(x)g_2(x)g_3'(x)\ldots g_{n-1}(x)g_n(x) + \ldots + g_1(x)g_2(x)g_3(x)\ldots g_{n-1}(x)g_n'(x)$

b. $\dfrac{n(n+1)}{2}$

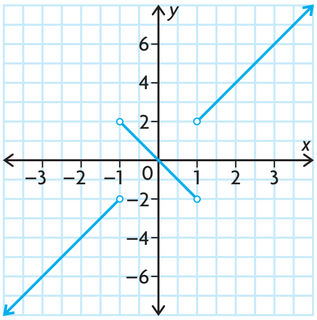
12. $y = 3x^2 + 6x - 5$

13.



a. $x = 1$ or $x = -1$

b. $f'(x) = -2x, -1 < x < 1$



c. $f'(-2) = -4, f'(0) = 0, f'(3) = 6$

14. $y = \dfrac{16}{x^2} - 1$

$\dfrac{dy}{dx} = -\dfrac{32}{x^3}$

Slope of this line is 4.

$-\dfrac{32}{x^3} = 4$

$x = -2$

$y = 3$

Point is at $(-2, 3)$.

Find intersection of line and curve:

$y = 4x + 11$

Substitute,

$4x + 11 = \dfrac{16}{x^2} - 1$

$4x^3 + 11x^2 - 16 - x^2$ or

$4x^3 + 12x^2 - 16 = 0$

Let $x = -2$

RS $= 4(-2) + 12(-2)^2 - 16$
$= 0$

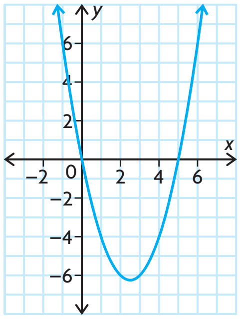
Since $x = -2$ satisfies the equation, therefore it is a solution.

When $x = -2, y = 4(-2) + 11 = 3$.

Intersection point is $(-2, 3)$. Therefore, the line is tangent to the curve.

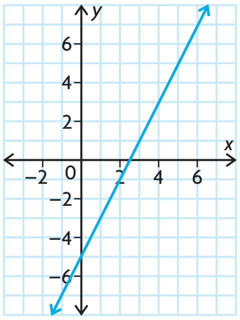
Mid-Chapter Review, pp. 92–93

1. a.



b. $f'(0) = -5, f'(1) = -3, f'(2) = -1, f'(3) = 1, f'(4) = 3, f'(5) = 5$

c.



d. $f(x)$ is quadratic; $f'(x)$ is linear.

2. a. 6

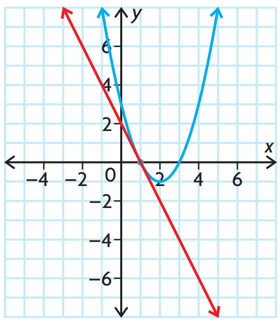
b. $4x$

c. $\dfrac{-5}{(x-5)^2}$

d. $\dfrac{1}{2\sqrt{x-2}}$

3. a. $y = -2x + 2$

b.



4. a. $24x^3$

b. $\dfrac{5}{\sqrt{x}}$

c. $-\dfrac{6}{x^4}$

d. $5 - \dfrac{6}{x^3}$

e. $242t + 22$

f. $\dfrac{1}{x^2}$

5. $y = x - \dfrac{3}{8}$

6. a. $8x - 7$

b. $-6x^2 + 8x + 5$

c. $-\dfrac{10}{x^3} + \dfrac{9}{x^4}$

d. $\dfrac{1}{2x^{\frac{1}{2}}} + \dfrac{1}{3x^{\frac{2}{3}}}$

e. $-\dfrac{14}{x^3} - \dfrac{3}{2x^{\frac{1}{2}}}$

f. $\dfrac{4}{x^2} + 5$

7. a. $y = 7$

b. $y = -\dfrac{1}{3}x$

c. $y = -128x + 297$

8. a. $48x^3 - 81x^2 + 40x - 45$

b. $-36t^2 - 50t + 39$

c. $24x^3 + 24x^2 - 78x - 36$

d. $-162x^2 + 216x^5 - 72x^8$

9. $76x - y - 28 = 0$

10. $(3, 8)$

11. $10x - 8$

12. a. $\dfrac{500}{9}$ L

b. $-\dfrac{200}{27}$ L/min

c. $-\dfrac{200}{27}$ L/min

13. a. $\dfrac{1900}{3}\pi$ cm^3/cm

b. $256\,\pi$ cm^3/cm

14. This statement is always true. A cubic polynomial function will have the form $f(x) = ax^3 + bx^2 + cx + d, a \neq 0$. So, the derivative of this cubic is $f'(x) = 3ax^2 + 2bx + c$ and since $3a \neq 0$, this derivative is a quadratic polynomial function. For example, if $f'(x) = x^3 + x^2 + 1$, we get $f'(x) = 3x^2 + 2x$, and if $f(x) = 2x^3 + 3x^2 + 6x + 2$, we get $f'(x) = 6x^2 + 6x + 6$.

15. $y = \dfrac{x^{2a+3b}}{x^{a-b}}, a, b \in I$

Simplifying,

$y = x^{2a+3b-(a-b)} = x^{a+4b-1}$

Then,

$y'(a + 4b)^{a+4b-1}$

16. a. -188

b. $f'(3)$ is the slope of the tangent line to $f(x)$ at $x = 3$ and the rate of change in the value of $f(x)$ with respect to x at $x = 3$.

17. **a.** 100 bacteria
b. 1200 bacteria
c. 370 bacteria/h

18. $C'(t) = -\frac{100}{t^2}$; The values of the derivative are the rates of change of the percent with respect to time at 5, 50, and 100 min. The percent of carbon dioxide that is released per unit of time from the soft drink is decreasing. The soft drink is getting flat.

Section 2.4, pp. 97–98

1. For x, a, b real numbers,
$x^a x^b = x^{a+b}$
For example,
$x^9 x^{-6} = x^3$
Also,
$(x^a)^b = x^{ab}$
For example,
$(x^2)^3 = x^6$
Also,
$\frac{x^a}{x^b} = x^{a-b}, x \neq 0$
For example,
$\frac{x^5}{x^3} = x^2$

2.

Function	Rewrite	Differentiate and Simplify, if Necessary
$f(x) = \frac{x^2 + 3x}{x}$, $x \neq 0$	$f(x) = x + 3$	$f'(x) = 1$
$g(x) = \frac{3x^{\frac{5}{3}}}{x}$, $x \neq 0$	$g(x) = 3x^{\frac{2}{3}}$	$g'(x) = 2x^{-\frac{1}{3}}$
$h(x) = \frac{1}{10x^5}$, $x \neq 0$	$h(x) = \frac{1}{10}x^{-5}$	$h'(x) = \frac{-1}{2}x^{-6}$
$y = \frac{8x^3 + 6x}{2x}$, $x \neq 0$	$y = 4x^2 + 3$	$\frac{dy}{dx} = 8x$
$s = \frac{t^2 - 9}{t - 3}$, $t \neq 3$	$s = t + 3$	$\frac{ds}{dt} = 1$

3. In the previous problem, all of these rational examples could be differentiated via the power rule after a minor algebraic simplification. A second approach would be to rewrite a rational example
$h(x) = \frac{f(x)}{g(x)}$
using the exponent rules as
$h(x) = f(x)(g(x))^{-1}$, and then apply the product rule for differentiation (together with the power of a function rule) to find $h'(x)$. A third (an perhaps easiest) approach would be to just apply the quotient rule to find $h'(x)$.

4. **a.** $\frac{1}{(x + 1)^2}$
b. $\frac{13}{(t + 5)^2}$
c. $\frac{2x^4 - 3x^2}{(2x^2 - 1)^2}$
d. $\frac{-2x}{(x^2 + 3)^2}$
e. $\frac{5x^2 + 6x + 5}{(1 - x^2)^2}$
f. $\frac{x^2 + 4x - 3}{(x^2 + 3)^2}$

5. **a.** $\frac{13}{4}$ **c.** $\frac{200}{841}$
b. $\frac{7}{25}$ **d.** $-\frac{7}{3}$

6. -9

7. $\left(9, \frac{27}{5}\right)$ and $\left(-1, \frac{3}{5}\right)$

8. Since $(x + 2)^2$ is positive or zero for all $x \in \mathbf{R}$, $\frac{8}{(x + 2)^2} > 0$ for $x \neq -2$. Therefore, tangents to the graph of $f(x) = \frac{5x + 2}{x + 2}$ do not have a negative slope.

9. **a.** (0, 0) and (8, 32)
b. no horizontal tangents

10. 75.4 bacteria per hour at $t = 1$ and 63.1 bacteria per hour at $t = 2$

11. $5x - 12y - 4 = 0$

12. **a.** 20 m
b. $\frac{10}{9}$ m/s

13. **a.** **i.** 1 cm
ii. 1 s
iii. 0.25 cm/s
b. No, the radius will never reach 2 cm because $y = 2$ is a horizontal asymptote of the graph of the function. Therefore, the radius approaches but never equals 2 cm.

14. $a = 1, b = 0$

15. 1.87 h

16. 2.83 s

17. $ad - bc > 0$

Section 2.5, pp. 105–106

1. **a.** 0 **d.** $\sqrt{15}$
b. 0 **e.** $\sqrt{x^2 - 1}$
c. -1 **f.** $x - 1$

2. **a.** $(f \circ g) = x$,
$(g \circ f) = |x|$,
$\{x \geq 0\}$, $\{x \in \mathbf{R}\}$; not equal
b. $(f \circ g) = \frac{1}{(x^2 + 1)}$,
$(g \circ f) = \left(\frac{1}{x^2}\right) + 1$,
$\{x \neq 0\}$, $\{x \in \mathbf{R}\}$; not equal
c. $(f \circ g) = \frac{1}{\sqrt{x + 2}}$,
$(g \circ f) = \sqrt{\frac{1}{x} + 2}$,
$\{x > -2\}$, $\left\{x \leq -\frac{1}{2}f, x > 0\right\}$;
not equal

3. If $f(x)$ and $g(x)$ are two differentiable functions of x, and
$h(x) = (f \circ g)(x)$
$= f(g(x))$
is the composition of these two functions, then $h'(x) = f'(g(x)) \times g'(x)$.
This is known as the "chain rule" for differentiation of composite functions.
For example, if $f(x) = x^{10}$ and $g(x) = x^2 + 3x + 5$, then $h(x) = (x^2 + 3x + 5)^{10}$, and so
$h'(x) = f'(g(x)) \times g'(x)$
$= 10(x^2 + 3x + 5)^9(2x + 3)$
As another example, if $f(x) = x^{\frac{2}{3}}$ and $g(x) = x^2 + 1$, then $h(x) = (x^2 + 1)^{\frac{2}{3}}$, and so $h'(x) = \frac{2}{3}(x^2 + 1)^{-\frac{1}{3}}(2x)$.

4. **a.** $8(2x + 3)^2$
b. $6x(x^2 - 4)^2$
c. $4(2x^2 + 3x - 5)^3(4x + 3)$
d. $-6x(\pi^2 - x^2)^2$
e. $\frac{x}{\sqrt{x^2 - 3}}$
f. $\frac{-10x}{(x^2 - 16)^6}$

5. **a.** $-2x^{-3}$; $\frac{6}{x^4}$
b. $(x + 1)^{-1}$; $\frac{-1}{(x + 1)^2}$
c. $(x^2 - 4)^{-1}$; $\frac{-2x}{(x^2 - 4)^2}$
d. $3(9 - x^2)^{-1}$; $\frac{6x}{(9 - x^2)^2}$
e. $(5x^2 + x)^{-1}$; $-\frac{10x + 1}{(5x^2 + x)^2}$
f. $(x^2 + x + 1)^{-4}$; $-\frac{8x + 4}{(x^2 + x + 1)^5}$

6. $h(-1) = -4$; $h'(-1) = 35$

7. $-\frac{2}{x^2}\left(\frac{1}{x} - 3\right)$

8. a. $(x + 4)^2(x - 3)^5(9x + 15)$
b. $6x(x^2 + 3)^2(x^3 + 3)$
$(2x^3 + 3x + 3)$
c. $\dfrac{-2x^2 + 6x + 2}{(x^2 + 1)^2}$
d. $15x^2(3x - 5)(x - 1)$
e. $4x^3(1 - 4x^2)^2(1 - 10x^2)$
f. $\dfrac{48x(x^2 - 3)^3}{(x^2 + 3)^5}$

9. a. $\dfrac{91}{36}$ b. $-\dfrac{5\sqrt[3]{2}}{24\pi}$

10. $x = 0$ or $x = 1$

11. $\dfrac{1}{4}$

12. $60x - y - 119 = 0$

13. a. 52 b. 54 c. 878 d. 78

14. -6

15. 2222 L/min

16. 2.75 m/s

17. a. $p'(x)q(x)r(x) + p(x)q'(x)r(x)$
$+ p(x)q(x)r'(x)$
b. -344

18. $\dfrac{dy}{dx} = 3(x^2 + x - 2)^2(2x + 1)$
At the point (1, 3), slope of the tangent will be $3(1 + 1 - 2)^2(2 + 1) = 0$.
Equation of the tangent at (1, 3) is
$y = 3$
$(x^2 + x - 2)^3 + 3 = 3$
$(x + 2)^3(x - 1)^3 = 0$
$x = -2$ or $x = 1$
Since -2 and 1 are both triple roots, the line with equation $y = 3$ is also a tangent at $(-2, 3)$.

19. $-\dfrac{2x(x^2 + 3x - 1)(1 - x)^2}{(1 + x)^4}$

Review Exercise, pp. 110–113

1. To find the derivative $f'(x)$, the limit $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$ must be computed, provided it exists. If this limit does not exist, then the derivative of $f(x)$ does not exist at this particular value of x. As an alternative to this limit, we could also find $f'(x)$ from the definition by computing the equivalent limit $f'(x) = \lim\limits_{z \to x} \dfrac{f(z) - f(x)}{z - x}$. These two limits are seen to be equivalent by substituting $z = x + h$.

2. a. $4x - 5$ c. $\dfrac{4}{(4 - x)^2}$
b. $\dfrac{1}{2\sqrt{x - 6}}$

3. a. $2x - 5$
b. $\dfrac{3}{4x^{\frac{1}{4}}}$
c. $-\dfrac{28}{3x^5}$
d. $-\dfrac{2x}{(x^2 + 5)^2}$
e. $\dfrac{12x}{(3 - x^2)^3}$
f. $\dfrac{7x + 2}{\sqrt{7x^2 + 4x + 1}}$

4. a. $2 + \dfrac{2}{x^3}$
b. $\dfrac{\sqrt{x}}{2}(7x^2 - 3)$
c. $-\dfrac{5}{(3x - 5)^2}$
d. $\dfrac{3x - 1}{2\sqrt{x - 1}}$
e. $-\dfrac{1}{3\sqrt{x}(\sqrt{x} + 2)^{\frac{5}{3}}}$
f. 1

5. a. $20x^3(2x - 5)^5(x - 1)$
b. $\dfrac{x^2}{\sqrt{x^2 + 1}} + \sqrt{x^2 + 1}$
c. $\dfrac{(2x - 5)^3(2x + 23)}{(x + 1)^4}$
d. $\dfrac{318(10x - 1)^5}{(3x + 5)^7}$
e. $(x - 2)^2(x^2 + 9)^3$
$\times (11x^2 - 16x + 27)$
f. $-\dfrac{3(1 - x^2)^2(x^2 + 6x + 1)}{8(3 - x)^4}$

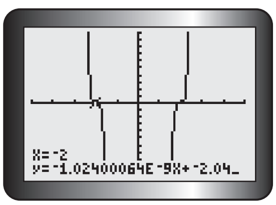
6. a. $f(x^2) \times 2x$
b. $2xf'(x) + 2f(x)$

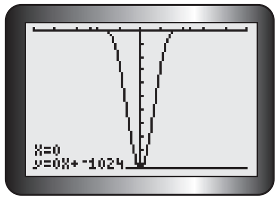
7. a. $-\dfrac{184}{9}$ b. $\dfrac{25}{289}$ c. $-\dfrac{8}{5}$

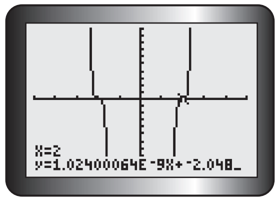
8. $-\dfrac{2}{3}$

9. $x = 2 \pm 2\sqrt{2}$;
$x = 5, x = -1$

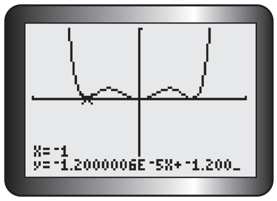
10. a. i. $x = 0, x = \pm 2$
ii. $x = 0, x = \pm 1, x = \pm\dfrac{\sqrt{3}}{3}$
b. i.

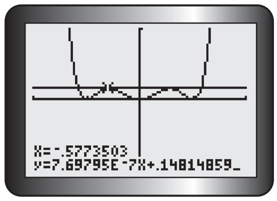


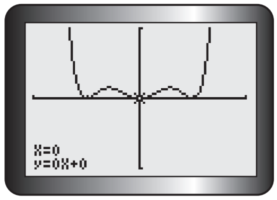


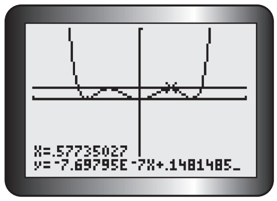


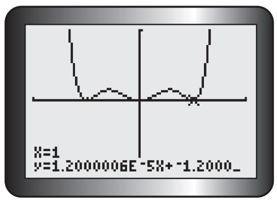
ii.









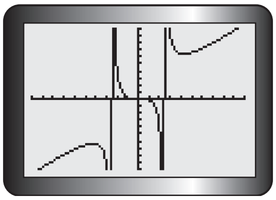


11. **a.** $160x - y + 16 = 0$
b. $60x + y - 61 = 0$

12. $5x - y - 7 = 0$

13. $(2, 8)$; $b = -8$

14. **a.**



b. $y = 0$, $y = 6.36$, $y = -6.36$
c. $(0, 0)$, $\left(3\sqrt{2}, \dfrac{9\sqrt{2}}{2}\right)$, $\left(-3\sqrt{2}, -\dfrac{9\sqrt{2}}{2}\right)$
d. -14

15. **a.** $\sqrt[3]{50}$
b. 1

16. **a.** When $t = 10$, 9; when $t = 15$, 19
b. At $t = 10$, the number of words memorized is increasing by 1.7 words/min. At $t = 15$, the number of words memorized is increasing by 2.325 words/min.

17. **a.** $\dfrac{30t}{(9 + t^2)^{\frac{1}{2}}}$
b. No; since $t > 0$, the derivative is always positive, meaning that the rate of change in the cashier's productivity is always increasing. However, these increases must be small, since, according to the model, the cashier's productivity can never exceed 20.

18. **a.** $x^2 + 40$
b. 6 gloves/week

19. **a.** $750 - \dfrac{x}{3} - 2x^2$
b. \$546.67

20. $-\dfrac{5}{4}$

21. **a.** $B(0) = 500$, $B(30) = 320$
b. $B'(0) = 0$, $B'(30) = -12$
c. $B(0) =$ blood sugar level with no insulin
$B(30) =$ blood sugar level with 30 mg of insulin
$B'(0) =$ rate of change in blood sugar level with no insulin
$B'(30) =$ rate of change in blood sugar level with 30 mg of insulin
d. $B'(50) = -20$, $B(50) = 0$
$B'(50) = -20$ means that the patient's blood sugar level is decreasing at 20 units/mg of insulin 1 h after 50 mg of insulin is injected.
$B(50) = 0$ means that the patient's blood sugar level is *zero* 1 h after 50 mg of insulin is injected. These values are not logical because a person's blood sugar level can never reach zero and continue to decrease.

22. **a.** $f(x)$ is not differentiable at $x = 1$ because it is not defined there (vertical asymptote at $x = 1$).
b. $g(x)$ is not differentiable at $x = 1$ because it is not defined there (hole at $x = 1$).
c. The graph has a cusp at $(2, 0)$ but is differentiable at $x = 1$.
d. The graph has a corner at $x = 1$, so $m(x)$ is not differentiable at $x = 1$.

23. **a.** $f(x)$ is not defined at $x = 0$ and $x = 0.25$. The graph has vertical asymptotes at $x = 0$ and $x = 0.25$. Therefore, $f(x)$ is not differentiable at $x = 0$ and $x = 0.25$.
b. $f(x)$ is not defined at $x = 3$ and $x = -3$. At $x = -3$, the graph has a vertical asymptote and at $x = 3$ it has a hole. Therefore, $f(x)$ is not differentiable at $x = 3$ and $x = -3$.
c. $f(x)$ is not defined for $1 < x < 6$. Therefore, $f(x)$ is not differentiable for $1 < x < 6$.

24. $\dfrac{25}{(t + 1)^2}$

25. Answers may vary. For example:

$$f(x) = 2x + 3$$
$$y = \frac{1}{2x + 3}$$
$$y' = \frac{(2x + 3)(0) - (1)(2)}{(2x + 3)^2}$$
$$= -\frac{2}{(2x + 3)^2}$$
$$f(x) = 5x + 10$$
$$y = \frac{1}{5x + 10}$$
$$y' = \frac{(5x + 10)(0) - (1)(5)}{(5x + 10)^2}$$
$$= -\frac{5}{(5x + 10)^2}$$

Rule: If $f(x) = ax + b$ and $y = \frac{1}{f(x)}$, then

$$y' = \frac{-a}{(ax + b)^2}$$
$$y' = \lim_{h \to 0} \frac{1}{h}\left[\frac{1}{a(x + h) + b} - \frac{1}{ax + b}\right]$$
$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{ax + b - [a(x + h) + b]}{[a(x + h) + b](ax + h)}\right]$$
$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{-ah}{[a(x + h) + b](ax + b)}\right]$$
$$= \lim_{h \to 0} \left[\frac{-a}{[a(x + h) + b](ax + b)}\right]$$
$$= \frac{-a}{(ax + b)^2}$$

26. **a.** $y = u + 5u^{-1}$
b. $2(1 - 5(2x - 3)^{-2})$

27. **a.** $y = \sqrt{u} + 5u$
b. $(2x - 3)^{-\frac{1}{2}} + 10$

28. **a.** $6(2x - 5)^2(3x^2 + 4)^4 \times (13x^2 - 25x + 4)$
b. $8x^2(4x^2 + 2x - 3)^4 (52x^2 + 16x - 9)$
c. $2(5 + x)(4 - 7x^3)^5 \times (4 - 315x^2 - 70x^3)$
d. $\dfrac{6(-9x + 7)}{(3x + 5)^5}$
e. $\dfrac{2(2x^2 - 5)^2(4x^2 + 48x + 5)}{(x + 8)^3}$
f. $\dfrac{-3x^3(7x - 16)}{(4x - 8)^{\frac{3}{2}}}$
g. $8\left(\dfrac{2x + 5}{6 - x^2}\right)^3\left(\dfrac{(x + 2)(x + 3)}{(6 - x^2)^2}\right)$
h. $-9(4x + x^2)^{-10}(4 + 2x)$

29. $a = -4$, $b = 32$, $c = 0$

30. **a.** $-3t^2 + 5$
b. -7000 ants/h
c. 75 000 ants
d. 9.27 h

Chapter 2 Test, p. 114

1. You need to use the chain rule when the derivative for a given function cannot be found using the sum, difference, product, or quotient rules or when writing the function in a form that would allow the use of these rules is tedious. The chain rule is used when a given function is a composition of two or more functions.

2. f is the blue graph (it's cubic). f' is the red graph (it is quadratic). The derivative of a polynomial function has degree one less than the derivative of the function. Since the red graph is a quadratic (degree 2) and the blue graph is cubic (degree 3), the blue graph is f and the red graph is f'.

3. $1 - 2x$

4. a. $x^2 + 15x^{-6}$
   b. $60(2x - 9)^4$
   c. $-x^{-\frac{3}{2}} + \frac{1}{\sqrt{3}} + 2x^{-\frac{2}{3}}$
   d. $\frac{5(x^2 + 6)^4(3x^2 + 8x - 18)}{(3x + 4)^6}$
   e. $2x(6x^2 - 7)^{-\frac{2}{3}}(8x^2 - 7)$
   f. $\frac{4x^5 - 18x + 8}{x^5}$

5. 14

6. $-\frac{40}{3}$

7. $60x + y - 61 = 0$

8. $\frac{75}{32}$ ppm/year

9. $\left(-\frac{1}{4}, \frac{1}{256}\right)$

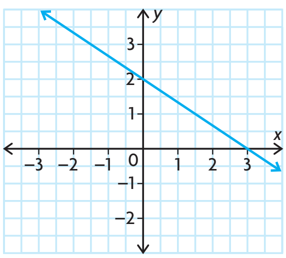
10. $\left(-\frac{1}{3}, \frac{32}{27}\right)$, $(1, 0)$

11. $a = 1, b = -1$

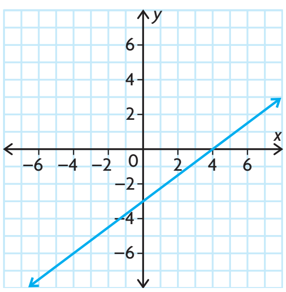
Chapter 3

Review of Prerequisite Skills, pp. 116–117

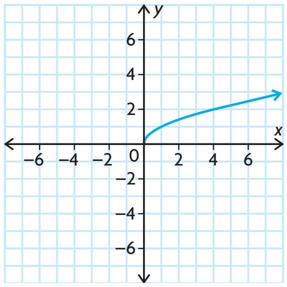
1. a.



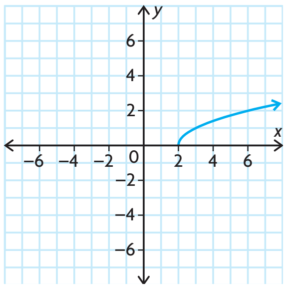
   b.



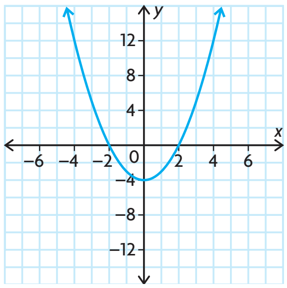
   c.



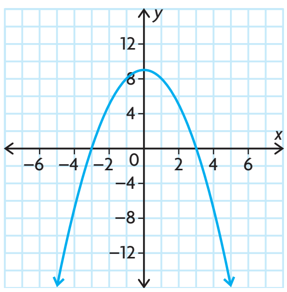
   d.



   e.



   f.



2. a. $x = \frac{14}{5}$
   b. $x = -13$
   c. $t = 3$ or $t = 1$
   d. $t = -\frac{1}{2}$ or $t = 3$
   e. $t = 3$ or $t = 6$
   f. $x = 0$ or $x = -3$ or $x = 1$
   g. $x = 0$ or $x = 4$
   h. $t = -3$ or $t = \frac{1}{2}$ or $t = -\frac{1}{2}$
   i. $t = \pm\frac{9}{4}$ or $t = \pm 1$

3. a. $x > 3$
   b. $x < 0$ or $x > 3$
   c. $0 < x < 4$

4. a. 25 cm^2
   b. 48 cm^2
   c. $49\pi \text{ cm}^2$
   d. $36\pi \text{ cm}^2$

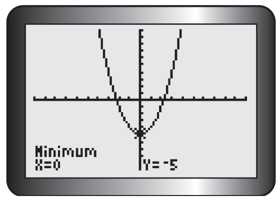
5. a. $SA = 56\pi \text{ cm}^2$, $V = 48\pi \text{ cm}^3$
   b. $h = 6$ cm, $SA = 80\pi \text{ cm}^2$
   c. $r = 6$ cm, $SA = 144\pi \text{ cm}^2$
   d. $h = 7$ cm, $V = 175\pi \text{ cm}^3$

6. a. $SA = 54 \text{ cm}^2$, $V = 27 \text{ cm}^3$
   b. $SA = 30 \text{ cm}^2$, $V = 5\sqrt{5} \text{ cm}^3$
   c. $SA = 72 \text{ cm}^2$, $V = 24\sqrt{3} \text{ cm}^3$
   d. $SA = 24k^2 \text{ cm}^2$, $V = 8k^3 \text{ cm}^3$

7. a. $(3, \infty)$
   b. $(-\infty, -2]$
   c. $(-\infty, 0)$
   d. $[-5, \infty)$
   e. $(-2, 8]$
   f. $(-4, 4)$

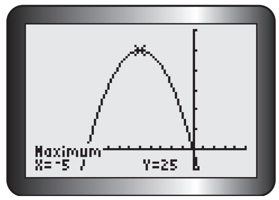
8. a. $\{x \in \mathbf{R} \mid x > 5\}$
   b. $\{x \in \mathbf{R} \mid x \le -1\}$
   c. $\{x \in \mathbf{R}\}$
   d. $\{x \in \mathbf{R} \mid -10 \le x \le 12\}$
   e. $\{x \in \mathbf{R} \mid -1 < x < 3\}$
   f. $\{x \in \mathbf{R} \mid 2 \le x < 20\}$

9. a.



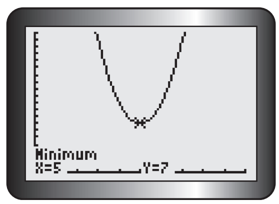
The function has a minimum value of -5 and no maximum value.

   b.



The function has a maximum value of 25 and no minimum value.

   c.



The function has a minimum value of 7 and no maximum value.

3. $1 - 2x$

4. a. $x^2 + 15x^{-6}$
 b. $60(2x - 9)^4$
 c. $-x^{-\frac{3}{2}} + \dfrac{1}{\sqrt{3}} + 2x^{-\frac{2}{3}}$
 d. $\dfrac{5(x^2 + 6)^4(3x^2 + 8x - 18)}{(3x + 4)^6}$
 e. $2x(6x^2 - 7)^{-\frac{2}{3}}(8x^2 - 7)$
 f. $\dfrac{4x^5 - 18x + 8}{x^5}$

5. 14

6. $-\dfrac{40}{3}$

7. $60x + y - 61 = 0$

8. $\dfrac{75}{32}$ ppm/year

9. $\left(-\dfrac{1}{4}, \dfrac{1}{256}\right)$

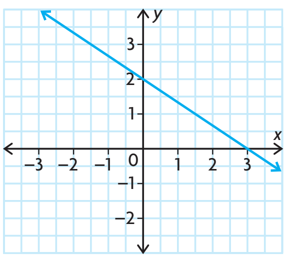
10. $\left(-\dfrac{1}{3}, \dfrac{32}{27}\right)$, $(1, 0)$

11. $a = 1, b = -1$

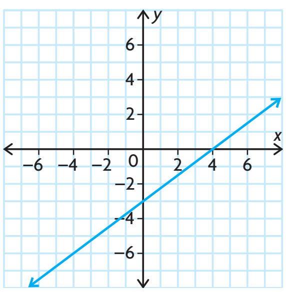
Chapter 3

Review of Prerequisite Skills, pp. 116–117

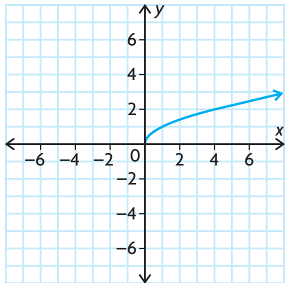
1. a.



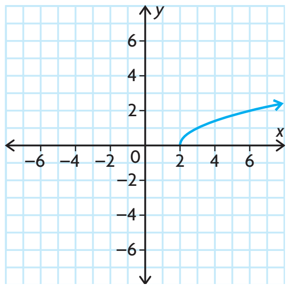
 b.



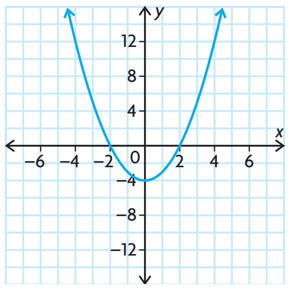
 c.



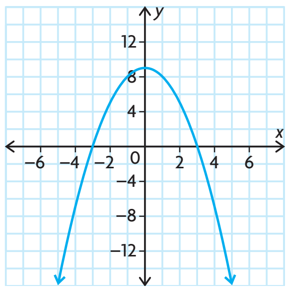
 d.



 e.



 f.



2. a. $x = \dfrac{14}{5}$
 b. $x = -13$
 c. $t = 3$ or $t = 1$
 d. $t = -\dfrac{1}{2}$ or $t = 3$
 e. $t = 3$ or $t = 6$
 f. $x = 0$ or $x = -3$ or $x = 1$
 g. $x = 0$ or $x = 4$
 h. $t = -3$ or $t = \dfrac{1}{2}$ or $t = -\dfrac{1}{2}$
 i. $t = \pm\dfrac{9}{4}$ or $t = \pm 1$

3. a. $x > 3$
 b. $x < 0$ or $x > 3$
 c. $0 < x < 4$

4. a. 25 cm^2
 b. 48 cm^2
 c. $49\pi\text{ cm}^2$
 d. $36\pi\text{ cm}^2$

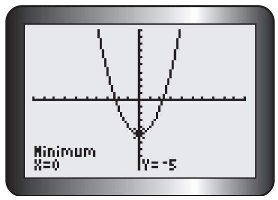
5. a. $SA = 56\pi\text{ cm}^2$, $V = 48\pi\text{ cm}^3$
 b. $h = 6$ cm, $SA = 80\pi\text{ cm}^2$
 c. $r = 6$ cm, $SA = 144\pi\text{ cm}^2$
 d. $h = 7$ cm, $V = 175\pi\text{ cm}^3$

6. a. $SA = 54\text{ cm}^2$, $V = 27\text{ cm}^3$
 b. $SA = 30\text{ cm}^2$, $V = 5\sqrt{5}\text{ cm}^3$
 c. $SA = 72\text{ cm}^2$, $V = 24\sqrt{3}\text{ cm}^3$
 d. $SA = 24k^2\text{ cm}^2$, $V = 8k^3\text{ cm}^3$

7. a. $(3, \infty)$
 b. $(-\infty, -2]$
 c. $(-\infty, 0)$
 d. $[-5, \infty)$
 e. $(-2, 8]$
 f. $(-4, 4)$

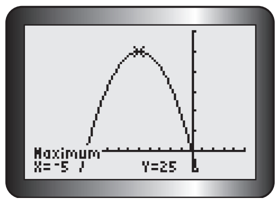
8. a. $\{x \in \mathbf{R} \mid x > 5\}$
 b. $\{x \in \mathbf{R} \mid x \leq -1\}$
 c. $\{x \in \mathbf{R}\}$
 d. $\{x \in \mathbf{R} \mid -10 \leq x \leq 12\}$
 e. $\{x \in \mathbf{R} \mid -1 < x < 3\}$
 f. $\{x \in \mathbf{R} \mid 2 \leq x < 20\}$

9. a.



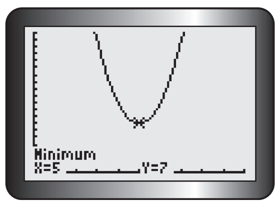
The function has a minimum value of -5 and no maximum value.

 b.



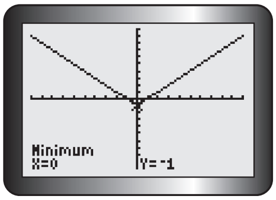
The function has a maximum value of 25 and no minimum value.

 c.



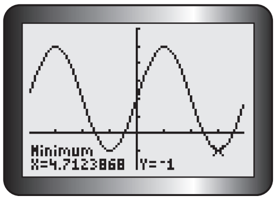
The function has a minimum value of 7 and no maximum value.

d.

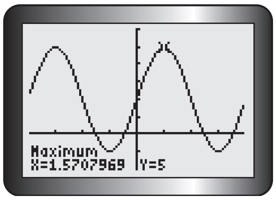


The function has a minimum value of -1 and no maximum value.

e.

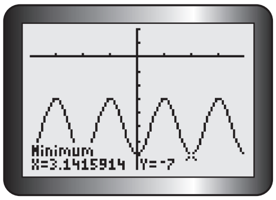


The function has a minimum value of -1.

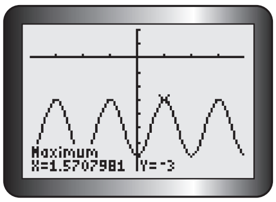


The function has a maximum value of 5.

f.



The function has a minimum value of -7.



The function has a maximum value of -3.

Section 3.1, pp. 127–129

1. At $t = 1$, the velocity is positive; this means that the object is moving in whatever is the positive direction for the scenario. At $t = 5$, the velocity is negative; this means that the object is moving in whatever is the negative direction for the scenario.

2. **a.** $y'' = 90x^8 + 90x^4$

b. $f''(x) = -\dfrac{1}{4}x^{-\frac{3}{2}}$

c. $y'' = 2$

d. $h''(x) = 36x^2 - 24x - 6$

e. $y'' = \dfrac{3}{\sqrt{x}} - \dfrac{6}{x^4}$

f. $f''(x) = \dfrac{-4x - 4}{(x + 1)^4}$

g. $y'' = 2 + \dfrac{6}{x^4}$

h. $g''(x) = -\dfrac{9}{4(3x - 6)^{\frac{3}{2}}}$

i. $y'' = 48x + 96$

j. $h''(x) = \dfrac{10}{9x^{\frac{1}{3}}}$

3. **a.** $v(t) = 10t - 3,$
$a(t) = 10$

b. $v(t) = 6t^2 + 36,$
$a(t) = 12t$

c. $v(t) = 1 - 6t^{-2},$
$a(t) = 12t^{-3}$

d. $v(t) = 2(t - 3),$
$a(t) = 2$

e. $v(t) = \dfrac{1}{2}(t + 1)^{\frac{1}{2}},$

$a(t) = -\dfrac{1}{4}(t + 1)^{\frac{3}{2}}$

f. $v(t) = \dfrac{27}{(t + 3)^2},$

$a(t) = -54(t + 3)^{-3}$

4. **a.** **i.** $t = 3$
ii. $1 < t < 3$
iii. $3 < t < 5$

b. **i.** $t = 3, t = 7$
ii. $1 < t < 3, 7 < t < 9$
iii. $3 < t < 7$

5. **a.** $v(t) = t^2 - 4t + 3,$
$a(t) = 2t - 4$

b. at $t = 1$ and $t = 3$

c. after 3 s

6. **a.** For $t = 1$, moving in a positive direction.
For $t = 4$, moving in a negative direction.

b. For $t = 1$, the object is stationary.
For $t = 4$, the object is moving in a positive direction.

c. For $t = 1$, the object is moving in a negative direction.
For $t = 4$, the object is moving in a positive direction.

7. **a.** $v(t) = 2t - 6$

b. $t = 3$ s

8. **a.** $t = 4$ s

b. $s(4) = 80$ m

9. **a.** $v(5) = 3$ m/s

b. $a(5) = 2$ m/s^2

10. **a.** $v(t) = \dfrac{35}{2}t^{\frac{3}{2}} - \dfrac{7}{2}t^{\frac{5}{2}},$

$a(t) = \dfrac{105}{2}t^{\frac{1}{2}} - \dfrac{35}{4}t^{\frac{3}{2}}$

b. 5 s

c. 5 s

d. $0 < t < 6$ s

e. after 7 s

11. **a.** 25 m/s

b. 31.25 m

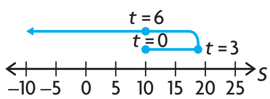
c. -25 m/s

12. **a.** $v(8) = 98$ m/s,
$a(8) = 12$ m/s^2

b. 38 m/s

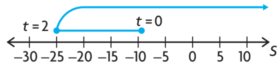
13. **a.** $s = 10 + 6t - t^2$
$v = 6 - 2t$
$= 2(3 - t)$
$a = -2$

The object moves to the right from its initial position of 10 m from the origin, 0, to the 19 m mark, slowing down at a rate of 2 m/s^2. It stops at the 19 m mark, then moves to the left, accelerating at 2 m/s^2 as it goes on its journey into the universe. It passes the origin after $(3 + \sqrt{19})$ s.



b. $s = t^3 - 12t - 9$
$v = 3t^2 - 12$
$= 3(t^2 - 4)$
$= 3(t - 2)(t + 2)$
$a = 6t$

The object begins at 9 m to the left of the origin, 0, and slows down to a stop after 2 s when it is 25 m to the left of the origin. Then, the object moves to the right, accelerating at faster rates as time increases. It passes the origin just before 4 s (approximately 3.7915) and continues to accelerate as time goes by on its journey into space.



14. $t = 1$ s; away

15. **a.** $s(t) = kt^2 + (6k^2 - 10k)t + 2k$
$v(t) = 2kt + (6k^2 - 10k)$
$a(t) = 2k + 0$
$= 2k$
Since $k \neq 0$ and $k \in \mathbf{R}$, then $a(t) = 2k \neq 0$ and an element of the real numbers. Therefore, the acceleration is constant.
b. $t = 5 - 3k, -9k^3 + 30k^2 - 23k.$

16. **a.** The acceleration is continuous at $t = 0$ if $\lim\limits_{t\to 0} a(t) = a(0)$.
For $t \geq 0$,
$$s(t) = \frac{t^3}{t^2 + 1}$$
and
$$v(t) = \frac{3t^2(t^2 + 1) - 2t(t^3)}{(t^2 + 1)^2}$$
$$= \frac{t^4 + 3t^2}{(t^2 + 1)^2}$$
and
$$a(t) = \frac{(4t^3 + 6t)(t^2 + 1)^2}{(t^2 + 1)^2} - \frac{2(t^2 + 1)(2t)(t^4 + 3t^2)}{(t^2 + 1)^2}$$
$$= \frac{(4t^3 + 6t)(t^2 + 1)}{(t^2 + 1)^3} - \frac{4t(t^4 + 3t^2)}{(t^2 + 1)^3}$$
$$= \frac{4t^5 + 6t^3 + 4t^3}{(t^2 + 1)^3} + \frac{6t - 4t^5 - 12t^3}{(t^2 + 1)^3}$$
$$= \frac{-2t^3 + 6t}{(t^2 + 1)^3}$$
Therefore,
$$a(t) = \begin{cases} 0, \text{ if } t < 0 \\ \dfrac{-2t^3 + 6t}{(t^2 + 1)^3}, \text{ if } t \geq 0 \end{cases}$$
and
$$v(t) = \begin{cases} 0, \text{ if } t < 0 \\ \dfrac{t^4 + 3t^2}{(t^2 + 1)^2}, \text{ if } t \geq 0 \end{cases}$$
$$\lim_{t\to 0^-} a(t) = 0, \quad \lim_{t\to 0^+} a(t) = \frac{0}{1}$$
$$= 0$$
Thus, $\lim\limits_{t\to 0} a(t) = 0.$
Also, $a(0) = \dfrac{0}{1}$
$= 0$
Therefore, $\lim\limits_{t\to 0} a(t) = a(0).$
Thus, the acceleration is continuous at $t = 0$.
b. velocity approaches 1, acceleration approaches 0

17. $$v = \sqrt{b^2 + 2gs}$$
$$v = (b^2 + 2gs)^{\frac{1}{2}}$$
$$\frac{dv}{dt} = \frac{1}{2}(b^2 + 2gs)^{\frac{1}{2}} \times \left(0 + 2g\frac{ds}{dt}\right)$$
$$a = \frac{1}{2v} \times 2gv$$
$$a = g$$
Since g is a constant, a is a constant, as required.
Note: $\dfrac{ds}{dt} = v$
$\dfrac{dv}{dt} = a$

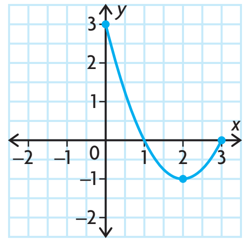
18. $$F = m_0 \frac{d}{dt}\left(\frac{v}{\sqrt{1 - (\frac{v}{c})^2}}\right)$$
Using the quotient rule,
$$= \frac{m_0 \frac{dv}{dt}\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}{1 - \frac{v^2}{c^2}} - \frac{\frac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\left(-\frac{2v\frac{dv}{dt}}{c^2}\right) \times v}{1 - \frac{v^2}{c^2}}$$
Since $\dfrac{dv}{dt} = a$,
$$= \frac{m_0\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\left[a\left(1 - \frac{v^2}{c^2}\right) + \frac{v^2 a}{c^2}\right]}{1 - \frac{v^2}{c^2}}$$
$$= \frac{m_0\left[\frac{ac^2 - av^2}{c^2} + \frac{v^2 a}{c^2}\right]}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}$$
$$= \frac{m_0 ac^2}{c^2(1 - \frac{v^2}{c^2})^{\frac{3}{2}}}$$
$$= \frac{m_0 a}{(1 - \frac{v^2}{c^2})^{\frac{3}{2}}}, \text{ as required.}$$

Section 3.2, pp. 135–138

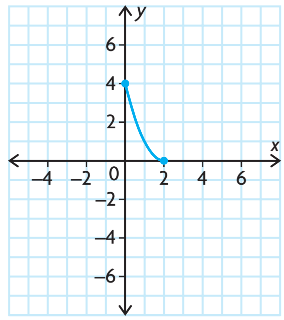
1. **a.** The algorithm can be used; the function is continuous.
b. The algorithm cannot be used; the function is discontinuous at $x = 2$.
c. The algorithm cannot be used; the function is discontinuous at $x = 2$.
d. The algorithm can be used; the function is continuous on the given domain.

2. **a.** max: 8, min: -12
b. max: 30, min: -5
c. max: 100, min: -100
d. max: 30, min: -20

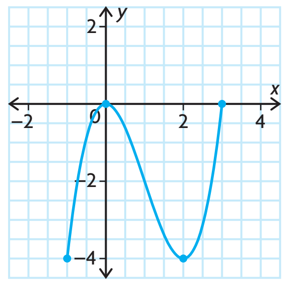
3. **a.** max is 3 at $x = 0$, min is -1 at $x = 2$



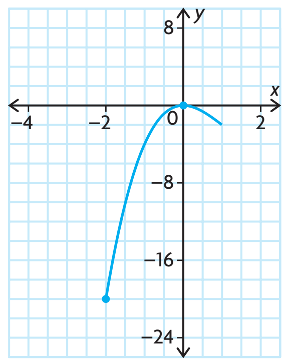
b. max is 4 at $x = 0$, min is 0 at $x = 2$



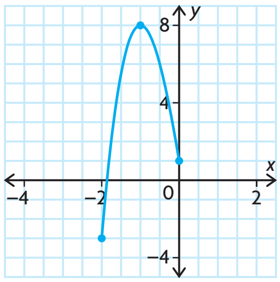
c. min is -4 at $x = -1, 2$, max is 0 at $x = 0, 3$



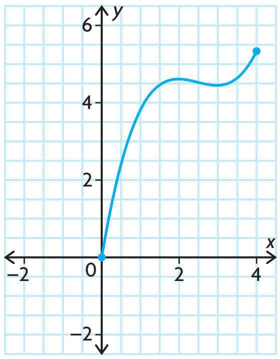
d. max is 0 at $x = 0$, min is -20 at $x = -2$



e. max is 8 at $x = -1$,
min is -3 at $x = -2$



f. max is $\frac{16}{3}$ at $x = 4$,
min is 0 at $x = 0$



4. **a.** min value of 4 when $x = 2$,
max value of 10.4 when $x = 10$
b. min value of 3 when $x = 9$,
max value of 4 when $x = 4$
c. max value of 1 when $x = 1$,
min value of $\frac{1}{2}$ when $x = 0, 2$
d. min value of -169 when $x = 3$,
max value of 47 when $x = -3$
e. max value of 2 when $x = 1$,
min value of -2 when $x = -1$
f. min value of 0.94 when $x = 4$,
max value of 1.6 when $x = 2$

5. **a.** max velocity is $\frac{4}{3}$ m/s,
min velocity is $\frac{4}{5}$ m/s
b. min velocity is 0 m/s, no maximum velocity, but $v(t) \to 4$ as $t \to \infty$

6. 20 bacteria/cm^3

7. **a.** 80 km/h
b. 50 km/h
c. $0 \le v < 80$
d. $80 < v \le 100$

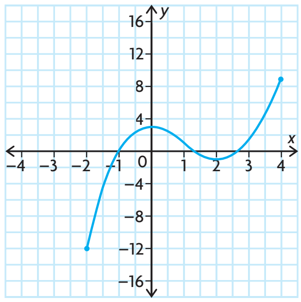
8. min concentration is at $t = 1$
max concentration is at $t = 3$

9. 0.05 years or approximately 18 days

10. 70 km/h; \$31.50

11. absolute max value = 42,
absolute min value = 10

12. **a.**



b. $-2 \le x \le 4$
c. increasing: $-2 \le x < 0$
$2 < x \le 4$
decreasing: $0 < x < 2$

13. Absolute max: Compare all local maxima and values of $f(a)$ and $f(b)$ when the domain of $f(x)$ is $a \le x \le b$. The one with the highest value is the absolute maximum.
Absolute min: We need to consider all local minima and the value of $f(a)$ and $f(b)$ when the domain of $f(x)$ is $a \le x \le b$. Compare them, and the one with the lowest value is the absolute minimum.
You need to check the endpoints because they are not necessarily critical points.

14. 245 units

15. 300 units

Mid-Chapter Review, pp. 139–140

1. **a.** $h''(x) = 36x^2 - 24x - 6$
b. $f''(x) = 48x - 120$
c. $y'' = \dfrac{30}{(x+3)^3}$
d. $g''(x) = -\dfrac{x^2}{(x^2+1)^{\frac{3}{2}}} + \dfrac{1}{(x^2+1)^{\frac{1}{2}}}$

2. **a.** 108 m
b. -45 m/s
c. -18 m/s^2

3. **a.** 6 m/s
b. $t \doteq 0.61$ s
c. $t = 1.50$ s
d. -8.67 m/s
e. -9.8 m/s^2, -9.8 m/s^2

4. **a.** Velocity is 0 m/s
Acceleration is 10 m/s
b. Object is stationary at time $t = \frac{1}{3}$ s and $t = 2$ s.
Before $t = \frac{1}{3}$, $v(t)$ is positive and therefore the object is moving to the right.
Between $t = \frac{1}{3}$ and $t = 2$, $v(t)$ is negative and therefore the object is moving to the left.
After $t = 2$, $v(t)$ is positive and therefore the object is moving to the right.
c. $t \doteq 1.2$ s; At that time, the object is neither accelerating nor decelerating.

5. **a.** min value is 1 when $x = 0$,
max value is 21 when $x = 2$
b. min value is 0 when $x = -2$,
max value is 25 when $x = 3$
c. min value is 0 when $x = 1$,
max value is 0.38 when $x = \sqrt{3}$

6. 3.96 °C

7. **a.** 105
b. 3
c. -6
d. -78
e. 3
f. 1448
g. $-\dfrac{202}{27}$
h. $-\dfrac{185}{6}$

8. -1.7 m/s^2

9. **a.** 189 m/s
b. 27 s
c. 2916 m
d. 6.2 m/s^2

10. 16 m; 4 s

11. **a.** $0 \le t \le 4.31$
b. 2.14 s
c. 22.95 m

Section 3.3, pp. 145–147

1. 25 cm by 25 cm

2. If the perimeter is fixed, then the figure will be a square.

3. 150 m by 300 m

4. height 8.8 cm, length 8.24 cm, and width 22.4 cm

5. 110 cm by 110 cm

6. 8 m by 8 m

7. 125 m by 166.67 m

8. 4 m by 6 m by 6 m

9. base 10 cm by 10 cm, height 10 cm

10. 100 square units when $5\sqrt{2}$

11. **a.** $r = 5.42$, $h = 10.84$
b. $\frac{h}{d} = \frac{1}{1}$; yes

12. **a.** 15 cm^2 when $W = 2.5$ cm and $L = 6$ cm
b. 30 cm^2 when $W = 4$ cm and $L = 7.5$ cm
c. The largest area occurs when the length and width are each equal to one-half of the sides adjacent to the right angle.

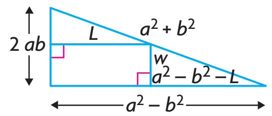
13. **a.** base is 20 cm and each side is 20 cm
b. approximately 260 000 cm^3

14. **a.** triangle side length 0.96 cm, rectangle 0.96 cm by 1.09 cm
b. Yes. All the wood would be used for the outer frame.

15. 0.36 h after the first train left the station

16. 1:02 p.m.; 3 km

17.



$$\frac{a^2 - b^2 - L}{a^2 - b^2} = \frac{W}{2ab}$$

$$W = \frac{2ab}{a^2 - b^2}(a^2 - b^2 - L)$$

$$A = LW = \frac{2ab}{a^2 - b^2}[a^2L - b^2L - L^2]$$

Let $\frac{dA}{dL} = a^2 - b^2 - 2L = 0,$

$L = \frac{a^2 - b^2}{2}$ and

$$W = \frac{2ab}{a^2 - b^2}\left[a^2 - b^2 - \frac{a^2 - b^2}{2}\right]$$
$$= ab$$

The hypothesis is proven.

18. Let the height be h and the radius r.

Then, $\pi r^2 h = k$, $h = \frac{k}{\pi r^2}$.

Let M represent the amount of material,

$M = 2\pi r^2 + 2\pi rh$

$$= 2\pi r^2 + 2\pi r\left(\frac{k}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{2k}{r}, 0 \le r \le \infty$$

Using the max min Algorithm,

$$\frac{dM}{dr} = 4\pi r - \frac{2k}{r^2}$$

Let $\frac{dM}{dr} = 0$, $r^3 = \frac{k}{2\pi}$, $r \neq 0$ or

$$r = \left(\frac{k}{2\pi}\right)^{\frac{1}{3}}.$$

When $r \to 0$, $M \to \infty$

$r \to \infty$, $M \to \infty$

$$r = \left(\frac{k}{2\pi}\right)^{\frac{1}{3}}$$

$$d = 2\left(\frac{k}{2\pi}\right)^{\frac{1}{3}}$$

$$h = \frac{k}{\pi\left(\frac{k}{2\pi}\right)^{\frac{2}{3}}} = \frac{k}{\pi} \cdot \frac{(2\pi)^{\frac{2}{3}}}{k^{\frac{2}{3}}} = \frac{k^{\frac{1}{3}}}{\pi^{\frac{1}{3}}} \times 2^{\frac{2}{3}}$$

Min amount of material is

$$M = 2\pi\left(\frac{k}{2\pi}\right)^{\frac{2}{3}} + 2k\left(\frac{2\pi}{k}\right)^{\frac{1}{3}}.$$

Ratio $\frac{h}{d} = \frac{\left(\frac{k}{\pi}\right)^{\frac{1}{3}} \times 2^{\frac{2}{3}}}{2\left(\frac{k}{2\pi}\right)^{\frac{1}{3}}} = \frac{\left(\frac{k}{\pi}\right)^{\frac{1}{3}} \times 2^{\frac{2}{3}}}{2^{\frac{2}{3}}\left(\frac{k}{\pi}\right)^{\frac{1}{3}}} = \frac{1}{1}$

19. **a.** no cut
b. 44 cm for circle; 56 cm for square

20. $\sqrt{17}$

21. Let point A have coordinates $(a^2, 2a)$. (Note that the x-coordinate of any point on the curve is positive, but that the y-coordinate can be positive or negative. By letting the x-coordinate be a^2, we eliminate this concern.)
Similarly, let B have coordinates $(b^2, 2b)$. The slope of AB is $\frac{2a - 2b}{a^2 - b^2} = \frac{2}{a + b}$.
Using the mid-point property, C has coordinates $\left(\frac{a^2 + b^2}{2}, a + b\right)$.
Since CD is parallel to the x-axis, the y-coordinate of D is also $a + b$. The slope of the tangent at D is given by $\frac{dy}{dx}$ for the expression $y^2 = 4x$.
Differentiating,

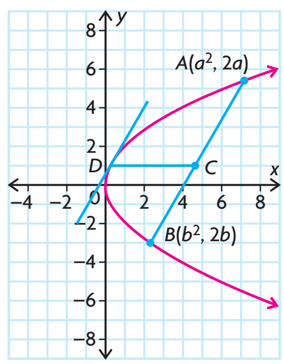
$$2y\frac{dy}{dx} = 4$$

$$\frac{dy}{dx} = \frac{2}{y}$$

And since at point D, $y = a + b$,

$$\frac{dy}{dx} = \frac{2}{a + b}$$

But this is the same as the slope of AB.
Then, the tangent at D is parallel to the chord AB.



22. when P is at the point (5, 2.5)

23. $\frac{2k}{\sqrt{3}}$ by $\frac{2}{3}k^2$

Section 3.4, pp. 151–154

1. **a.** \$1.80
b. \$1.07
c. 5625 L

2. **a.** 15 terms
b. 16 terms/h
c. 20 terms/h

3. **a.** $t = 1$
b. 1.5
c.

d. The level will be a maximum.
e. The level is decreasing.

4. \$6000/h when plane is flying at 15 000 m

5. 250 m by 375 m

6. \$1100 or \$1125

7. \$22.50

8. 6 nautical miles/h

9. 20.4 m by 40.8 m by 24.0 m

10. $r = 4.3$ cm, $h = 17.2$ cm

11. **a.** \$15
b. \$12.50, \$825
c. If you increase the price, the number sold will decrease. Profit in situations like this will increase for several price increases and then it will decrease because too many customers stop buying.

12. 12.1 cm by 18.2 cm by 18.2 cm

13. \$50

14. \$81.25

15. 19 704 units

16. $P(x) = R(x) - C(x)$
Marginal Revenue $= R'(x)$.
Marginal Cost $= C'(x)$.
Now $P'(x) = R'(x) - C'(x)$.
The critical point occurs when $P'(x) = 0$.
If $R'(x) = C'(x)$, then
$P'(x) = R'(x) - C'(x)$
$= 0$
Therefore, the instantaneous rate of change in profit is 0 when the marginal revenue equals the marginal cost.

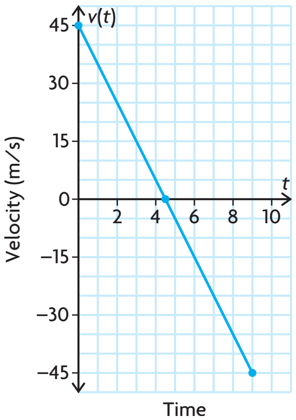
17. $r = 230$ cm and h is about 900 cm

18. 128.4 km/h

19. maximum velocity: $\frac{4}{27}r_0A$, radius: $\frac{2r_0}{3}$.

Review Exercise, pp. 156–159

1. $f'(x) = 4x^3 + 4x^{-5}$,
$f''(x) = 12x^2 - 20x^{-6}$
2. $\frac{d^2y}{dx^2} = 72x^7 - 42x$
3. $v = 2t + (2t - 3)^{\frac{1}{2}}$,
$a = 2 - (2t - 3)^{\frac{3}{2}}$
4. $v(t) = 1 - 5t^{-2}$,
$a(t) = 10t^{-3}$
5. The upward velocity is positive for $0 \le t \le 4.5$ s, zero for $t = 4.5$ s, and negative for $t > 4.5$ s.



6. **a.** min: −52,
max: 0
b. min: −65,
max: 16
c. min: 12,
max: 20
7. **a.** 62 m
b. Yes, 2 m beyond the stop sign
c. Stop signs are located two or more metres from an intersection. Since the car only went 2 m beyond the stop sign, it is unlikely the car would hit another vehicle travelling perpendicular.
8. min is 2, max is $2 + 3\sqrt{3}$
9. 250
10. **a.** **i.** \$2200
ii. \$5.50
iii. \$3.00; \$3.00
b. **i.** \$24 640
ii. \$61.60
iii. \$43.20; \$43.21
c. **i.** \$5020
ii. \$12.55
iii. \$0.03; \$0.03
d. **i.** \$2705
ii. \$6.88
iii. \$5.01; \$5.01
11. 2000
12. **a.** moving away from its starting point
b. moving away from the origin and towards its starting position
13. **a.** $t = \frac{2}{3}$ **b.** yes
14. 27.14 cm by 27.14 cm for the base and height 13.57 cm
15. length 190 m, width approximately 63 m
16. 31.6 cm by 11.6 cm by 4.2 cm
17. radius 4.3 cm, height 8.6 cm
18. Run the pipe 7.2 km along the river shore and then cross diagonally to the refinery.
19. 10:35 p.m.
20. \$204 or \$206
21. The pipeline meets the shore at a point C, 5.7 km from point A, directly across from P.
22. 11.35 cm by 17.02 cm
23. 34.4 m by 29.1 m
24. 2:23 p.m.
25. 3.2 km from point C
26. **a.** absolute maximum: $f(7) = 41$,
absolute minimum: $f(1) = 5$
b. absolute maximum: $f(3) = 36$,
absolute minimum: $f(-3) = -18$
c. absolute maximum: $f(5) = 67$,
absolute minimum: $f(-5) = -63$
d. absolute maximum: $f(4) = 2752$,
absolute minimum: $f(-2) = -56$
27. **a.** 62.9 m **c.** 3.6 m/s^2
b. 4.7 s
28. **a.** $f''(2) = 60$ **d.** $f''(1) = -\frac{5}{16}$
b. $f''(-1) = 26$ **e.** $f''(4) = -\frac{1}{108}$
c. $f''(0) = 192$ **f.** $f''(8) = -\frac{1}{72}$
29. **a.** position: 1, velocity: $\frac{1}{6}$,
acceleration: $-\left(\frac{1}{18}\right)$, speed: $\frac{1}{6}$
b. position: $\frac{8}{3}$, velocity: $\frac{4}{9}$,
acceleration: $\frac{10}{27}$, speed: $\frac{4}{9}$
30. **a.** $v(t) = \frac{2}{3}(t^2 + t)^{-\frac{1}{3}}(2t + 1)$,
$a(t) = \frac{2}{9}(t^2 + t)^{-\frac{4}{3}}(2t^2 + 2t - 1)$
b. 1.931 m/s
c. 2.36 m/s
d. undefined
e. 0.141 m/s^2

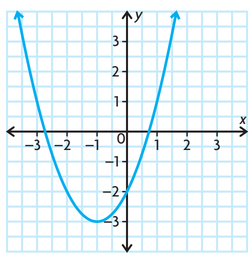
Chapter 3 Test, p. 160

1. **a.** $y'' = 14$
b. $f''(x) = -180x^3 - 24x$
c. $y'' = 60x^{-5} + 60x$
d. $f''(x) = 96(4x - 8)$
2. **a.** $v(3) = -57$,
$a(3) = -44$
b. $v(2) = 6$,
$a(2) = -24$
3. **a.** $v(t) = 2t - 3$,
$a(t) = 2$
b. −0.25 m
c. 1 m/s, 1 m/s
d. between $t = 0$ s and $t = 1.5$ s
e. 2 m/s^2
4. **a.** min: −63, max: 67
b. min: 7.5, max: 10
5. **a.** 2.1 s
b. about 22.9
6. 250 m by 166.7 m
7. 162 mm by 324 mm by 190 mm
8. \$850/month

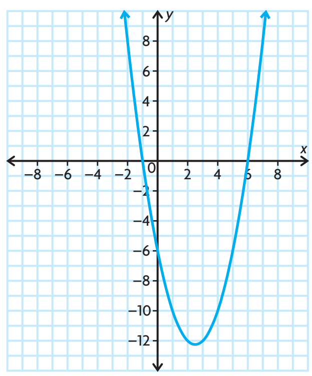
Chapter 4

Review of Prerequisite Skills, pp. 162–163

1. **a.** $y = -\frac{3}{2}$ or $y = 1$
b. $x = 7$ or $x = -2$
c. $x = -\frac{5}{2}$
d. $y = 1$ or $y = -3$ or $y = -2$
2. **a.** $x < -\frac{7}{3}$
b. $x \le 2$
c. $-1 < t < 3$
d. $x < -4$ or $x > 1$
3. **a.**

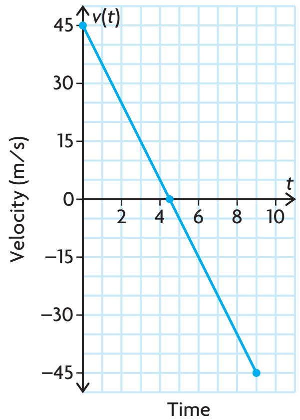


b.



Review Exercise, pp. 156–159

1. $f'(x) = 4x^3 + 4x^{-5}$,
   $f''(x) = 12x^2 - 20x^{-6}$
2. $\frac{d^2y}{dx^2} = 72x^7 - 42x$
3. $v = 2t + (2t - 3)^{\frac{1}{2}}$,
   $a = 2 - (2t - 3)^{\frac{3}{2}}$
4. $v(t) = 1 - 5t^{-2}$,
   $a(t) = 10t^{-3}$
5. The upward velocity is positive for $0 \le t \le 4.5$ s, zero for $t = 4.5$ s, and negative for $t > 4.5$ s.



6. **a.** min: -52,
   max: 0
   b. min: -65,
   max: 16
   c. min: 12,
   max: 20
7. **a.** 62 m
   b. Yes, 2 m beyond the stop sign
   c. Stop signs are located two or more metres from an intersection. Since the car only went 2 m beyond the stop sign, it is unlikely the car would hit another vehicle travelling perpendicular.
8. min is 2, max is $2 + 3\sqrt{3}$
9. 250
10. **a.** **i.** \$2200
    ii. \$5.50
    iii. \$3.00; \$3.00
    b. **i.** \$24 640
    ii. \$61.60
    iii. \$43.20; \$43.21
    c. **i.** \$5020
    ii. \$12.55
    iii. \$0.03; \$0.03
    d. **i.** \$2705
    ii. \$6.88
    iii. \$5.01; \$5.01
11. 2000
12. **a.** moving away from its starting point
    b. moving away from the origin and towards its starting position
13. **a.** $t = \frac{2}{3}$ **b.** yes
14. 27.14 cm by 27.14 cm for the base and height 13.57 cm
15. length 190 m, width approximately 63 m
16. 31.6 cm by 11.6 cm by 4.2 cm
17. radius 4.3 cm, height 8.6 cm
18. Run the pipe 7.2 km along the river shore and then cross diagonally to the refinery.
19. 10:35 p.m.
20. \$204 or \$206
21. The pipeline meets the shore at a point C, 5.7 km from point A, directly across from P.
22. 11.35 cm by 17.02 cm
23. 34.4 m by 29.1 m
24. 2:23 p.m.
25. 3.2 km from point C
26. **a.** absolute maximum: $f(7) = 41$,
    absolute minimum: $f(1) = 5$
    b. absolute maximum: $f(3) = 36$,
    absolute minimum: $f(-3) = -18$
    c. absolute maximum: $f(5) = 67$,
    absolute minimum: $f(-5) = -63$
    d. absolute maximum: $f(4) = 2752$,
    absolute minimum: $f(-2) = -56$
27. **a.** 62.9 m **c.** 3.6 m/s^2
    b. 4.7 s
28. **a.** $f''(2) = 60$ **d.** $f''(1) = -\frac{5}{16}$
    b. $f''(-1) = 26$ **e.** $f''(4) = -\frac{1}{108}$
    c. $f''(0) = 192$ **f.** $f''(8) = -\frac{1}{72}$
29. **a.** position: 1, velocity: $\frac{1}{6}$,
    acceleration: $-\left(\frac{1}{18}\right)$, speed: $\frac{1}{6}$
    b. position: $\frac{8}{3}$, velocity: $\frac{4}{9}$,
    acceleration: $\frac{10}{27}$, speed: $\frac{4}{9}$
30. **a.** $v(t) = \frac{2}{3}(t^2 + t)^{-\frac{1}{3}}(2t + 1)$,
    $a(t) = \frac{2}{9}(t^2 + t)^{-\frac{4}{3}}(2t^2 + 2t - 1)$
    b. 1.931 m/s
    c. 2.36 m/s
    d. undefined
    e. 0.141 m/s^2

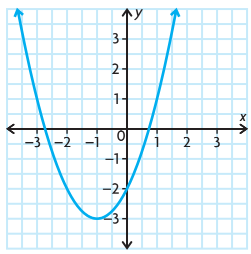
Chapter 3 Test, p. 160

1. **a.** $y'' = 14$
   b. $f''(x) = -180x^3 - 24x$
   c. $y'' = 60x^{-5} + 60x$
   d. $f''(x) = 96(4x - 8)$
2. **a.** $v(3) = -57$,
   $a(3) = -44$
   b. $v(2) = 6$,
   $a(2) = -24$
3. **a.** $v(t) = 2t - 3$,
   $a(t) = 2$
   b. -0.25 m
   c. 1 m/s, 1 m/s
   d. between $t = 0$ s and $t = 1.5$ s
   e. 2 m/s^2
4. **a.** min: -63, max: 67
   b. min: 7.5, max: 10
5. **a.** 2.1 s
   b. about 22.9
6. 250 m by 166.7 m
7. 162 mm by 324 mm by 190 mm
8. \$850/month

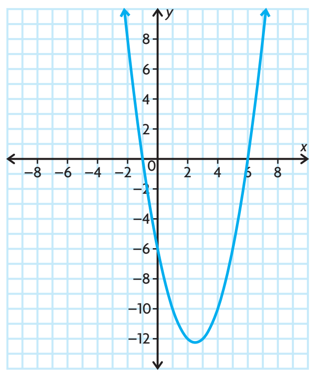
Chapter 4

Review of Prerequisite Skills, pp. 162–163

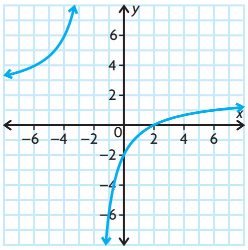
1. **a.** $y = -\frac{3}{2}$ or $y = 1$
   b. $x = 7$ or $x = -2$
   c. $x = -\frac{5}{2}$
   d. $y = 1$ or $y = -3$ or $y = -2$
2. **a.** $x < -\frac{7}{3}$
   b. $x \le 2$
   c. $-1 < t < 3$
   d. $x < -4$ or $x > 1$
3. **a.**



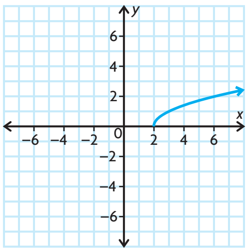
b.



c.



d.



4. **a.** 0
b. 7
c. 27
d. 3

5. **a.** $x^3 + 6x + x^{-2}$
b. $-\dfrac{x^2 + 2x + 3}{(x^2 - 3)^2}$
c. $2(3x^2 - 6x)(6x - 6)$
d. $\dfrac{t - 8}{(t - 4)^{\frac{3}{2}}}$

6. **a.** $x - 8 + \dfrac{28}{x + 3}$
b. $x + 7 - \dfrac{2}{x - 1}$

7. $\left(\dfrac{2}{3}, 2.19\right), (-1, 4.5)$

8. **a.** If $f(x) = x^n$, where n is a real number, then $f'(x) = nx^{n-1}$.
b. If $f(x) = k$, where k is a constant, then $f'(x) = 0$.
c. If $k(x) = f(x)g(x)$, then $k'(x) = f'(x)g(x) - f(x)g'(x)$
d. If $h(x) = \dfrac{f(x)}{g(x)}$, then
$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2},$$
$g(x) \neq 0$.
e. If f and g are functions that have derivatives, then the composite function $h(x) = f(g(x))$ has a derivative given by $h'(x) = f'(g(x)) \cdot g'(x)$.
f. If u is a function of x, and n is a positive integer, then $\dfrac{d}{dx}(u^n) = nu^{n-1}\dfrac{du}{dx}$.

9. **a.** As $x \rightarrow \pm\infty, f(x) \rightarrow \infty$.
b. As $x \rightarrow -\infty, f(x) \rightarrow -\infty$.
As $x \rightarrow \infty, f(x) \rightarrow \infty$.
c. As $x \rightarrow -\infty, f(x) \rightarrow -\infty$.
As $x \rightarrow \infty, f(x) \rightarrow -\infty$.

10. **a.** $\dfrac{1}{2x}$; $x = 0$
b. $\dfrac{1}{-x + 3}$; $x = 3$
c. $\dfrac{1}{(x + 4)^2 + 1}$; no vertical asymptote
d. $\dfrac{1}{(x + 3)^2}$; $x = -3$

11. **a.** $y = 0$
b. $y = 4$
c. $y = \frac{1}{2}$
d. $y = 2$

12. **a.** **i.** no x-intercept; (0, 5)
ii. (0, 0); (0, 0)
iii. $\left(\dfrac{5}{3}, 0\right); \left(0, \dfrac{5}{3}\right)$
iv. $\left(\dfrac{2}{5}, 0\right)$; no y-intercept
b. **i.** Domain: $\{x \in \mathbf{R} \mid x \neq -1\}$,
Range: $\{y \in \mathbf{R} \mid y \neq 0\}$
ii. Domain: $\{x \in \mathbf{R} \mid x \neq 2\}$,
Range: $\{y \in \mathbf{R} \mid y \neq 4\}$
iii. Domain: $\left\{x \in \mathbf{R} \mid x \neq \dfrac{1}{2}\right\}$,
Range: $\left\{y \in \mathbf{R} \mid y \neq \dfrac{1}{2}\right\}$
iv. Domain: $\{x \in \mathbf{R} \mid x \neq 0\}$,
Range: $\{y \in \mathbf{R} \mid y \neq 2\}$

Section 4.1, pp. 169–171

1. **a.** (0, 1), (−4, 33)
b. (0, 2)
c. $\left(\dfrac{1}{2}, 0\right)$, (2.25, −48.2), (−2, −125)
d. $\left(1, \dfrac{5}{2}\right), \left(-1, -\dfrac{5}{2}\right)$

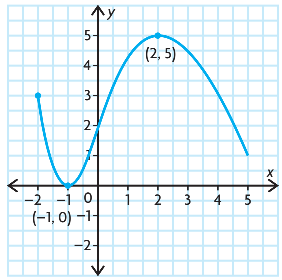
2. A function is increasing when $f'(x) > 0$ and is decreasing when $f'(x) < 0$.

3. **a.** **i.** $x < -1, x > 2$
ii. $-1 < x < 2$
iii. (−1, 4), (2, −1)
b. **i.** $-1 < x < 1$
ii. $x < -1, x > 1$
iii. (−1, 2), (2, 4)
c. **i.** $x < -2$
ii. $-2 < x < 2, 2 < x$
iii. none
d. **i.** $-1 < x < 2, 3 < x$
ii. $x < -1, 2 < x < 3$
iii. (2, 3)

4. **a.** increasing: $x < -2, x > 0$;
decreasing: $-2 < x < 0$
b. increasing: $x < 0, x > 4$;
decreasing: $0 < x < 4$
c. increasing: $x < -1, x > 1$;
decreasing: $-1 < x < 0$,
$0 < x < 1$
d. increasing: $-1 < x < 3$;
decreasing: $x < -1, x > 3$
e. increasing: $-2 < x < 0, x > 1$;
decreasing: $x < -2, 0 < x < 1$
f. increasing: $x > 0$;
decreasing: $x < 0$

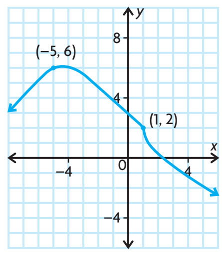
5. increasing: $-3 < x < -2, x > 1$;
decreasing: $x < -3, -2 < x < 1$

6.

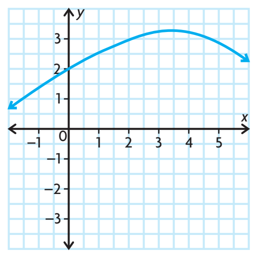


7. $a = 3, b = -9, c = -9$

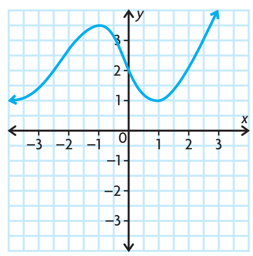
8.



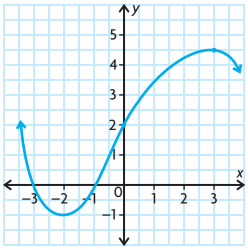
9. **a.** **i.** $x < 4$
ii. $x > 4$
iii. $x = 4$



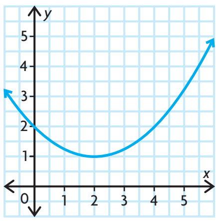
b. **i.** $x < -1, x > 1$
ii. $-1 < x < 1$
iii. $x = -1, x = 1$



c. i. $-2 < x < 3$
ii. $x < -2, x > 3$
iii. $x = -2, x = 3$



d. i. $x > 2$
ii. $x < 2$
iii. $x = 2$



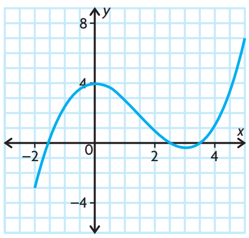
10. $f(x) = ax^2 + bx + c$
$f'(x) = 2ax + b$

Let $f'(x) = 0$, then $x = \frac{-b}{2a}$.

If $x < \frac{-b}{2a}$, $f'(x) < 0$, therefore the function is decreasing.
If $x > \frac{-b}{2a}$, $f'(x) > 0$, therefore the function is increasing.

11. $f'(x) = 0$ for $x = 2$,
increasing: $x > 2$,
decreasing: $x < 2$,
local minimum: $(2, -44)$

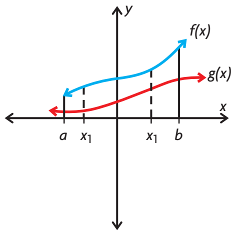
12.



13. Let $y = f(x)$ and $u = g(x)$.
Let x_1 and x_2 be any two values in the interval $a \leq x \leq b$ so that $x_1 < x_2$.
Since $x_1 < x_2$, both functions are increasing:

$f(x_2) > f(x_1)$ (1)
$g(x_2) > g(x_1)$ (2)
$yu = f(x) \cdot g(x)$

$(1) \times (2)$ results in
$f(x_2) \cdot g(x_2) > f(x_1)g(x_1)$
The function yu or $f(x) \cdot g(x)$ is strictly increasing.

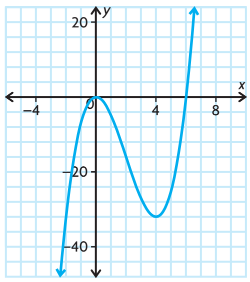


14. strictly decreasing

Section 4.2, pp. 178–180

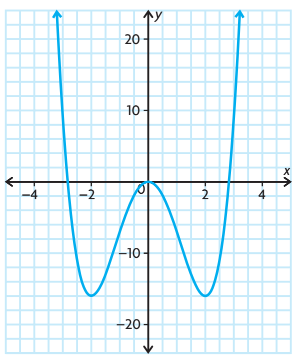
1. Determining the points on the graph of the function for which the derivative of the function at the x-coordinate is 0

2. a. Take the derivative of the function. Set the derivative equal to 0. Solve for x. Evaluate the original function for the values of x. The (x, y) pairs are the critical points.
b. $(0, 0), (4, -32)$

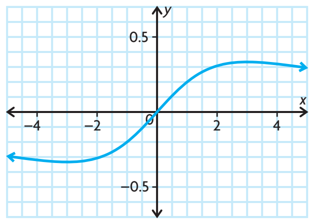


3. a. local minima: $(-2, -16), (2, -16)$, local maximum: $(0, 0)$
b. local minimum: $(-3, -0.3)$, local maximum: $(3, 0.3)$
c. local minimum: $(-2, 5)$, local maximum: $(0, 1)$

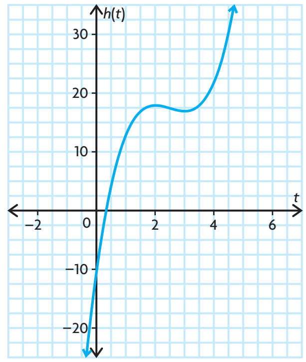
4. a. $(0, 0), (2\sqrt{2}, 0), (-2\sqrt{2}, 0)$



b. $(0, 0)$

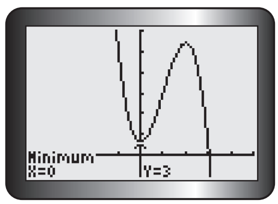


c. $(-3, 1, 0), (0, 1)$

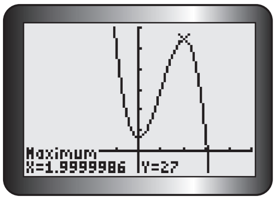


5. a. local minimum: $(0, 3)$, local maximum: $(2, 27)$, Tangent is parallel to the horizontal axis for both.
b. $(0, 0)$ neither maximum nor minimum, Tangent is parallel to the horizontal axis.
c. $(5, 0)$; neither maximum nor minimum, Tangent is not parallel to the horizontal axis.
d. local minimum: $(0, -1)$, Tangent is parallel to the horizontal axis. $(-1, 0)$ and $(1, 0)$ are neither maxima or minima. Tangent is not parallel to the horizontal axis for either.

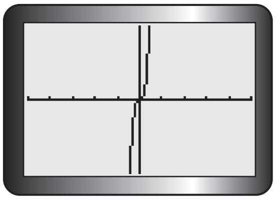
6. a.



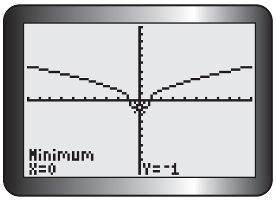
b.



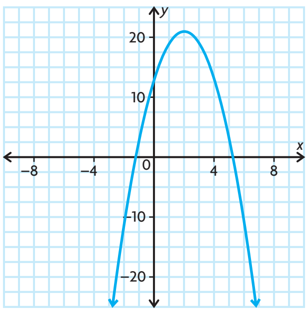
c.



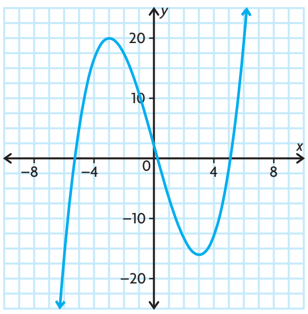
d.



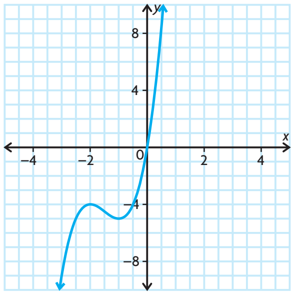
7. **a.** $(2, 21)$ local maximum



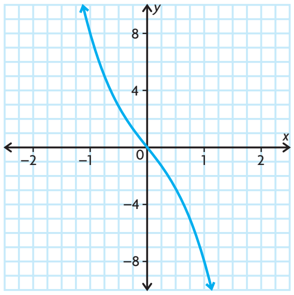
b. $(-3, 20)$ local maximum, $(3, -16)$ local minimum



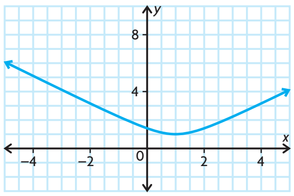
c. $(-2, -4)$ local maximum, $(-1, -5)$ local minimum



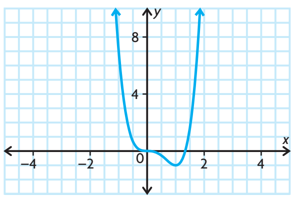
d. no critical points



e. $(1, 1)$ local minimum

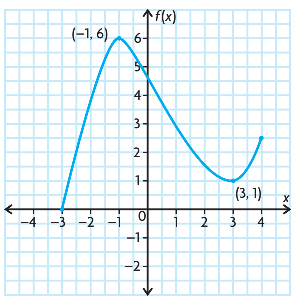


f. $(0, 0)$ neither maximum nor minimum, $(1, -1)$ local minimum



8. local minima at $x = -6$ and $x = 2$; local maximum at $x = -1$

9.



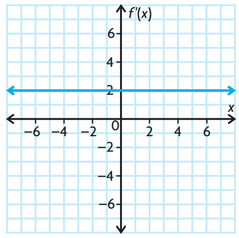
10. $a = -\frac{11}{9}, b = \frac{22}{3}, c = 1$

11. $p = -2, q = 6$
minimum; the derivative is negative to the left and positive to the right

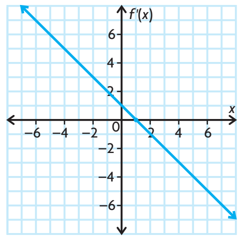
12. **a.** $k < 0$
b. $k = 0$
c. $k > 0$

13. $a = -1, b = 3, c = 0, d = 0$

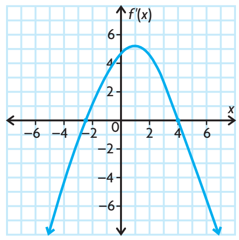
14. **a.**



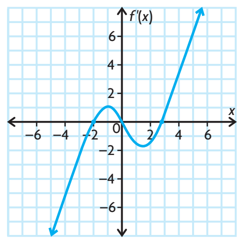
b.



c.

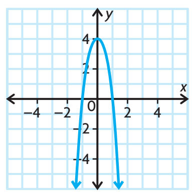


d.

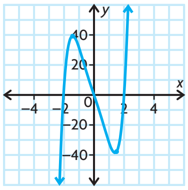


15. **a.** $a = -4, b = -36, c = 0$
b. $(3, -198)$
c. local minimum: $(-2, -73)$ and $(3, -198)$,
local maximum: $(0, -9)$

16. **a.** local maximum: $(0, 4)$



b. local minimum: $(1.41, -39.6)$, local maximum: $(1.41, 39.6)$



17. $h(x) = \dfrac{f(x)}{g(x)}$

Since $f(x)$ has a local maximum at $x = c$, then $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$. Since $g(x)$ has a local minimum at $x = c$, then $g'(x) < 0$ for $x < c$ and $g'(x) > 0$ for $x > c$.

$$h(x) = \frac{f(x)}{g(x)}$$

$$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$

If $x < c, f'(x) > 0$ and $g'(x) < 0$, then $h'(x) > 0$.
If $x > c, f'(x) < 0$ and $g'(x) > 0$, then $h'(x) < 0$.
Since for $x < c, h'(x) > 0$ and for $x > c, h'(x) < 0$. Therefore, $h(x)$ has a local maximum at $x = c$.

Section 4.3, pp. 193–195

1. **a.** vertical asymptotes at $x = -2$ and $x = 2$; horizontal asymptote at $y = 1$
b. vertical asymptote at $x = 0$; horizontal asymptote at $y = 0$

2. $f(x) = \dfrac{g(x)}{h(x)}$

Conditions for a vertical asymptote: $h(x) = 0$ must have at least one solution s, and $\lim\limits_{x \to \infty} f(x) = \infty$.

Conditions for a horizontal asymptote: $\lim\limits_{x \to \infty} f(x) = k$, where $k \in \mathbf{R}$, or $\lim\limits_{x \to \infty} f(x) = k$, where $k \in \mathbf{R}$.

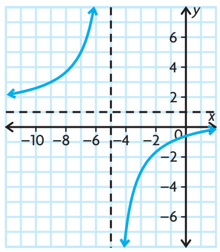
Condition for an oblique asymptote: The highest power of $g(x)$ must be one more than the highest power of $k(x)$.

3. **a.** 2
b. 5
c. $-\dfrac{5}{2}$
d. ∞

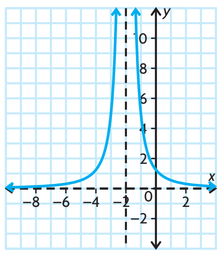
4. **a.** $x = -5$; large and positive to left of asymptote, large and negative to right of asymptote
b. $x = 2$; large and negative to left of asymptote, large and positive to right of asymptote
c. $x = 3$; large and positive to left of asymptote, large and positive to right of asymptote
d. hole at $x = 3$, no vertical asymptote
e. $x = -3$; large and positive to left of asymptote, large and negative to right of asymptote
$x = 1$; large and negative to left of asymptote, large and positive to right of asymptote
f. $x = -1$; large and positive to left of asymptote, large and negative to right of asymptote
$x = 1$; large and negative to left of asymptote, large and positive to right of asymptote

5. **a.** $y = 1$; large negative: approaches from above, large positive: approaches from below
b. $y = 0$; large negative: approaches from below, large positive: approaches from above
c. $y = 3$; large negative: approaches from above, large positive: approaches from above
d. no horizontal asymptotes

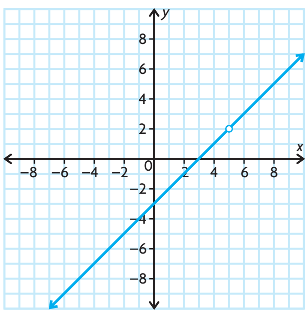
6. **a.**



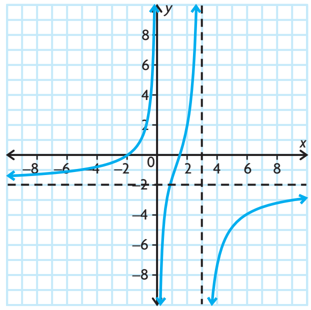
b.



c.



d.

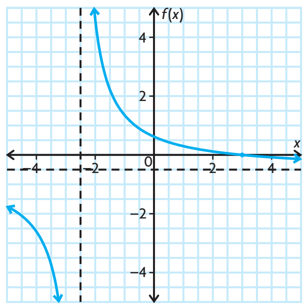


7. **a.** $y = 3x - 7$
b. $y = x + 3$
c. $y = x - 2$
d. $y = x + 3$

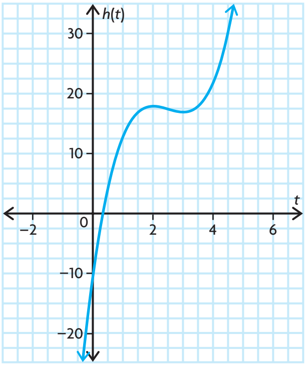
8. **a.** large negative: approaches from below, large positive: approaches from above
b. large negative: approaches from above, large positive: approaches from below

9. **a.** $x = -5$; large and positive to left of asymptote, large and negative to right of asymptote
$y = 3$
b. $x = 1$; large and positive to left of asymptote, large and positive to right of asymptote
$y = 1$
c. $x = -2$; large and negative to left of asymptote, large and positive to right of asymptote
$y = 1$
d. $x = 2$; large and negative to left of asymptote, large and positive to right of asymptote
$y = 1$

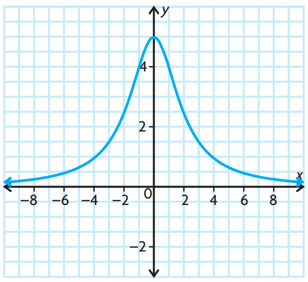
10. **a.**



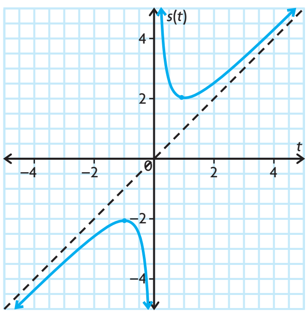
b.



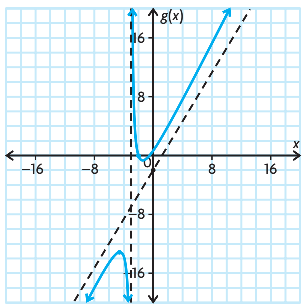
c.



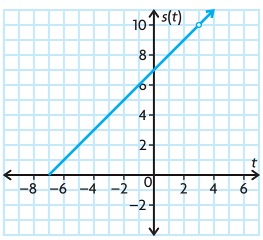
d.



e.



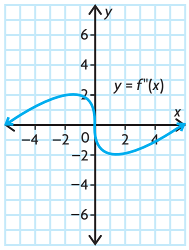
f.



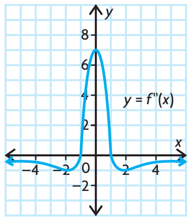
11. **a.** $y = \frac{a}{c}$

b. $x = -\frac{d}{c}$

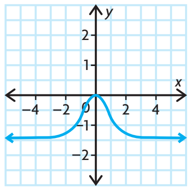
12. **a.**



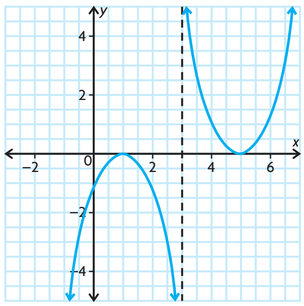
b.



13. **a.**



b.



14. **a.** $f(x)$ and $r(x)$: $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} r(x)$ exist.

b. $h(x)$: the highest degree of x in the numerator is exactly one degree higher than the highest degree of x in the denominator.

c. $h(x)$: the denominator is defined for all $x \in \mathbf{R}$.

$f(x) = \frac{-x - 3}{(x - 7)(x + 2)}$ has vertical Asymptotes at $x = 7$ and $x = -2$.

As $x \to -2^-$, $f(x) \to -\infty$.
As $x \to -2^+$, $f(x) \to \infty$.
As $x \to 7^-$, $f(x) \to \infty$.
As $x \to 7^+$, $f(x) \to -\infty$.

$f(x)$ has a horizontal asymptote at $y = 0$.

$g(x)$ has a vertical asymptote at $x = 3$.
As $x \to 3^-$, $g(x) \to \infty$.
As $x \to 3^+$, $g(x) \to -\infty$.

$y = x$ is an oblique asymptote for $h(x)$.

$r(x) = \frac{(x + 3)(x - 2)}{(x - 4)(x + 4)}$ has vertical asymptotes at $x = -4$ and $x = 4$.

As $x \to -4^-$, $r(x) \to \infty$.
As $x \to -4^+$, $r(x) \to -\infty$.
As $x \to 4^-$, $r(x) \to -\infty$.
As $x \to 4^+$, $r(x) \to \infty$.

$r(x)$ has a horizontal asymptote at $y = 1$.

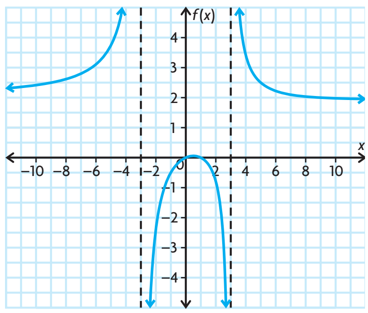
15. $a = \frac{9}{5}, b = \frac{3}{5}$

16. **a.** $\lim_{x\to\infty} \frac{x^2 + 1}{x + 1} = \lim_{x\to\infty} \frac{x + \frac{1}{x}}{1 + \frac{1}{x}}$
$= \infty$

$$\lim_{x\to\infty} \frac{x^2 + 2x + 1}{x + 1} = \lim_{x\to\infty} \frac{(x + 1)(x + 1)}{(x + 1)} = \lim_{x\to\infty} (x + 1) = \infty$$

b. $$\lim_{x\to\infty}\left[\frac{x^2 + 1}{x + 1} - \frac{x^2 + 2x + 1}{x + 1}\right] = \lim_{x\to\infty} \frac{x^2 + 1 - x^2 - 2x - 1}{x + 1} = \lim_{x\to\infty} \frac{-2x}{x + 1} = \lim_{x\to\infty} \frac{-2}{} = -2$$

17.



Mid-Chapter Review, pp. 196–197

1. **a.** decreasing: $(-\infty, 2)$, increasing: $(2, \infty)$

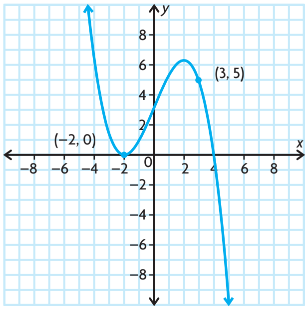
b. decreasing: $(0, 2)$, increasing: $(-\infty, 0)$, $(2, \infty)$

c. increasing: $(-\infty, -3)$, $(3, \infty)$

d. decreasing: $(-\infty, 0)$, increasing: $(0, \infty)$

2. increasing: $x < -1$ and $x > 2$, decreasing: $-1 < x < 2$

3.



4. **a.** -4 **b.** $0, \pm 3\sqrt{3}$ **c.** $0, \pm\sqrt{2}$ **d.** $\pm 1, \pm 2$ **e.** 0 **f.** $\pm\sqrt{2}$

5. **a.** $x = 1$ local maximum, $x = 2$ local minimum
b. $x = -\frac{2}{3}$ local maximum, $x = 2$ local minimum

6. ± 2

7. $x = 2$ local minimum, increasing: $x > 2$, decreasing: $x < 2$

8. **a.** $x = -2$; large and positive to left of asymptote, large and negative to right of asymptote
b. $x = -3$; large and negative to left of asymptote, large and positive to right of asymptote
$x = 3$; large and positive to left of asymptote, large and negative to right of asymptote
c. $x = -3$; large and negative to left of asymptote, large and positive to right of asymptote
d. $x = -\frac{2}{3}$; large and positive to left of asymptote, large and negative to right of asymptote
$x = 5$; large and positive to left of asymptote, large and negative to right of asymptote

9. **a.** $y = 3$; large negative: approaches from above, large positive: approaches from below
b. $y = 1$; large negative: approaches from below, large positive: approaches from above

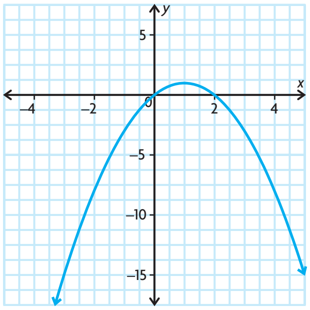
10. **a.** $x = 5$; large and positive to left of asymptote, large and positive to right of asymptote
b. no discontinuities
c. $x = 6 + 2\sqrt{6}$; large and negative to left of asymptote, large and positive to right of asymptote
$x = 6 - 2\sqrt{6}$; large and negative to left of asymptote, large and positive to right of asymptote

11. **a.** $f(x)$ is increasing.
b. $f(x)$ is decreasing.

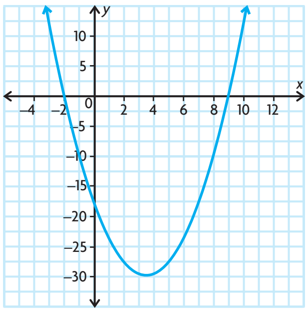
12. increasing: $0 < t < 0.97$, decreasing: $t > 0.97$

13. increasing: $t > 2.5198$, decreasing: $t < 2.5198$

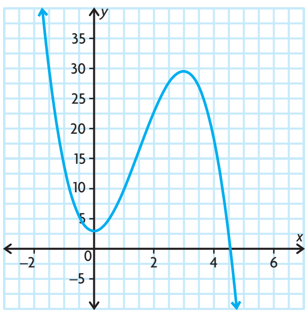
14.



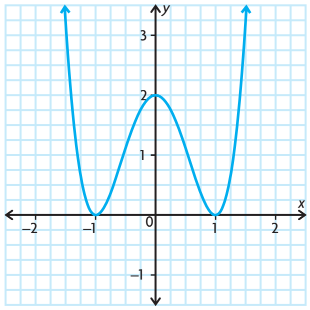
15. **a.** **i.** $x = \frac{7}{2}$
ii. increasing: $x > \frac{7}{2}$, decreasing: $x < \frac{7}{2}$
iii. local minimum at $x = \frac{7}{2}$
iv.



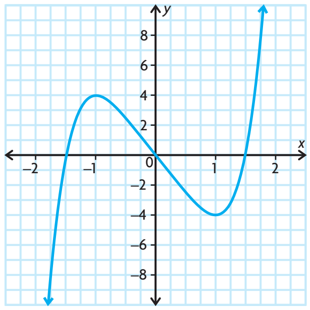
b. **i.** $x = 0, x = 3$
ii. increasing: $0 < x < 3$, decreasing: $x < 0, x > 3$
iii. local minimum at $x = 0$, local maximum at $x = 3$
iv.



c. **i.** $x = -1, x = 0, x = 1$
ii. increasing: $-1 < x < 0, x > 1$; decreasing: $x < -1, 0 < x < 1$
iii. local maximum at $x = 0$, local minimum at $x = 1, -1$
iv.



d. **i.** $x = -1, x = 1$
ii. increasing: $x < -1, x > 1$; decreasing: $-1 < x < 1$
iii. local maximum at $x = -1$, local minimum at $x = 1$
iv.

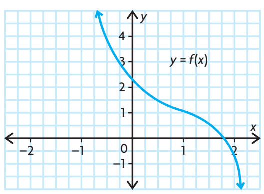


16. **a.** $x = -\frac{1}{2}$; large and positive to left of asymptote, large and negative to right of asymptote; $y = \frac{1}{2}$
b. $x = -2$; large and positive to left of asymptote, large and positive to right of asymptote; $y = 1$
c. $x = -3$; large and positive to left of asymptote, large and negative to right of asymptote; $y = -1$
d. $x = -4$; large and negative to left of asymptote, large and positive to right of asymptote; $y = 2$

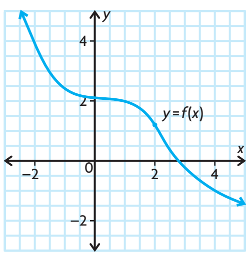
17. **a.** $-\frac{2}{3}$ **b.** $\frac{1}{6}$ **c.** -3 **d.** 0 **e.** ∞ **f.** 1 **g.** 1 **h.** ∞

Section 4.4, pp. 205–206

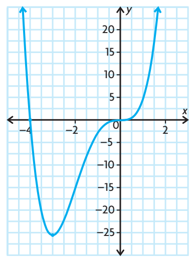
1. a. A: negative, B: negative, C: positive, D: positive
   b. A: negative, B: negative, C: positive, D: negative
2. a. local minimum: $(5, -105)$, local maximum: $(-1, 20)$
   b. local maximum: $\left(0, \frac{25}{48}\right)$
   c. local maximum: $(-1, -2)$, local minimum: $(1, 2)$
   d. $(3, 8)$ is neither a local maximum or minimum.
3. a. $\left(\frac{4}{3}, -14\frac{20}{27}\right)$
   b. $\left(-4, \frac{25}{64}\right)\left(4, \frac{25}{64}\right)$
   c. no points of inflection
   d. $(3, 8)$
4. a. 24; above
   b. 4; above
   c. $-\frac{9}{100\sqrt{10}}$; below
   d. $-\frac{2}{27}$; below
5. a. i. concave up on $x < 1$, concave down on $x > 1$
   ii. $x = 1$
   iii.



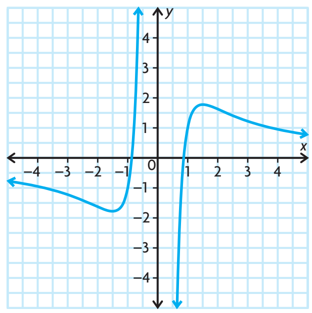
   b. i. concave up on $x < 0$ or $x > 2$, concave down on $0 < x < 2$
   ii. $x = 0$ and $x = 2$
   iii.



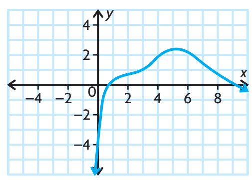
6. For any function $y = f(x)$, find the critical points, i.e., the values of x such that $f'(x) = 0$ or $f'(x)$ does not exist. Evaluate $f''(x)$ for each critical value. If the value of the second derivative at a critical point is positive, the point is a local minimum. If the value of the second derivative at a critical point is negative, the point is a local maximum.
7. Use the first derivative test or the second derivative test to determine the type of critical points that may be present.
8. a. i. $(-2, -16)$, $(0, 0)$
   ii.



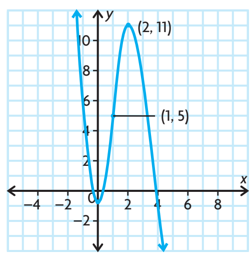
   b. i. $\left(-\frac{3}{\sqrt{2}}, -\frac{8\sqrt{2}}{9}\right), \left(\frac{3}{\sqrt{2}}, \frac{8\sqrt{2}}{9}\right)$
   ii.



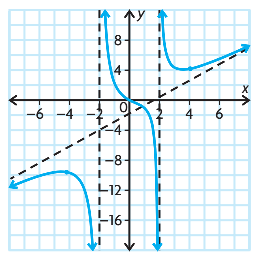
9.



10. $a = -3, b = 9, c = -1$



11. $\frac{27}{64}$
12. $f(x) = ax^4 + bx^3$
    $f'(x) = 4ax^3 + 3bx^2$
    $f''(x) = 12ax^2 + 6bx$
    For possible points of inflection, we solve $f''(x) = 0$:
    $12ax^2 + 6bx = 0$
    $6x(2ax + b) = 0$
    $x = 0$ or $x = -\frac{b}{2a}$.
    The graph of $y = f''(x)$ is a parabola with x-intercepts 0 and $-\frac{b}{2a}$.
    We know the values of $f''(x)$ have opposite signs when passing through a root. Thus, at $x = 0$ and at $x = -\frac{b}{2a}$, the concavity changes as the graph goes through these points. Thus, $f(x)$ has points of inflection at $x = 0$ and $x = -\frac{b}{2a}$.
    To find the x-intercepts, we solve
    $f(x) = 0$
    $x^3(ax + b) = 0$
    $x = 0$ or $x = -\frac{b}{a}$
    The point midway between the x-intercepts has x-coordinate $-\frac{b}{2a}$.
    The points of inflection are $(0, 0)$ and $\left(-\frac{b}{2a}, -\frac{b^4}{16a^3}\right)$.
13. a.



   b. Answers may vary. For example, there is a section of the graph that lies between the two sections of the graph that approaches the asymptote.
14. $n = 1, n = 2$: no inflection points; $n = 3, n = 4$: inflection point at $x = c$; The graph of f has an inflection point at $x = c$ when $n \geq 3$.

Section 4.5, pp. 212–213

1. A cubic polynomial that has a local minimum must also have a local maximum. If the local minimum is to the left of the local maximum, then $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$. If the local minimum is to the right of the local maximum, then $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.
2. A polynomial of degree three has at most two local extremes. A polynomial of degree four has at most three local extremes. Since each local maximum and minimum of a function corresponds

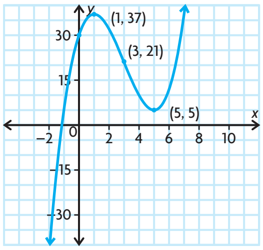
to a zero of its derivative, the number of zeros of the derivative is the maximum number of local extreme values that the function can have. For a polynomial of degree n, the derivative has degree $n - 1$, so it has at most $n - 1$ zeros, and thus at most $n - 1$ local extremes.

3. **a.** $x = -3$ or $x = -1$

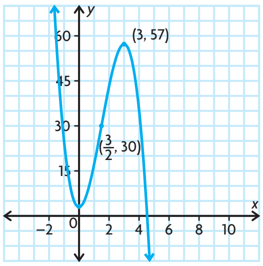
b. no vertical asymptotes

c. $x = 3$

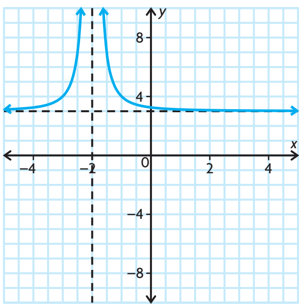
4. **a.**



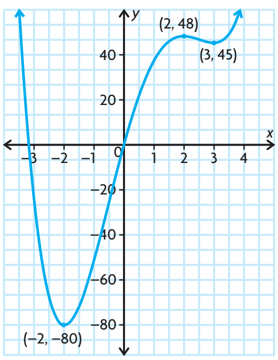
b.



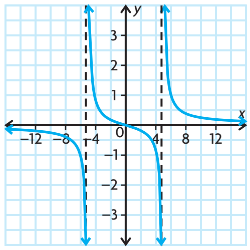
c.



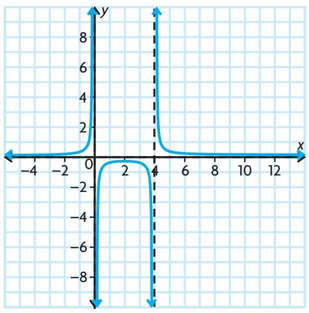
d.



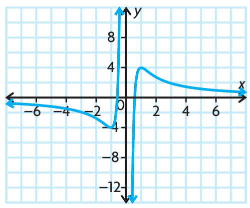
e.



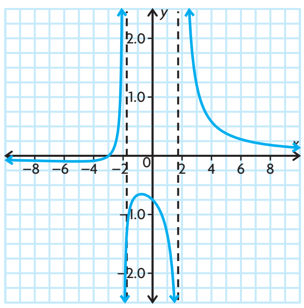
f.



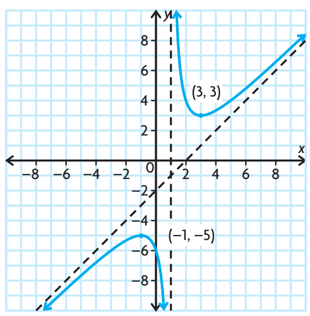
g.



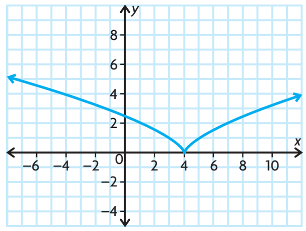
h.



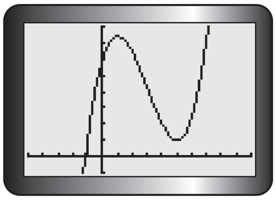
i.



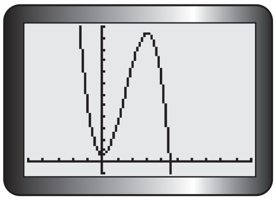
j.



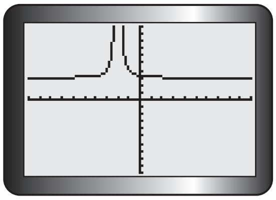
5. **a.**



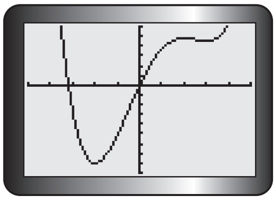
b.



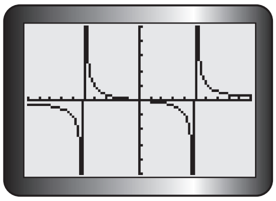
c.



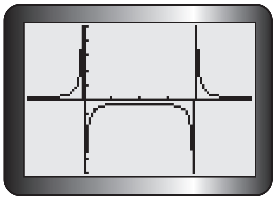
d.



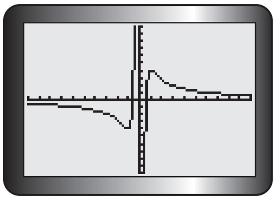
e.



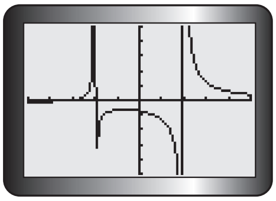
f.



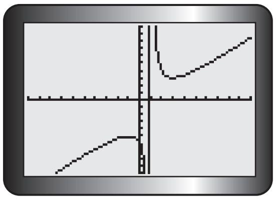
g.



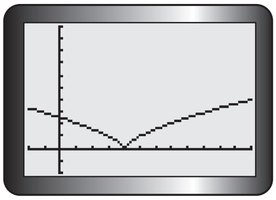
h.



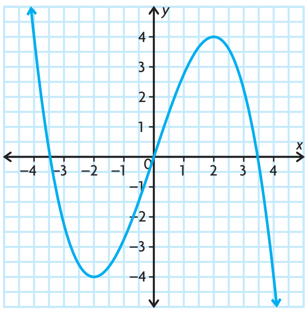
i.



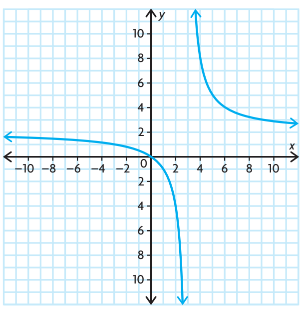
j.



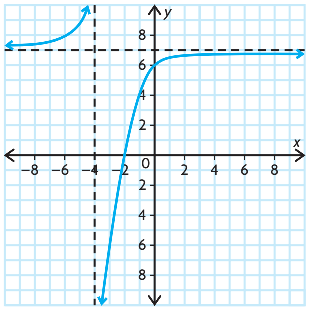
6. $a = -\frac{1}{4}, b = 0, c = 3, d = 0$



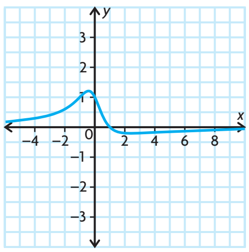
7. a. Answers may vary. For example:



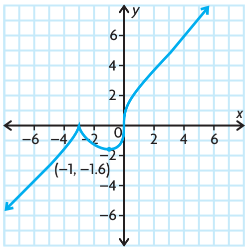
b. Answers may vary. For example:



8.



9.



10. a. $y = 1$ is a horizontal asymptote to the right-hand branch of the graph. $y = -1$ is a horizontal asymptote to the left-hand branch of the graph.

b. $y = \frac{3}{2}$ and $y = -\frac{3}{2}$ are horizontal asymptotes.

11. $y = ax^3 + bx^2 + cx + d$

$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$

$$\frac{d^2y}{dx^2} = 6ax + 2b = 6a\left(x + \frac{b}{3a}\right)$$

For possible points of inflection, we solve $\frac{d^2y}{dx^2} = 0$:

$$x = -\frac{b}{3a}$$

The sign of $\frac{d^2y}{dx^2}$ changes as x goes from values less than $\frac{-b}{3a}$ to values greater than $\frac{-b}{3a}$. Thus, there is a point of inflection at $x = \frac{-b}{3a}$.

At $x = \frac{b}{3a}, \frac{dy}{dx} = 3a\left(\frac{-b}{3a}\right)^2 + 2b\left(\frac{-b}{3a}\right) + c = c - \frac{b^2}{3a}$

Review Exercise, pp. 216–219

1. a. i. $x < 1$
ii. $x > 1$
iii. $(1, 20)$
b. i. $x < -3, -3 < x < 1, x > 6.5$
ii. $1 < x < 3, 3 < x < 6.5$
iii. $(1, -1), (6.5, -1)$

2. No, a counter example is sufficient to justify the conclusion. The function $f(x) = x^3$ is always increasing, yet the graph is concave down for $x < 0$ and concave up for $x > 0$.

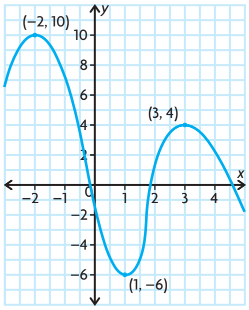
3. a. $(0, 20)$, local minimum; tangent is horizontal
b. $(0, 6)$, local maximum; tangent is horizontal
$(3, 33)$, neither local maximum nor minimum; tangent is horizontal
c. $\left(-1, -\frac{1}{2}\right)$, local minimum;
$\left(7, \frac{1}{14}\right)$, local maximum; tangents at both points are parallel
d. $(1, 0)$, neither local maximum nor minimum; tangent is not horizontal

4. a. $a < x < b, x > e$
b. $b < x < c$
c. $x < a, d < x < e$
d. $c < x < d$

5. a. $x = 3$; large and negative to left of asymptote, large and positive to right of asymptote
b. $x = -5$; large and positive to left of asymptote, large and negative to right of asymptote
c. hole at $x = -3$
d. $x = -4$; large and positive to left of asymptote, large and negative to right of asymptote
$x = 5$; large and negative to left of asymptote, large and positive to right of asymptote

6. $(0, 5)$; Since the derivative is 0 at $x = 0$, the tangent line is parallel to the x-axis at that point. Because the derivative is always positive, the function is always increasing and, therefore, must cross the tangent line instead of just touching it.

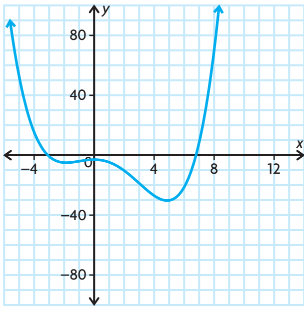
7.



8. a. i. concave up: $-1 < x < 3$; concave down: $x < -1$, $3 < x$

ii. points of inflection: $x = -1$, $x = 5$

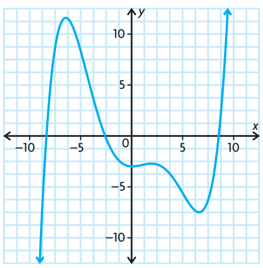
iii.



b. i. concave up: $-4.5 < x < 1$, $5 < x$
concave down: $x < -4.5$, $1 < x < 5$

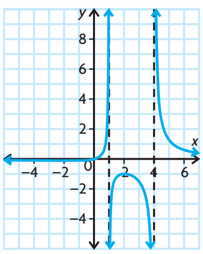
ii. points of inflection: $x = -4.5$, $x = 1$, and $x = 5$

iii.

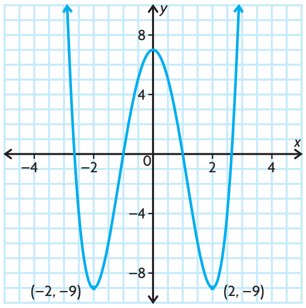


9. a. $a = 1, b = 0$

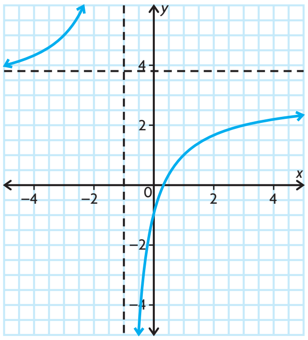
b.



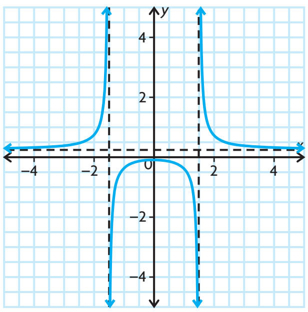
10. a.



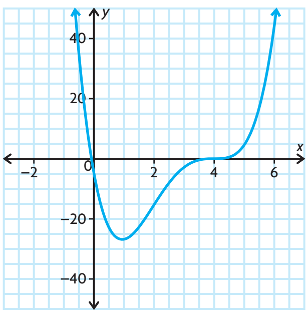
b.



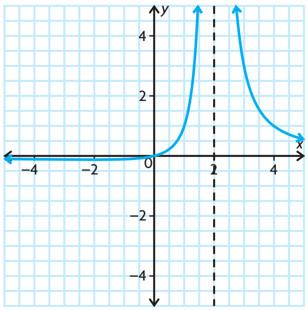
c.



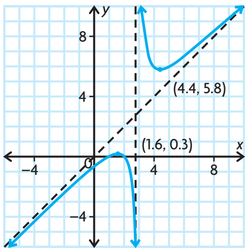
d.



e.



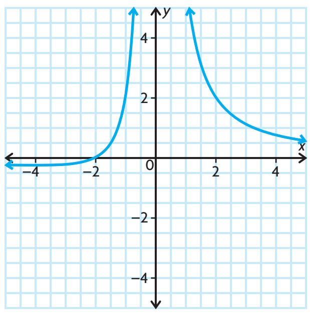
f.



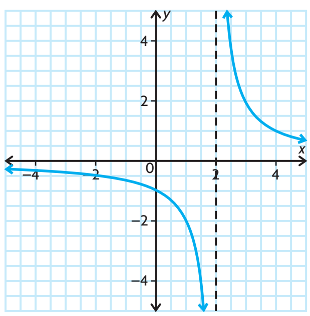
11. a. $-2 \le k \le 2, x \ne \pm k$

b. There are three different graphs that result for values of k chosen.

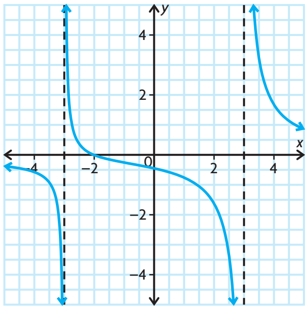
$k = 0$



$k = 2$



For all other values of k, the graph will be similar to the graph below.



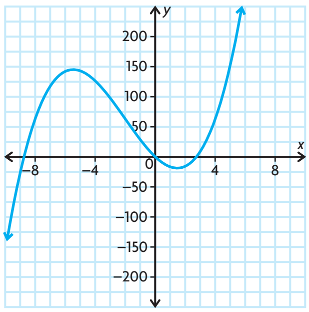
12. **a.** $y = x - 3$

b. $y = 4x + 11$

13. $x = -2, x = 0, x = 2$;
increasing: $-2 < x < 0, x > 2$;
decreasing: $x < -2, 0 < x < 2$

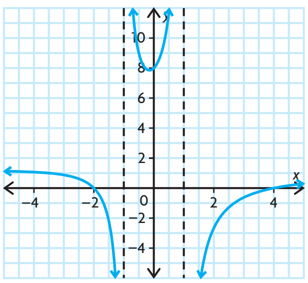
14. local maximum: $(-2.107, 17.054)$,
local minimum: $(1.107, 0.446)$,
absolute maximum: $(3, 24.5)$,
absolute minimum: $(-4, -7)$

15.

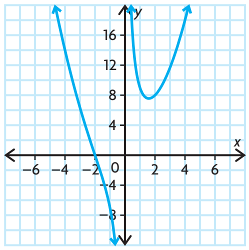


16. **a.** $p(x)$: oblique asymptote, $y = 0.75x$
$q(x)$: vertical asymptotes at $x = -1$ and $x = 3$; horizontal asymptote at $y = 0$
$r(x)$: vertical asymptotes at $x = -1$ and $x = 1$; horizontal asymptote at $y = 1$
$s(x)$: vertical asymptote at $y = 2$

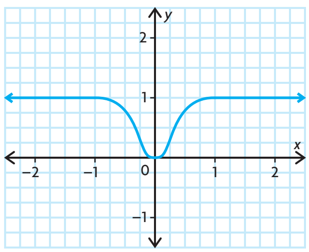
b.



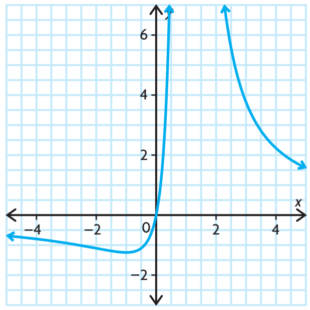
17. Domain: $\{x \in \mathbf{R} | x \neq 0\}$;
x-intercept: -2;
y-intercept: 8;
vertical asymptote: $x = 0$; large and negative to the left of the asymptote, large and positive to the right of the asymptote;
no horizontal or oblique asymptote;
increasing: $x > 1.59$;
decreasing: $x < 0, 0 < x < 1.59$;
concave up: $x < -2, x > 0$;
concave down: $-2 < x < 0$;
local minimum at $(1.59, 7.56)$;
point of inflection at $(-2, 0)$



18. If $f(x)$ is increasing, then $f'(x) > 0$. From the graph of f', $f'(x) > 0$ for $x > 0$. If $f(x)$ is decreasing, then $f'(x) < 0$. From the graph of f', $f'(x) < 0$ for $x < 0$. At a stationary point, $x = 0$. From the graph, the zero for $f'(x)$ occurs at $x = 0$. At $x = 0$, $f'(x)$ changes from negative to positve, so f has a local minimum point there. If the graph of f is concave up, then f'' is positive. From the slope of f', the graph of f is concave up for $-0.6 < x < 0.6$. If the graph of f is concave down, then f'' is negative. From the slope of f', the graph of f is concave down for $x < -0.6$ and $x > 0.6$. Graphs will vary slightly.



19. domain: $\{x \in \mathbf{R} | x \neq 1\}$;
x-intercept and y-intercept: $(0, 0)$;
vertical asymptote: $x = 1$; large and positive on either side of the asymptote;
horizontal asymptote: $y = 0$;
increasing: $-1 < x < 1$;
decreasing: $x < -1, x > 1$;
concave down: $x < -2$;
concave up: $-2 < x < 1, x > 1$;
local minimum at $(-1, -1.25)$;
point of inflection: $(-2, -1.11)$



20. **a.** Graph A is f, graph C is f', and graph B is f''. We know this because when you take the derivative, the degree of the denominator increases by one. Graph A has a squared term in the denominator, graph C has a cubic term in the denominator, and graph B has a term to the power of four in the denominator.

b. Graph F is f, graph E is f' and graph D is f''. We know this because the degree of the denominator increases by one degree when the derivative is taken.

Chapter 4 Test, p. 220

1. **a.** $x < -9$ or $-6 < x < -3$ or $0 < x < 4$ or $x > 8$

b. $-9 < x < -6$ or $-3 < x < 0$ or $4 < x < 8$

c. $(-9, 1), (-6, -2), (0, 1), (8, -2)$

d. $x = -3, x = 4$

e. $f''(x) > 0$

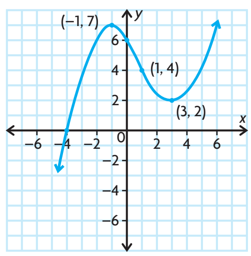
f. $-3 < x < 0$ or $4 < x < 8$

g. $(-8, 0), (10, -3)$

2. **a.** $x = 3$ or $x = -\frac{1}{2}$ or $x = \frac{1}{2}$

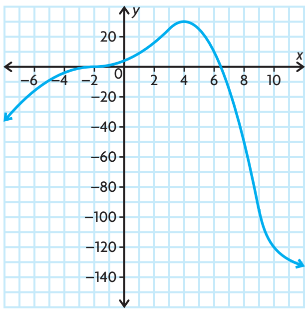
b. $\left(-\frac{1}{2}, -\frac{17}{8}\right)$: local maximum
$\left(\frac{1}{2}, \frac{15}{8}\right)$: local maximum
$(3, -45)$: local minimum

3.



4. hole at $x = -2$; large and negative to left of asymptote, large and positive to right of asymptote;
$y = 1$;
Domain: $\{x \in \mathbf{R} | x \neq -2, x \neq 3\}$

5.



6. There are discontinuities at $x = -3$ and $x = 3$.

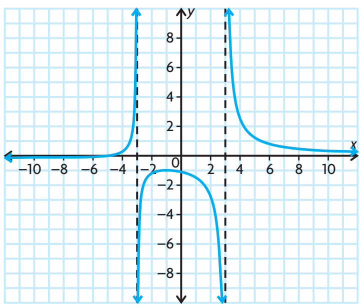
$$\left.\begin{aligned}\lim_{x\to 3^-} f(x) &= \infty\\ \lim_{x\to 3^+} f(x) &= -\infty\end{aligned}\right\} x = -3 \text{ is a vertical asymptote.}$$

$$\left.\begin{aligned}\lim_{x\to 3^-} f(x) &= -\infty\\ \lim_{x\to 3^+} f(x) &= \infty\end{aligned}\right\} x = 3 \text{ is a vertical asymptote.}$$

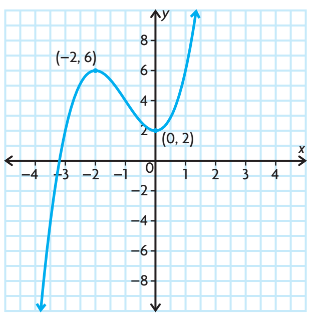
The y-intercept is $-\frac{10}{9}$ and x-intercept is -5.

$\left(-9, -\frac{1}{9}\right)$ is a local minimum and $(-1, -1)$ is a local maximum.

$y = 0$ is a horizontal asymptote.



7. $b = 3, c = 2$



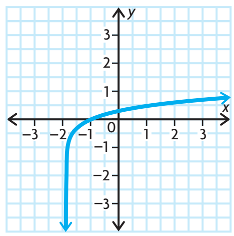
Chapter 5

Review of Prerequisite Skills, pp. 224–225

1. **a.** $\frac{1}{9}$ **b.** 4 **c.** $\frac{1}{9}$ **d.** $\frac{9}{4}$

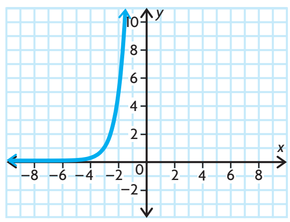
2. **a.** $\log_5 625 = 4$
b. $\log_4 \frac{1}{16} = -2$
c. $\log_x 3 = 3$
d. $\log_{10} 450 = w$
e. $\log_3 z = 8$
f. $\log_a T = b$

3. **a.**



x-intercept: $(-1, 0)$

b.



no x-intercept

4. **a.** $\frac{y}{r}$ **b.** $\frac{x}{r}$ **c.** $\frac{y}{x}$

5. **a.** 2π **b.** $\frac{\pi}{4}$ **c.** $-\frac{\pi}{2}$ **d.** $\frac{\pi}{6}$ **e.** $\frac{3\pi}{2}$ **f.** $-\frac{2\pi}{3}$ **g.** $\frac{5\pi}{4}$ **h.** $\frac{11\pi}{6}$

6. **a.** b **b.** $\frac{b}{a}$ **c.** a **d.** a **e.** b **f.** $-b$

7. **a.** $\cos\theta = -\frac{12}{13}$,
$\tan\theta = -\frac{5}{12}$

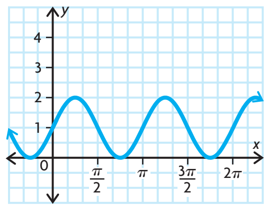
b. $\sin\theta = -\frac{\sqrt{5}}{3}$,
$\tan\theta = \frac{\sqrt{5}}{2}$

c. $\sin\theta = -\frac{2}{\sqrt{5}}$,
$\cos\theta = \frac{1}{\sqrt{5}}$

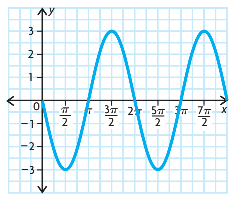
d. $\cos\frac{\pi}{2} = 0$,
$\tan\frac{\pi}{2}$ is undefined

8. **a.** period: π, amplitude: 1
b. period: 4π, amplitude: 2
c. period: 2, amplitude: 3
d. period: $\frac{\pi}{6}$, amplitude: $\frac{2}{7}$
e. period: 2π, amplitude: 5
f. period: 2π, amplitude: $\frac{3}{2}$

9. **a.**



b.

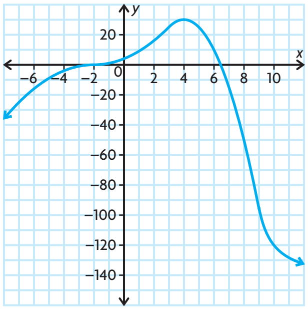


10. **a.** $\tan x + \cot x = \sec x \csc x$

$$\begin{aligned}\text{LS} &= \tan x + \cot x\\ &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\\ &= \frac{\sin^2 x + \cos^2 x}{\cos x + \sin x}\\ &= \frac{1}{\cos x + \sin x}\\ \text{RS} &= \sec x \csc x\\ &= \frac{1}{\cos x} \times \frac{1}{\sin x}\\ &= \frac{1}{\cos x \sin x}\end{aligned}$$

Therefore, $\tan x + \cot x = \sec x \csc x$.

4. hole at $x = -2$; large and negative to left of asymptote, large and positive to right of asymptote;
$y = 1$;
Domain: $\{x \in \mathbf{R} | x \neq -2, x \neq 3\}$

5. 

6. There are discontinuities at $x = -3$ and $x = 3$.

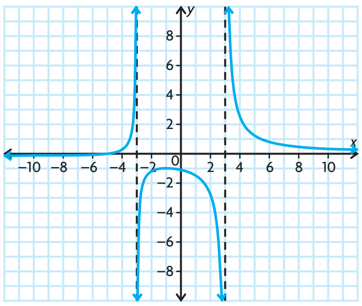
$$\left.\begin{array}{l}\lim_{x\to 3^-} f(x) = \infty \\ \lim_{x\to 3^+} f(x) = -\infty\end{array}\right\} x = -3 \text{ is a vertical asymptote.}$$

$$\left.\begin{array}{l}\lim_{x\to 3^-} f(x) = -\infty \\ \lim_{x\to 3^+} f(x) = \infty\end{array}\right\} x = 3 \text{ is a vertical asymptote.}$$

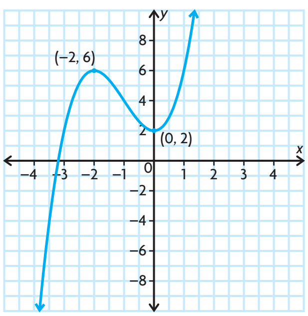
The y-intercept is $-\frac{10}{9}$ and x-intercept is -5.

$\left(-9, -\frac{1}{9}\right)$ is a local minimum and $(-1, -1)$ is a local maximum.

$y = 0$ is a horizontal asymptote.



7. $b = 3, c = 2$

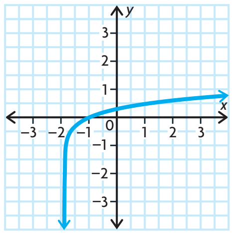


Chapter 5

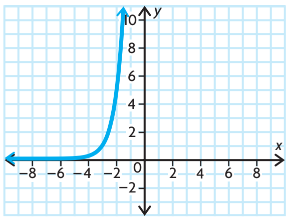
Review of Prerequisite Skills, pp. 224–225

1. a. $\frac{1}{9}$ c. $\frac{1}{9}$
b. 4 d. $\frac{9}{4}$

2. a. $\log_5 625 = 4$
b. $\log_4 \frac{1}{16} = -2$
c. $\log_x 3 = 3$
d. $\log_{10} 450 = w$
e. $\log_3 z = 8$
f. $\log_a T = b$

3. a. 

x-intercept: $(-1, 0)$

b. 

no x-intercept

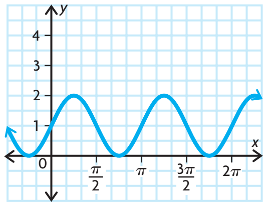
4. a. $\frac{y}{r}$ b. $\frac{x}{r}$ c. $\frac{y}{x}$

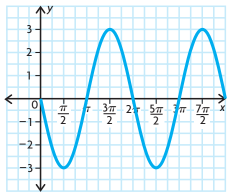
5. a. 2π e. $\frac{3\pi}{2}$
b. $\frac{\pi}{4}$ f. $-\frac{2\pi}{3}$
c. $-\frac{\pi}{2}$ g. $\frac{5\pi}{4}$
d. $\frac{\pi}{6}$ h. $\frac{11\pi}{6}$

6. a. b c. a e. b
b. $\frac{b}{a}$ d. a f. $-b$

7. a. $\cos\theta = -\frac{12}{13}$,
$\tan\theta = -\frac{5}{12}$
b. $\sin\theta = -\frac{\sqrt{5}}{3}$,
$\tan\theta = \frac{\sqrt{5}}{2}$
c. $\sin\theta = -\frac{2}{\sqrt{5}}$,
$\cos\theta = \frac{1}{\sqrt{5}}$
d. $\cos\frac{\pi}{2} = 0$,
$\tan\frac{\pi}{2}$ is undefined

8. a. period: π,
amplitude: 1
b. period: 4π,
amplitude: 2
c. period: 2,
amplitude: 3
d. period: $\frac{\pi}{6}$,
amplitude: $\frac{2}{7}$
e. period: 2π,
amplitude: 5
f. period: 2π,
amplitude: $\frac{3}{2}$

9. a. 

b. 

10. a. $\tan x + \cot x = \sec x \csc x$

$$\begin{aligned}
\text{LS} &= \tan x + \cot x \\
&= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \\
&= \frac{\sin^2 x + \cos^2 x}{\cos x + \sin x} \\
&= \frac{1}{\cos x + \sin x} \\
\text{RS} &= \sec x \csc x \\
&= \frac{1}{\cos x} \times \frac{1}{\sin x} \\
&= \frac{1}{\cos x \sin x}
\end{aligned}$$

Therefore, $\tan x + \cot x = \sec x \csc x$.

b. $\dfrac{\sin x}{1-\sin^2 x} = \tan x + \sec x$

$$\begin{aligned}
\text{LS} &= \frac{\sin x}{1-\sin^2 x}\\
&= \frac{\sin x}{\cos^2 x}\\
\text{RS} &= \tan x \sec x\\
&= \frac{\sin x}{\cos x} \times \frac{1}{\cos x}\\
&= \frac{\sin x}{\cos^2 x}
\end{aligned}$$

Therefore, $\dfrac{\sin x}{1-\sin^2 x} = \tan x \sec x$.

11. **a.** $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

b. $x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

Section 5.1, pp. 232–234

1. You can only use the power rule when the term containing variables is in the base of the exponential expression. In the case of $y = e^x$, the exponent contains a variable.

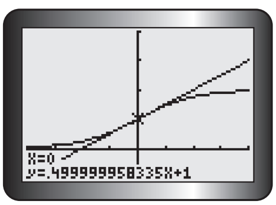
2. **a.** $3e^{3x}$
b. $3e^{3t-5}$
c. $20e^{10t}$
d. $-3e^{-3x}$
e. $(-6+2x)e^{5-6x+x^2}$
f. $\dfrac{1}{2\sqrt{x}}e^{\sqrt{x}}$

3. **a.** $6x^2e^{x^3}$
b. $e^{3x}(3x+1)$
c. $\dfrac{-3x^2e^{-x^3}(x) - e^{-x^3}}{x^2}$
d. $\sqrt{x}e^x + e^x\left(\dfrac{1}{2\sqrt{x}}\right)$
e. $2te^{t^2} - 3e^{-t}$
f. $\dfrac{2e^{2t}}{(1+e^{2t})^2}$

4. **a.** $e^3 - e^{-3}$
b. $\dfrac{1}{e}$
c. $-2-3e$

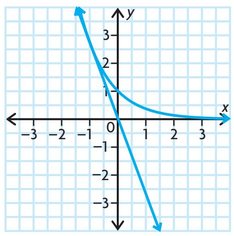
5. **a.** $y = \dfrac{1}{2}x + 1$

b.



c. The answers agree very well; the calculator does not show a slope of exactly 0.5, due to internal rounding.

6. $ex + y = 0$



7. $y = \dfrac{1}{e}$

8. $(0, 0)$ and $\left(2, \dfrac{4}{e^2}\right)$

9. If $y = \dfrac{5}{2}(e^{\frac{x}{5}} + e^{-\frac{x}{5}})$, then

$$y' = \frac{5}{2}\left(\frac{1}{5}e^{\frac{x}{5}} - \frac{1}{5}e^{-\frac{x}{5}}\right), \text{ and}$$

$$\begin{aligned}
y'' &= \frac{5}{2}\left(\frac{1}{25}e^{\frac{x}{5}} + \frac{1}{25}e^{-\frac{x}{5}}\right)\\
&= \frac{1}{25}\left[\frac{5}{2}\left(e^{\frac{x}{5}} + e^{-\frac{x}{5}}\right)\right]\\
&= \frac{1}{25}y
\end{aligned}$$

10. **a.** $\dfrac{dy}{dx} = -3e^{-3x}$,
$\dfrac{d^2y}{dx^2} = 9e^{-3x}$,
$\dfrac{d^3y}{dx^3} = -27e^{-3x}$

b. $\dfrac{d^n y}{dx^n} = (-1)^n(3^n)\,e^{-3x}$

11. **a.** $\dfrac{dy}{dx} = -3e^{3x}$,
$\dfrac{d^2y}{dx^2} = -3e^x$

b. $\dfrac{dy}{dx} = e^{2x}(2x+1)$,
$\dfrac{d^2y}{dx^2} = 4xe^{2x} + 4e^{2x}$

c. $\dfrac{dy}{dx} = e^x(3-x)$,
$\dfrac{d^2y}{dx^2} = e^x(2-x)$

12. **a.** 31 000
b. $-\dfrac{100}{3}e^{-\frac{t}{30}}$
c. -17 bacteria/h
d. 31 000 at time $t = 0$
e. The number of bacteria is constantly decreasing as time passes.

13. **a.** $40\left(1 - e^{-\frac{t}{4}}\right)$

b. $a = \dfrac{dv}{dt} = 40\left(\dfrac{1}{4} - e^{-\frac{t}{4}}\right) = 10e^{-\frac{t}{4}}$

From **a**, $v = 40\left(1 - e^{-\frac{t}{4}}\right)$, which gives $e^{\frac{t}{4}} = 1 - \dfrac{v}{40}$. Thus,

$a = 10\left(1 - \dfrac{v}{40}\right) = 10 - \dfrac{1}{4}v$.

c. 40 m/s
d. about 12 s, about 327.3 m

14. **a.** **i.** e
ii. e
b. The limits have the same value because as $x \to \infty, \frac{1}{x} \to 0$.

15. **a.** 1
b. e^2

16. $m = -3$ or $m = 2$

17. **a.**
$$\begin{aligned}
\frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left[\frac{1}{2}(e^x - e^{-x})\right]\\
&= \frac{1}{2}(e^x + e^{-x})\\
&= \cosh x
\end{aligned}$$

b.
$$\begin{aligned}
\frac{d}{dx}(\cosh x) &= \frac{1}{2}(e^t - e^{-t})\\
&= \sinh x
\end{aligned}$$

c. Since $\tanh x = \dfrac{\sinh x}{\cosh x}$,

$$\begin{aligned}
\frac{d}{dx}(\tanh x) &= \frac{\left(\frac{d}{dx}\sinh x\right)(\cosh x)}{(\cosh x)^2} - \frac{(\sinh x)\left(\frac{d}{dx}\cosh x\right)}{(\cosh x)^2}\\
&= \frac{\frac{1}{2}(e^x + e^{-x})\left(\frac{1}{2}\right)(e^x + e^{-x})}{(\cosh x)^2} - \frac{\frac{1}{2}(e^x - e^{-x})\left(\frac{1}{2}\right)(e^x - e^{-x})}{(\cosh x)^2}\\
&= \frac{\frac{1}{4}[(e^{2x} + 2 + e^{-2x})}{(\cosh x)^2} - \frac{(e^{2t} - 2 + e^{-2t})]}{(\cosh x)^2}\\
&= \frac{\frac{1}{4}(4)}{(\cosh x)^2}\\
&= \frac{1}{(\cosh x)^2}
\end{aligned}$$

18. **a.** Four terms: $2.666\,66\overline{6}$
Five terms: $2.708\,33\overline{3}$
Six terms: $2.716\,66\overline{6}$
Seven terms: $2.718\,05\overline{5}$

b. The expression for e in part a is a special case of $e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ in that it is the case when $x = 1$. Then $e^x = e^1 = e$ is in fact $e^1 = e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$. The value of x is 1.

Section 5.2, p. 240

1. **a.** $3(2^{3x})\ln 2$
b. $\ln 3.1(3.1)^x + 3x^2$
c. $3(10^{3t-5})\ln 10$
d. $(-6 + 2n)(10^{5-6n+n^2})\ln 10$
e. $2x(3^{x^2+2})\ln 3$
f. $400(2)^{x+3}\ln 2$

2. **a.** $5^x[(x^5 \times \ln 5) + 5x^4]$
b. $(3)^{x^2}[(2x^2 \ln 3) + 1]$
c. $-\dfrac{2^t}{t^2} + \dfrac{2^t \ln 2}{t}$
d. $\dfrac{3^{\frac{x}{2}}[x \ln 3 - 4]}{x^3}$

3. $-\dfrac{3 \ln 10}{4}$

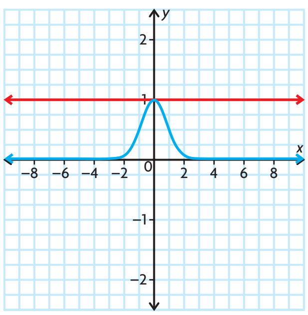
4. $-16.64x + y + 25.92 = 0$

5. $-23.03x + y + 13.03 = 0$

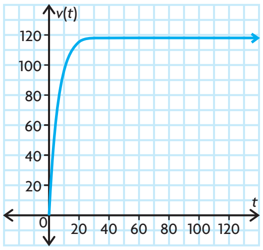
6. **a.** about 3.80 years
b. about -9.12%/year

7. **a.** In 1978, the rate of increase of debt payments was \$904 670 000/annum compared to \$122 250 000/annum in 1968. The rate of increase for 1978 is 7.4 times larger than that for 1968.
b. The rate of increase for 1998 is 7.4 times larger than that for 1988.
c. Answers may vary. For example, data from the past are not necessarily good indicators of what will happen in the future. Interest rates change, borrowing may decrease, principal may be paid off early.

8. $y = 1$



9.



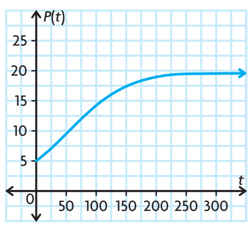
From the graph, the values of $v(t)$ quickly rise in the range of about $0 \le t \le 15$. The slope for these values is positive and steep. Then as the graph nears $t = 20$, the steepness of the slope decreases and seems to get very close to 0. One can reason that the car quickly accelerates for the first 20 units of time. Then, it seems to maintain a constant acceleration for the rest of the time. To verify this, one could differentiate and look at values where $v'(t)$ is increasing.

Section 5.3, pp. 245–247

1. **a.** absolute max: about 0.3849, absolute min: 0
b. absolute max: about 10.043, absolute min: about -5961.916

2. **a.** $f(x)$: max: 0.3849, min: 0; $m(x)$: max: about 10, min: about -5961
b. The graphing approach seems to be easier to use for the functions. It is quicker and it gives the graphs of the functions in a good viewing rectangle. The only problem may come in the second function, $m(x)$, because for $x < 1.5$, the function quickly approaches values in the negative thousands.

3. **a.** 500 squirrels
b. 2000 squirrels
c. (54.9, 10)
d.



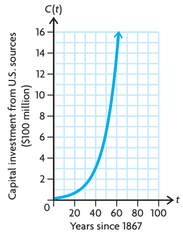
e. P grows exponentially until the point of inflection, then the growth rate decreases and the curve becomes concave down.

4. **a.** 1001 items
b. 500 items

5. 500 units

6. 47.2%; 0.462 h

7. **a.**



b. The growth rate of capital investment grew from 468 million dollars per year in 1947 to 2.112 billion dollars per year in 1967.
c. 7.5%
d. $C = 59.537 \times 10^9$ dollars, $\dfrac{dC}{dt} = 4.4849 \times 10^9$ dollars/year
e. Statistics Canada data shows the actual amount of U.S. investment in 1977 was 62.5×10^9 dollars. The error in the model is 3.5%.
f. $C = 570.490 \times 10^9$ dollars, $\frac{dC}{dt} = 42.975 \times 10^9$ dollars/year

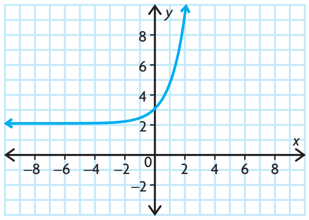
8. **a.** 478 158; 38.2 min after the drug was introduced
b. 42.72 min after the drug was introduced

9. 10 h of study should be assigned to the first exam and 20 h of study for the second exam.

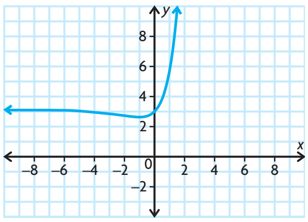
10. Use the algorithm for finding extreme values. First, find the derivate $f'(x)$. Then find any critical points by setting $f'(x) = 0$ and solving for x. Also find the values of x for which $f'(x)$ is undefined. Together these are the critical values. Now evaluate $f(x)$ for the critical values and the endpoints 2 and -2. The highest value will be the absolute maximum on the interval, and the lowest value will be the absolute minimum on the interval.

11. **a.** $f(x)$ is increasing on the intervals $(-\infty, -2)$ and $(0, \infty)$. Also, $f(x)$ is decreasing on the interval $(-2, 0)$.
b. 0

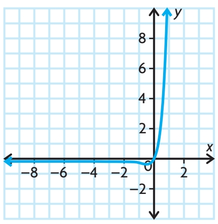
12. **a.** no maximum or minimum value



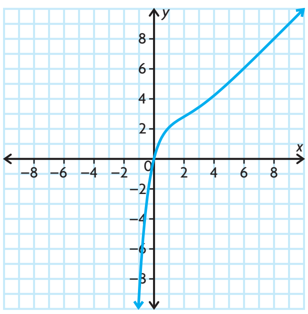
b. min: $-e^{-1} + 3 \doteq 2.63$, no max



c. min: $-e^{-1} \doteq -0.37$, no max

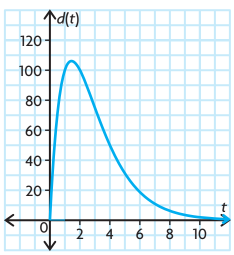


d. no max or min



13. about 0.61

14. **a.**



b. increasing when $0 < t < \frac{1}{\ln 2}$ and decreasing when $t > \frac{1}{\ln 2}$

c. about 106.15°/s

d. $t > 10$ s

15. The solution starts in a similar way to that of question 9. The effectiveness function is

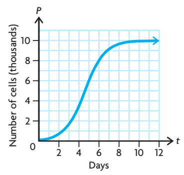
$$E(t) = 0.5\left(10 + te^{-\frac{t}{10}}\right) + 0.6\left(9 + (25 - t)e^{-\frac{25-t}{20}}\right)$$

The derivative simplifies to

$$E'(t) = 0.05e^{-\frac{t}{10}}(10 - t) + 0.03e^{-\frac{25-t}{10}}(5 - t)$$

This expression is very difficult to solve analytically. By calculation on a graphing calculator, we can determine that the maximum effectiveness occurs when $t = 8.16$ h.

16. **a.**

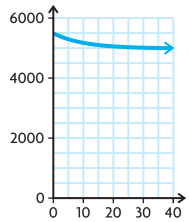


b. after 4.6 days, 5012

c. The rate of growth is slowing down as the colony is getting closer to its limiting value.

Mid-Chapter Review, pp. 248–249

1. **a.** $-15e^{-3x}$
   b. $e^{\frac{1}{3}x}$
   c. $e^{-2x}(-2x^3 + 3x^2)$
   d. $(e^x)(x^2 - 1)$
   e. $2(x + xe^{-x} - e^{-x} - e^{-2x})$
   f. $\dfrac{4}{(e^x + e^{-x})^2}$
2. **a.** $-500e^{-5t}$
   b. -2.5
3. $x + y - 2 = 0$
4. **a.** $y' = -3e^x$, $y'' = -3e^x$
   b. $y' = 2xe^{2x} + e^{2x}$, $y'' = 4xe^{2x} + 4e^{2x}$
   c. $y' = 3e^x - xe^x$, $y'' = 2e^x - xe^x$
5. **a.** $2(\ln 8)(8^{2x+5})$
   b. $0.64(\ln 10)((10)^{2x})$
   c. $2^x((\ln 2)(x^2) + 2x)$
   d. $900(\ln 5)(5)^{3x-1}$
   e. $(\ln 1.9)(1.9)^x + 1.9x^{0.9}$
   f. $4^x((\ln 4)(x - 2)^2 + 2x - 4)$
6. **a.** 5500
   b. $-50\left(e^{-\frac{t}{10}}\right)$
   c. decreasing by about 15 rabbits/month
   d. 5500
   e.



The graph is constantly decreasing. The y-intercept is (0, 5500). Rabbit populations normally grow exponentially, but this population is shrinking exponentially. Perhaps a large number of rabbit predators, such as snakes, recently began to appear in the forest. A large number of predators would quickly shrink the rabbit population.

7. at about 0.41 h
8. The original function represents growth when $ck > 0$, meaning that c and k must have the same sign. The original function represents decay when c and k have opposite signs.
9. **a.** 5000
   b. 5751
   c. 9111
10. **a.** 406.80 mm Hg
    b. 316.82 mm Hg
    c. 246.74 mm Hg
11. 15% per year
12. $f(x) = xe^x$

$$f'(x) = xe^x + (1)e^x = e^x(x + 1)$$

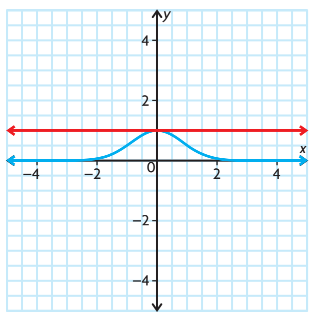
So, $e^x > 0$

$x + 1 > 0$

$x > -1$

This means that the function is increasing when $x > -1$.

13.



14. **a.** $A(t) = 1000(1.06)^t$
    b. $A'(t) = 1000(1.06)^t \ln 1.06$
    c. $A'(2) = \$65.47$, $A'(5) = \$77.98$, $A'(10) = \$104.35$
    d. No

e. $\dfrac{A'(2)}{A(2)} = \ln 1.06,$
$\dfrac{A'(5)}{A(5)} = \ln 1.06,$
$\dfrac{A'(10)}{A(10)} = \ln 1.06$

f. All the ratios are equivalent (they equal $\ln 1.06$, which is about 0.058 27), which means that $\frac{A'(t)}{A(t)}$ is constant.

15. $y = ce^x$
$y' = c(e^x) + (0)(e^x)$
$= ce^x$
$y = y' = ce^x$

Section 5.4, pp. 256–257

1. a. $2\cos 2x$
b. $-6\sin 3x$
c. $(3x^2 - 2)(\cos(x^3 - 2x + 4))$
d. $8\sin(-4x)$
e. $3\cos(3x) + 4\sin(4x)$
f. $2^x(\ln 2) + 2\cos x + 2\sin x$
g. $e^x\cos(e^x)$
h. $9\cos(3x + 2\pi)$
i. $2x - \sin x$
j. $-\dfrac{1}{x^2}\cos\left(\dfrac{1}{x}\right)$

2. a. $2\cos(2x)$
b. $\dfrac{2\sin 2x}{x} - \dfrac{\cos 2x}{x^2}$
c. $-\sin(\sin 2x) \times 2\cos 2x$
d. $\dfrac{1}{1 + \cos x}$
e. $e^x(2\cos x)$
f. $2x^3\cos x + 6x^2\sin x$
$+ 3x\sin x - 3\cos x$

3. a. $-x + 2y + \left(\dfrac{\pi}{3} - \sqrt{3}\right) = 0$
b. $-2x + y = 0$
c. $y = -1$
d. $y = -3\left(x - \dfrac{\pi}{2}\right)$
e. $y + \dfrac{\sqrt{3}}{2} = -\left(x - \dfrac{\pi}{4}\right)$
f. $2x + y - \pi = 0$

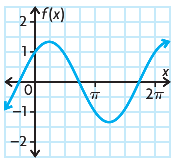
4. a. One could easily find $f'(x)$ and $g'(x)$ to see that they both equal $2(\sin x)(\cos x)$. However, it is easier to notice a fundamental trigonometric identity. It is known that $\sin^2 x + \cos^2 x = 1$. So, $\sin^2 x = 1 - \cos^2 x$.
Therefore, $f(x)$ is in fact equal to $g(x)$. So, because $f(x) = g(x)$, $f'(x) = g'(x)$.

b. $f'(x)$ and $g'(x)$ are each others' negative. That is,
$f'(x) = (\sin x)(\cos x)$, while
$g'(x) = -2(\sin x)(\cos x)$.

5. a. $v'(t) = \dfrac{\sin(\sqrt{t})\cos(\sqrt{t})}{\sqrt{t}}$
b. $v'(t) = \dfrac{-\sin t + 2(\sin t)(\cos t)}{2\sqrt{1 + \cos t + \sin^2 t}}$
c. $h'(x) = 3\sin x\sin 2x\cos 3x$
$+ 2\sin x\sin 3x\cos 2x$
$+ \sin 2x\sin 3x\cos x$
d. $m'(x) = 3(x^2 + \cos^2 x)^2$
$\times (2x - 2\sin x\cos x)$

6. a. absolute max: $\sqrt{2}$,
absolute min: $-\sqrt{2}$
b. absolute max: 2.26,
absolute min: -5.14
c. absolute max: $\sqrt{2}$,
absolute min: $-\sqrt{2}$
d. absolute max: 5,
absolute min: -5

7. a. $t = \frac{\pi}{4} + \pi k, \frac{3\pi}{4} + \pi k$ for positive integers k
b. 8

8. a.


b. $\left(\dfrac{\pi}{4}, \sqrt{2}\right)$

9. $(\csc x)' = -\csc x\cot x$,
$(\sec x)' = \sec x\tan x$

10. $-\sqrt{3}$

11. a. $t = \frac{\pi}{4} + \pi k, \frac{3\pi}{4} + \pi k$ for positive integers k
b. 4
c. minimum: 0 maximum: 4

12. $\theta = \dfrac{\pi}{3}$

13. $\theta = \dfrac{\pi}{6}$

14. First find y''.
$y = A\cos kt + B\sin kt$
$y' = -kA\sin kt + kB\cos kt$
$y'' = -k^2A\cos kt - k^2B\sin kt$
So, $y'' + k^2y$
$= -k^2A\cos kt - k^2B\sin kt$
$+ k^2(A\cos kt + B\sin kt)$
$= -k^2A\cos kt - k^2B\sin kt$
$+ k^2A\cos kt + k^2B\sin kt$
$= 0$
Therefore, $y'' + k^2y = 0$.

Section 5.5, p. 260

1. a. $3\sec^2 3x$
b. $2\sec^2 x - 2\sec 2x$
c. $6x^2\tan(x^2)\sec^2(x^3)$
d. $\dfrac{x(2\tan\pi x - \pi x\sec^2\pi x)}{\tan^2\pi x}$
e. $2x\sec^2(x^2) - 2\tan x\sec^2 x$
f. $15(\tan 5x\cos 5x + \sin 5x\sec^2 5x)$

2. a. $y = 2\left(x - \dfrac{\pi}{4}\right)$
b. $y = -2x$

3. a. $\cos x\sec^2(\sin x)$
b. $-4x[\tan(x^2 - 1)]^{-3}\sec^2(x^2 - 1)$
c. $-2\tan(\cos x)\sec^2(\cos x)\sin x$
d. $2(\tan x + \cos x)(\sec^2 x - \sin x)$
e. $\sin^2 x(3\tan x\cos x + \sin x\sec^2 x)$
f. $\dfrac{1}{2\sqrt{x}}e^{\tan\sqrt{x}}\sec^2 2\sqrt{x}$

4. a. $\cos x + \sec x + \dfrac{2\sin^2 x}{\cos^3 x}$
b. $2\sec^2 x(1 + 3\tan^2 x)$

5. $x = 0, \pi$, and 2π

6. $\left(\dfrac{\pi}{4}, 0.57\right)$

7. $y = \sec x + \tan x$
$= \dfrac{1}{\cos x} + \dfrac{\sin x}{\cos x}$
$= \dfrac{1 + \sin x}{\cos x}$
$\dfrac{dy}{dx} = \dfrac{\cos^2 x - (1 + \sin x)(-\sin x)}{\cos^2 x}$
$= \dfrac{\cos^2 x - (-\sin x - \sin^2 x)}{\cos^2 x}$
$= \dfrac{\cos^2 x + \sin x + \sin^2 x}{\cos^2 x}$
$= \dfrac{1 + \sin x}{\cos^2 x}$
The denominator is never negative.
$1 + \sin x > 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, since $\sin x$ reaches its minimum of -1 at $x = \frac{\pi}{2}$. Since the derivative of the original function is always positive in the specified interval, the function is always increasing in that interval.

8. $-4x + y - (2 - \pi) = 0$

9. Write $\tan x = \frac{\sin x}{\cos x}$ and use the quotient rule to derive the derivative of the tangent function.

10. $-\csc^2 x$

11. $f''(x) = 8\csc^2 x\cot x$

Review Exercise, pp. 263–265

1. a. $-e^x$
   b. $2 + 3e^x$
   c. $2e^{2x+3}$
   d. $(-6x + 5)e^{-3x^2+5x}$
   e. $e^x(x + 1)$
   f. $\dfrac{2e^t}{(e^t + 1)^2}$
2. a. $10^x \ln 10$
   b. $6x(4^{3x^2})\ln 4$
   c. $5 \times 5^x(x \ln 5 + 1)$
   d. $x^3 \times 2^x(x \ln 2 + 4)$
   e. $\dfrac{4 - 4x \ln 4}{4^x}$
   f. $5^{\sqrt{x}}\left(-\dfrac{1}{x^2} + \dfrac{\ln 5}{2x\sqrt{x}}\right)$
3. a. $6 \cos (2x) + 8 \sin (2x)$
   b. $3 \sec^2 (3x)$
   c. $-\dfrac{\sin x}{(2 - \cos x)^2}$
   d. $2x \sec^2 (2x) + \tan 2x$
   e. $e^{3x}(3 \sin 2x + 2 \cos 2x)$
   f. $-4 \cos (2x)\sin (2x)$
4. a. $x = 1$
   b. The function has a horizontal tangent at $(1, e)$. So this point could be possible local max or min.
5. a. 0
   b. The slope of the tangent to $f(x)$ at the point with x-coordinate $\frac{1}{2}$ is 0.
6. a. $e^x(x + 1)$
   b. $20e^{10x}(5x + 1)$
7. $y = \dfrac{e^{2x} - 1}{e^{2x} + 1}$

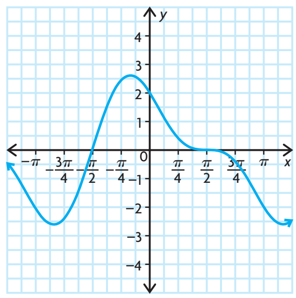
$$\frac{dy}{dx} = \frac{2e^{2x}(e^{2x} + 1) - (e^{2x} - 1)(2e^{2x})}{(e^{2x} + 1)^2}$$
$$= \frac{2e^{4x} + 2e^{2x} - 2e^{4x} + 2e^{2x}}{(e^{2x} + 1)^2}$$
$$= \frac{4e^{2x}}{(e^{2x} + 1)^2}$$

Now, $1 - y^2 = 1 - \dfrac{e^{4x} - 2e^{2x} + 1}{(e^{2x} + 1)^2}$
$$= \frac{e^{4x} + 2e^{2x} + 1 - e^{4x} + 2e^{2x} - 1}{(e^{2x} + 1)^2}$$
$$= \frac{4e^{2x}}{(3^{2x} + 1)^2} = \frac{dy}{dx}$$

8. $3x - y + 2\ln 2 - 2 = 0$
9. $-x + y = 0$
10. about 0.3928 m per unit of time
11. a. $t = 20$
    b. After 10 days, about 0.1156 mice are infected per day. Essentially, almost 0 mice are infected per day when $t = 10$.
12. a. c_2
    b. c_1
13. a. $-9e^{-x}(2 + 3e^{-x})^2$
    b. ex^{e-1}
    c. e^{x+e^x}
    d. $-25e^{5x}(1 - e^{5x})^4$
14. a. $5^x \ln 5$
    b. $(0.47)^x \ln (0.47)$
    c. $2(52)^{2x} \ln 52$
    d. $5(2)^x \ln 2$
    e. $4e^x$
    f. $-6(10)^{3x} \ln 10$
15. a. $2^x \ln 2 \cos 2^x$
    b. $x^2 \cos x + 2x \sin x$
    c. $-\cos \left(\dfrac{\pi}{2} - x\right)$
    d. $\cos^2 x - \sin^2 x$
    e. $-2 \cos x \sin x$
    f. $2 \sin x \cos^2 x - \sin^3 x$
16. $x + y - \dfrac{\pi}{2} = 0$
17. $v = \frac{ds}{dt}$;
    Thus, $v = 8(\cos (10\pi t))(10\pi)$
    $= 80\pi \cos (10\pi t)$
    The acceleration at any time t is
    $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.
    Hence, $a = 80\pi(-\sin (10\pi t))(10\pi)$
    $= -800\pi^2\sin (10\pi t)$. Now,
    $\frac{d^2s}{dt^2} + 100\pi^2 s = -800\pi^2 \sin (10\pi t)$
    $+ 100\pi^2(8 \sin (10\pi t)) = 0.$
18. displacement: 5,
    velocity: 10,
    acceleration: 20
19. each angle $\frac{\pi}{4}$ rad, or 45°
20. 4.5 m
21. 2.5 m
22. 5.19 ft
23. a. $f''(x) = -8 \sin^2 (x - 2)$
    $+ 8 \cos^2 (x - 2)$
    b. $f''(x) = (4 \cos x)(\sec^2 x \tan x)$
    $- 2 \sin x(\sec x)^2$

Chapter 5 Test, p. 266

1. a. $-4xe^{-2x^2}$
   b. $3e^{x^2+3x} \cdot \ln 3 \cdot (2x + 3)$
   c. $\frac{3}{2}[e^{3x} - e^{-3x}]$
   d. $2\cos x + 15\sin 5x$
   e. $6x\sin^2 (x^2)\cos (x^2)$
   f. $-\dfrac{\sec^2 \sqrt{1 - x}}{2\sqrt{1 - x}}$
2. $-6x + y = 2$,
   The tangent line is the given line.
3. $-2x + y = 1$
4. a. $a(t) = v'(t) = -10ke^{-kt}$
   $= -k(10e^{-kt})$
   $= -kv(t)$
   Thus, the acceleration is a constant multiple of the velocity. As the velocity of the particle decreases, the acceleration increases by a factor of k.
   b. 10 cm/s
   c. $\dfrac{\ln 2}{k}$; $-5k$
5. a. $f''(x) = 2(\sin^2 x - \cos^2 x)$
   b. $f''(x) = \csc x \cot^2 x$
   $+ \csc^3 x + \sin x$
6. absolute max: 1,
   absolute min: 0
7. 40.24
8. minimum: $\left(-4, -\frac{1}{e^4}\right)$, no maximum
9. a. $x = -\dfrac{\pi}{6}, -\dfrac{5\pi}{6}, \dfrac{\pi}{2}$
   b. increasing: $-\frac{5\pi}{6} < x < -\frac{\pi}{6}$;
   decreasing: $-\pi \le x < -\frac{5\pi}{6}$ and $-\frac{\pi}{6} < x < \pi$
   c. local maximum at $x = -\frac{\pi}{6}$; local minimum at $x = -\frac{5\pi}{6}$
   d.



Cumulative Review of Calculus, pp. 267–270

1. a. 16
   b. −2
   c. $\dfrac{1}{6}$
   d. $160\ln 2$
2. a. 13 m/s
   b. 15 m/s
3. $f(x) = x^3$
4. a. 19.6 m/s
   b. 19.6 m/s
   c. 53.655 m/s

Review Exercise, pp. 263–265

1. a. $-e^x$
   b. $2 + 3e^x$
   c. $2e^{2x+3}$
   d. $(-6x + 5)e^{-3x^2+5x}$
   e. $e^x(x + 1)$
   f. $\dfrac{2e^t}{(e^t + 1)^2}$
2. a. $10^x \ln 10$
   b. $6x(4^{3x^2})\ln 4$
   c. $5 \times 5^x(x \ln 5 + 1)$
   d. $x^3 \times 2^x(x \ln 2 + 4)$
   e. $\dfrac{4 - 4x \ln 4}{4^x}$
   f. $5^{\sqrt{x}}\left(-\dfrac{1}{x^2} + \dfrac{\ln 5}{2x\sqrt{x}}\right)$
3. a. $6 \cos(2x) + 8 \sin(2x)$
   b. $3 \sec^2(3x)$
   c. $-\dfrac{\sin x}{(2 - \cos x)^2}$
   d. $2x \sec^2(2x) + \tan 2x$
   e. $e^{3x}(3 \sin 2x + 2 \cos 2x)$
   f. $-4 \cos(2x)\sin(2x)$
4. a. $x = 1$
   b. The function has a horizontal tangent at $(1, e)$. So this point could be possible local max or min.
5. a. 0
   b. The slope of the tangent to $f(x)$ at the point with x-coordinate $\frac{1}{2}$ is 0.
6. a. $e^x(x + 1)$
   b. $20e^{10x}(5x + 1)$
7. $y = \dfrac{e^{2x} - 1}{e^{2x} + 1}$

   $\dfrac{dy}{dx} = \dfrac{2e^{2x}(e^{2x} + 1) - (e^{2x} - 1)(2e^{2x})}{(e^{2x} + 1)^2}$

   $= \dfrac{2e^{4x} + 2e^{2x} - 2e^{4x} + 2e^{2x}}{(e^{2x} + 1)^2}$

   $= \dfrac{4e^{2x}}{(e^{2x} + 1)^2}$

   Now, $1 - y^2 = 1 - \dfrac{e^{4x} - 2e^{2x} + 1}{(e^{2x} + 1)^2}$

   $= \dfrac{e^{4x} + 2e^{2x} + 1 - e^{4x} + 2e^{2x} - 1}{(e^{2x} + 1)^2}$

   $= \dfrac{4e^{2x}}{(3^{2x} + 1)^2} = \dfrac{dy}{dx}$
8. $3x - y + 2\ln 2 - 2 = 0$
9. $-x + y = 0$
10. about 0.3928 m per unit of time
11. a. $t = 20$
    b. After 10 days, about 0.1156 mice are infected per day. Essentially, almost 0 mice are infected per day when $t = 10$.
12. a. c_2
    b. c_1
13. a. $-9e^{-x}(2 + 3e^{-x})^2$
    b. ex^{e-1}
    c. e^{x+e^x}
    d. $-25e^{5x}(1 - e^{5x})^4$
14. a. $5^x \ln 5$
    b. $(0.47)^x \ln(0.47)$
    c. $2(52)^{2x} \ln 52$
    d. $5(2)^x \ln 2$
    e. $4e^x$
    f. $-6(10)^{3x} \ln 10$
15. a. $2^x \ln 2 \cos 2^x$
    b. $x^2 \cos x + 2x \sin x$
    c. $-\cos\left(\dfrac{\pi}{2} - x\right)$
    d. $\cos^2 x - \sin^2 x$
    e. $-2 \cos x \sin x$
    f. $2 \sin x \cos^2 x - \sin^3 x$
16. $x + y - \dfrac{\pi}{2} = 0$
17. $v = \frac{ds}{dt}$;

    Thus, $v = 8(\cos(10\pi t))(10\pi)$
    $= 80\pi \cos(10\pi t)$

    The acceleration at any time t is

    $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

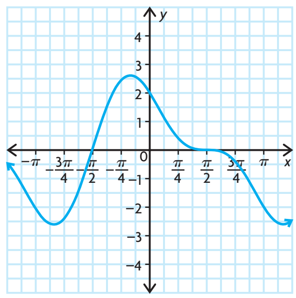
    Hence, $a = 80\pi(-\sin(10\pi t))(10\pi)$
    $= -800\pi^2 \sin(10\pi t)$. Now,

    $\frac{d^2s}{dt^2} + 100\pi^2 s = -800\pi^2 \sin(10\pi t)$
    $+ 100\pi^2(8 \sin(10\pi t)) = 0$.
18. displacement: 5,
    velocity: 10,
    acceleration: 20
19. each angle $\frac{\pi}{4}$ rad, or 45°
20. 4.5 m
21. 2.5 m
22. 5.19 ft
23. a. $f''(x) = -8 \sin^2(x - 2)$
    $+ 8 \cos^2(x - 2)$
    b. $f''(x) = (4 \cos x)(\sec^2 x \tan x)$
    $- 2 \sin x(\sec x)^2$

Chapter 5 Test, p. 266

1. a. $-4xe^{-2x^2}$
   b. $3e^{x^2+3x} \cdot \ln 3 \cdot (2x + 3)$
   c. $\frac{3}{2}[e^{3x} - e^{-3x}]$
   d. $2\cos x + 15\sin 5x$
   e. $6x \sin^2(x^2)\cos(x^2)$
   f. $-\dfrac{\sec^2\sqrt{1 - x}}{2\sqrt{1 - x}}$
2. $-6x + y = 2$,
   The tangent line is the given line.
3. $-2x + y = 1$
4. a. $a(t) = v'(t) = -10ke^{-kt}$
   $= -k(10e^{-kt})$
   $= -kv(t)$

   Thus, the acceleration is a constant multiple of the velocity. As the velocity of the particle decreases, the acceleration increases by a factor of k.
   b. 10 cm/s
   c. $\dfrac{\ln 2}{k}$; $-5k$
5. a. $f''(x) = 2(\sin^2 x - \cos^2 x)$
   b. $f''(x) = \csc x \cot^2 x$
   $+ \csc^3 x + \sin x$
6. absolute max: 1,
   absolute min: 0
7. 40.24
8. minimum: $\left(-4, -\frac{1}{e^4}\right)$, no maximum
9. a. $x = -\dfrac{\pi}{6}, -\dfrac{5\pi}{6}, \dfrac{\pi}{2}$
   b. increasing: $-\frac{5\pi}{6} < x < -\frac{\pi}{6}$;
   decreasing: $-\pi \le x < -\frac{5\pi}{6}$ and $-\frac{\pi}{6} < x < \pi$
   c. local maximum at $x = -\frac{\pi}{6}$; local minimum at $x = -\frac{5\pi}{6}$
   d.



Cumulative Review of Calculus, pp. 267–270

1. a. 16
   b. −2
   c. $\dfrac{1}{6}$
   d. $160\ln 2$
2. a. 13 m/s
   b. 15 m/s
3. $f(x) = x^3$
4. a. 19.6 m/s
   b. 19.6 m/s
   c. 53.655 m/s

5. **a.** 19 000 fish/year
   b. 23 000 fish/year

6. **a.** **i.** 3
   ii. 1
   iii. 3
   iv. 2
   b. No, $\lim_{x\to4} f(x)$ does not exist. In order for the limit to exist, $\lim_{x\to4^-} f(x)$ and $\lim_{x\to4^+} f(x)$ must exist and they must be the same. In this case, $\lim_{x\to4^-} f(x) = \infty$, but $\lim_{x\to4^+} f(x) = -\infty$, so $\lim_{x\to4} f(x)$ does not exist.

7. $f(x)$ is discontinuous at $x = 2$. $\lim_{x\to2^-} f(x) = 5$, but $\lim_{x\to2^+} f(x) = 3$.

8. **a.** $-\frac{1}{5}$ **d.** $\frac{4}{3}$
   b. 6 **e.** $\frac{1}{12}$
   c. $-\frac{1}{9}$ **f.** $\frac{1}{2}$

9. **a.** $6x + 1$
   b. $-\frac{1}{x^2}$

10. **a.** $3x^2 - 8x + 5$
    b. $\frac{3x^2}{\sqrt{2x^3 + 1}}$
    c. $\frac{6}{(x + 3)^2}$
    d. $4x(x^2 + 3)(4x^5 + 5x + 1) + (x^2 + 3)^2(20x^4 + 5)$
    e. $\frac{(4x^2 + 1)^4(84x^2 - 80x - 9)}{(3x - 2)^4}$
    f. $5[x^2 + (2x + 1)^3]^4 \times [2x + 6(2x + 1)^2]$

11. $4x + 3y - 10 = 0$

12. 3

13. **a.** $p'(t) = 4t + 6$
    b. 46 people per year
    c. 2006

14. **a.** $f'(x) = 5x^4 - 15x^2 + 1$; $f''(x) = 20x^3 - 30x$
    b. $f'(x) = \frac{4}{x^3}$; $f''(x) = -\frac{12}{x^4}$
    c. $f'(x) = -\frac{2}{\sqrt{x^3}}$; $f''(x) = \frac{3}{\sqrt{x^5}}$
    d. $f'(x) = 4x^3 + \frac{4}{x^5}$; $f''(x) = 12x^2 - \frac{20}{x^6}$

15. **a.** maximum: 82, minimum: 6
    b. maximum: $9\frac{1}{3}$, minimum: 2
    c. maximum: $\frac{e^4}{1 + e^4}$, minimum: $\frac{1}{2}$
    d. maximum: 5, minimum: 1

16. **a.** $v(t) = 9t^2 - 81t + 162$, $a(t) = 18t - 81$
    b. stationary when $t = 6$ or $t = 3$, advancing when $v(t) > 0$, and retreating when $v(t) < 0$
    c. $t = 4.5$
    d. $0 \le t < 4.5$
    e. $4.5 < t \le 8$

17. 14 062.5 m^2

18. $r \doteq 4.3$ cm, $h \doteq 8.6$ cm

19. $r = 6.8$ cm, $h = 27.5$ cm

20. **a.** $140 - 2x$
    b. 101 629.5 cm^3; 46.7 cm by 46.7 cm by 46.6 cm

21. $x = 4$

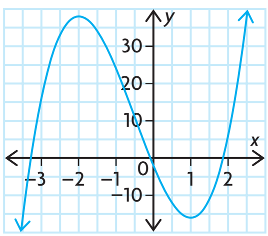
22. \$70 or \$80

23. \$1140

24. **a.** $\frac{dy}{dx} = -10x + 20$,
    $x = 2$ is critical number,
    Increase: $x < 2$,
    Decrease: $x > 2$
    b. $\frac{dy}{dx} = 12x + 16$,
    $x = -\frac{4}{3}$ is critical number,
    Increase: $x > -\frac{4}{3}$,
    Decrease: $x < -\frac{4}{3}$
    c. $\frac{dy}{dx} = 6x^2 - 24$,
    $x = \pm2$ are critical numbers,
    Increase: $x < -2$, $x > 2$,
    Decrease: $-2 < x < 2$
    d. $\frac{dy}{dx} = -\frac{2}{(x - 2)^2}$. The function has no critical numbers. The function is decreasing everywhere it is defined, that is, $x \ne 2$.

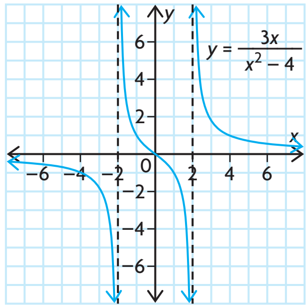
25. **a.** $y = 0$ is a horizontal asymptote.
    $x = \pm3$ are the vertical asymptotes.
    There is no oblique asymptote.
    $\left(0, -\frac{8}{9}\right)$ is a local maximum.
    b. There are no horizontal asymptotes.
    $x = \pm1$ are the vertical asymptotes.
    $y = 4x$ is an oblique asymptote.
    $(-\sqrt{3}, -6\sqrt{3})$ is a local maximum, $(\sqrt{3}, 6\sqrt{3})$ is a local minimum.

26. **a.**



$y = 4x^3 + 6x^2 - 24x - 2$

b.



27. **a.** $(-20)e^{5x+1}$
    b. $e^{3x}(3x + 1)$
    c. $(3\ln 6)6^{3x-8}$
    d. $(\cos x)e^{\sin x}$

28. $y = 2e(x - 1) + e$

29. **a.** 5 days
    b. 27

30. **a.** $2\cos x + 15\sin 5x$
    b. $8\cos 2x(\sin 2x + 1)^3$
    c. $\frac{2x + 3\cos 3x}{2\sqrt{x^2 + \sin 3x}}$
    d. $\frac{1 + 2\cos x}{(\cos x + 2)^2}$
    e. $2x\sec^2 x^2 - 2\tan x \sec^2 x$
    f. $-2x\sin x^2 \cos(\cos x^2)$

31. about 4.8 m

32. about 8.5 m

Chapter 6

Review of Prerequisite Skills, p. 273

1. **a.** $\frac{\sqrt{3}}{2}$ **d.** $\frac{\sqrt{3}}{2}$
   b. $-\sqrt{3}$ **e.** $\frac{\sqrt{2}}{2}$
   c. $\frac{1}{2}$ **f.** 1

2. $\frac{4}{3}$

3. **a.** $AB \doteq 29.7$, $\angle B \doteq 36.5°$, $\angle C \doteq 53.5°$
   b. $\angle A \doteq 97.9°$, $\angle B \doteq 29.7°$, $\angle C \doteq 52.4°$

4. $\angle Z \doteq 50°$, $XZ \doteq 7.36$, $YZ \doteq 6.78$

5. $\angle R \doteq 44°$, $\angle S \doteq 102°$, $\angle T \doteq 34°$

6. 5.82 km

7. 8.66 km

8. 21.1 km

9. 59.4 cm^2

5. **a.** 19 000 fish/year
b. 23 000 fish/year

6. **a.** **i.** 3
ii. 1
iii. 3
iv. 2
b. No, $\lim_{x \to 4} f(x)$ does not exist. In order for the limit to exist, $\lim_{x \to 4^-} f(x)$ and $\lim_{x \to 4^+} f(x)$ must exist and they must be the same. In this case, $\lim_{x \to 4^-} f(x) = \infty$, but $\lim_{x \to 4^+} f(x) = -\infty$, so $\lim_{x \to 4} f(x)$ does not exist.

7. $f(x)$ is discontinuous at $x = 2$. $\lim_{x \to 2^-} f(x) = 5$, but $\lim_{x \to 2^+} f(x) = 3$.

8. **a.** $-\frac{1}{5}$ **d.** $\frac{4}{3}$
b. 6 **e.** $\frac{1}{12}$
c. $-\frac{1}{9}$ **f.** $\frac{1}{2}$

9. **a.** $6x + 1$
b. $-\frac{1}{x^2}$

10. **a.** $3x^2 - 8x + 5$
b. $\frac{3x^2}{\sqrt{2x^3 + 1}}$
c. $\frac{6}{(x + 3)^2}$
d. $4x(x^2 + 3)(4x^5 + 5x + 1) + (x^2 + 3)^2(20x^4 + 5)$
e. $\frac{(4x^2 + 1)^4(84x^2 - 80x - 9)}{(3x - 2)^4}$
f. $5[x^2 + (2x + 1)^3]^4 \times [2x + 6(2x + 1)^2]$

11. $4x + 3y - 10 = 0$

12. 3

13. **a.** $p'(t) = 4t + 6$
b. 46 people per year
c. 2006

14. **a.** $f'(x) = 5x^4 - 15x^2 + 1$; $f''(x) = 20x^3 - 30x$
b. $f'(x) = \frac{4}{x^3}$; $f''(x) = -\frac{12}{x^4}$
c. $f'(x) = -\frac{2}{\sqrt{x^3}}$; $f''(x) = \frac{3}{\sqrt{x^5}}$
d. $f'(x) = 4x^3 + \frac{4}{x^5}$; $f''(x) = 12x^2 - \frac{20}{x^6}$

15. **a.** maximum: 82, minimum: 6
b. maximum: $9\frac{1}{3}$, minimum: 2
c. maximum: $\frac{e^4}{1 + e^4}$, minimum: $\frac{1}{2}$
d. maximum: 5, minimum: 1

16. **a.** $v(t) = 9t^2 - 81t + 162$, $a(t) = 18t - 81$
b. stationary when $t = 6$ or $t = 3$, advancing when $v(t) > 0$, and retreating when $v(t) < 0$
c. $t = 4.5$
d. $0 \le t < 4.5$
e. $4.5 < t \le 8$

17. 14 062.5 m^2

18. $r \doteq 4.3$ cm, $h \doteq 8.6$ cm

19. $r = 6.8$ cm, $h = 27.5$ cm

20. **a.** $140 - 2x$
b. 101 629.5 cm^3; 46.7 cm by 46.7 cm by 46.6 cm

21. $x = 4$

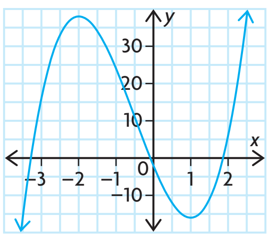
22. \$70 or \$80

23. \$1140

24. **a.** $\frac{dy}{dx} = -10x + 20$,
$x = 2$ is critical number,
Increase: $x < 2$,
Decrease: $x > 2$
b. $\frac{dy}{dx} = 12x + 16$,
$x = -\frac{4}{3}$ is critical number,
Increase: $x > -\frac{4}{3}$,
Decrease: $x < -\frac{4}{3}$
c. $\frac{dy}{dx} = 6x^2 - 24$,
$x = \pm 2$ are critical numbers,
Increase: $x < -2$, $x > 2$,
Decrease: $-2 < x < 2$
d. $\frac{dy}{dx} = -\frac{2}{(x - 2)^2}$. The function has no critical numbers. The function is decreasing everywhere it is defined, that is, $x \ne 2$.

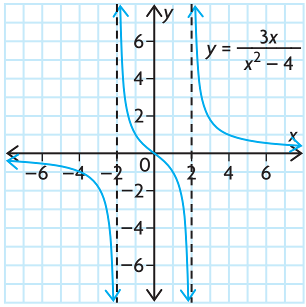
25. **a.** $y = 0$ is a horizontal asymptote.
$x = \pm 3$ are the vertical asymptotes.
There is no oblique asymptote.
$\left(0, -\frac{8}{9}\right)$ is a local maximum.
b. There are no horizontal asymptotes.
$x = \pm 1$ are the vertical asymptotes.
$y = 4x$ is an oblique asymptote.
$(-\sqrt{3}, -6\sqrt{3})$ is a local maximum, $(\sqrt{3}, 6\sqrt{3})$ is a local minimum.

26. **a.**



$y = 4x^3 + 6x^2 - 24x - 2$

b.



27. **a.** $(-20)e^{5x+1}$
b. $e^{3x}(3x + 1)$
c. $(3\ln 6)6^{3x-8}$
d. $(\cos x)e^{\sin x}$

28. $y = 2e(x - 1) + e$

29. **a.** 5 days
b. 27

30. **a.** $2\cos x + 15\sin 5x$
b. $8\cos 2x(\sin 2x + 1)^3$
c. $\frac{2x + 3\cos 3x}{2\sqrt{x^2 + \sin 3x}}$
d. $\frac{1 + 2\cos x}{(\cos x + 2)^2}$
e. $2x\sec^2 x^2 - 2\tan x \sec^2 x$
f. $-2x\sin x^2 \cos(\cos x^2)$

31. about 4.8 m

32. about 8.5 m

Chapter 6

Review of Prerequisite Skills, p. 273

1. **a.** $\frac{\sqrt{3}}{2}$ **d.** $\frac{\sqrt{3}}{2}$
b. $-\sqrt{3}$ **e.** $\frac{\sqrt{2}}{2}$
c. $\frac{1}{2}$ **f.** 1

2. $\frac{4}{3}$

3. **a.** $AB \doteq 29.7$, $\angle B \doteq 36.5°$, $\angle C \doteq 53.5°$
b. $\angle A \doteq 97.9°$, $\angle B \doteq 29.7°$, $\angle C \doteq 52.4°$

4. $\angle Z \doteq 50°$, $XZ \doteq 7.36$, $YZ \doteq 6.78$

5. $\angle R \doteq 44°$, $\angle S \doteq 102°$, $\angle T \doteq 34°$

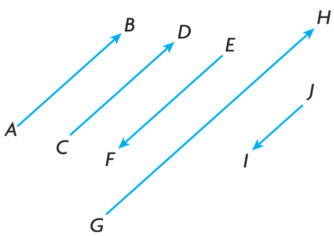
6. 5.82 km

7. 8.66 km

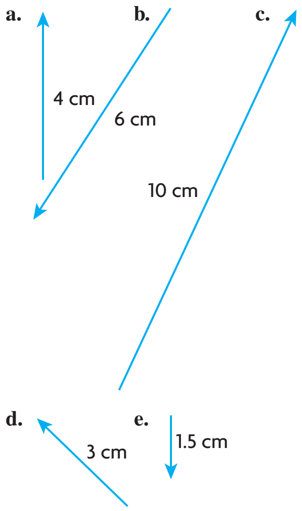
8. 21.1 km

9. 59.4 cm^2

Section 6.1, pp. 279–281

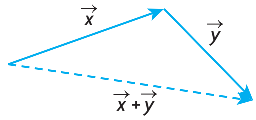
1. a. False; two vectors with the same magnitude can have different directions, so they are not equal.
   b. True; equal vectors have the same direction and the same magnitude.
   c. False; equal or opposite vectors must be parallel and have the same magnitude. If two parallel vectors have different magnitude, they cannot be equal or opposite.
   d. False; equal or opposite vectors must be parallel and have the same magnitude. Two vectors with the same magnitude can have directions that are not parallel, so they are not equal or opposite.
2. The following are scalars: height, temperature, mass, area, volume, distance, and speed. There is not a direction associated with any of these qualities.
   The following are vectors: weight, displacement, force, and velocity. There is a direction associated with each of these qualities.
3. Answers may vary. For example:
   A rolling ball stops due to friction, which resists the direction of motion.
   A swinging pendulum stops due to friction resisting the swinging pendulum.
4. Answers may vary. For example:
   a. $\overrightarrow{AD} = \overrightarrow{BC}$; $\overrightarrow{AB} = \overrightarrow{DC}$; $\overrightarrow{AE} = \overrightarrow{EC}$; $\overrightarrow{DE} = \overrightarrow{EB}$
   b. $\overrightarrow{AD} = -\overrightarrow{CB}$; $\overrightarrow{AB} = -\overrightarrow{CD}$; $\overrightarrow{AE} = -\overrightarrow{CE}$; $\overrightarrow{ED} = -\overrightarrow{EB}$; $\overrightarrow{DA} = -\overrightarrow{BC}$
   c. $\overrightarrow{AC}$ & $\overrightarrow{DB}$; $\overrightarrow{AE}$ & $\overrightarrow{EB}$; $\overrightarrow{EC}$ & $\overrightarrow{DE}$; $\overrightarrow{AB}$ & $\overrightarrow{CB}$
5. 

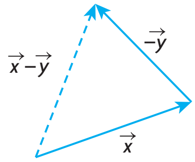
   a. $\overrightarrow{AB} = \overrightarrow{CD}$
   b. $\overrightarrow{AB} = -\overrightarrow{EF}$
   c. $|\overrightarrow{AB}| = |\overrightarrow{EF}|$ but $\overrightarrow{AB} \neq \overrightarrow{EF}$
   d. $\overrightarrow{GH} = 2\overrightarrow{AB}$
   e. $\overrightarrow{AB} = -2\overrightarrow{JI}$

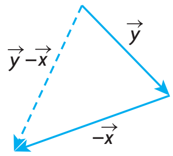
6. 

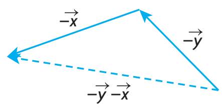
7. a. 100 km/h, south
   b. 50 km/h, west
   c. 100 km/h, northeast
   d. 25 km/h, northwest
   e. 60 km/h, east
8. a. 400 km/h, due south
   b. 70 km/h, southwesterly
   c. 30 km/h southeasterly
   d. 25 km/h, due east
9. a. i. False; they have equal magnitude, but opposite direction.
   ii. True; they have equal magnitude.
   iii. True; the base has sides of equal length, so the vectors have equal magnitude.
   iv. True; they have equal magnitude and direction.
   b. $|\overrightarrow{BD}| = \sqrt{18}$, $|\overrightarrow{BE}| = \sqrt{73}$, $|\overrightarrow{BH}| = \sqrt{82}$
10. a. The tangent vector describes James's velocity at that moment. At point A, his speed is 15 km/h and he is heading north. The tangent vector shows his velocity is 15 km/h, north.
    b. James's speed
    c. The magnitude of James's velocity (his speed) is constant, but the direction of his velocity changes at every point.
    d. C
    e. 3.5 min
    f. southwest
11. a. $\sqrt{10}$ or 3.16
    b. $(-3, 1)$
    c. $(0, -3)$
    d. $(0, 0)$

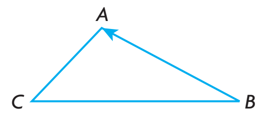
Section 6.2, pp. 290–292

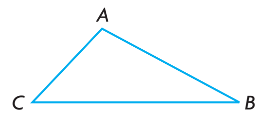
1. a. 

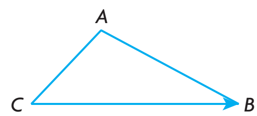
   b. 

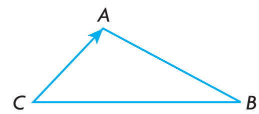
   c. 

   d. 

2. a. 

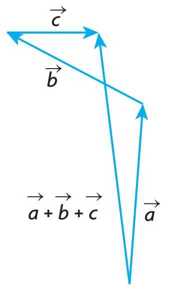
   $\overrightarrow{BA}$
   b. 

   $\vec{0}$
   c. 

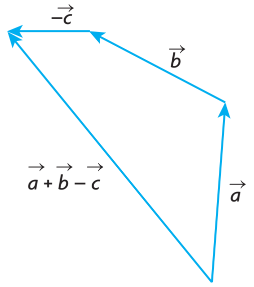
   $\overrightarrow{CB}$
   d. 

   $\overrightarrow{CA}$

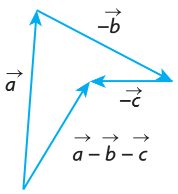
3. **a.**



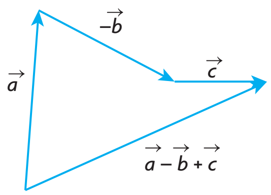
b.



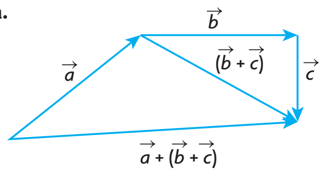
c.



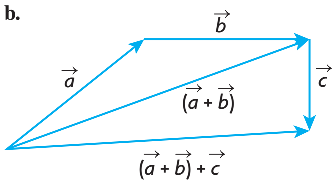
d.



4. **a.**



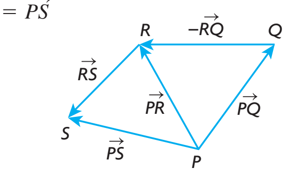
b.



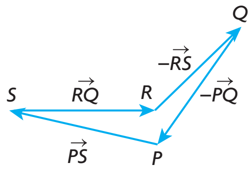
c. The resultant vectors are the same. The order in which you add vectors does not matter.

$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$

5. **a.** $=\overrightarrow{PS}$



b. $\vec{0}$



6. $\vec{x}+\vec{y}=\overrightarrow{MR}+\overrightarrow{RS}$

$=\overrightarrow{MS}$

$\vec{z}+\vec{t}=\overrightarrow{ST}+\overrightarrow{TQ}$

$=\overrightarrow{SQ}$

so

$(\vec{x}+\vec{y})+(\vec{z}+\vec{t})=\overrightarrow{MS}+\overrightarrow{SQ}$

$=\overrightarrow{MQ}$

7. **a.** $-\vec{x}$

b. $\vec{y}$

c. $\vec{x}+\vec{y}$

d. $-\vec{x}+\vec{y}$

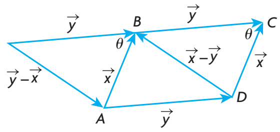
e. $\vec{x}+\vec{y}+\vec{z}$

f. $-\vec{x}-\vec{y}$

g. $-\vec{x}+\vec{y}+\vec{z}$

h. $-\vec{x}-\vec{z}$

8. **a.**

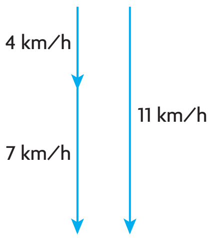


b. See the figure in part a. for the drawn vectors.

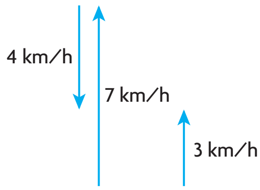
$|\vec{y}-\vec{x}|^2=|\vec{y}|^2+|\vec{x}|^2-$
$2|\vec{y}||-\vec{x}|\cos(\theta)$ and $|-\vec{x}|=|\vec{x}|$,
so $|\vec{y}-\vec{x}|^2=|\vec{x}-\vec{y}|^2$

9. **a.** 11 km/h

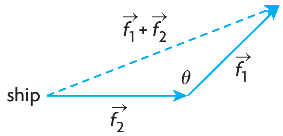
b.



c. 3 km/h



10. **a.**



b. The vectors form a triangle with side lengths $|\vec{f_1}|$, $|\vec{f_2}|$, and $|\vec{f_1}+\vec{f_2}|$. Find $|\vec{f_1}+\vec{f_2}|$ using the cosine law.

$|\vec{f_1}+\vec{f_2}|^2=|\vec{f_1}|^2+|\vec{f_2}|^2$
$-2|\vec{f_1}||\vec{f_2}|\cos(\theta)$

$|\vec{f_1}+\vec{f_2}|$
$=\sqrt{|\vec{f_1}|^2+|\vec{f_2}|^2-2|\vec{f_1}||\vec{f_2}|\cos(\theta)}$

11. 170 km/h, N28.1°W

12. $|\vec{x}+\vec{y}|=25$, $\theta \doteq 73.7°$

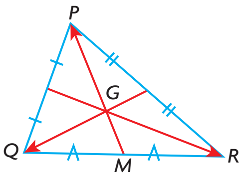
13. 0.52

14. The diagonals of a parallelogram bisect each other. So, $\overrightarrow{EA}=-\overrightarrow{EC}$ and $\overrightarrow{ED}=-\overrightarrow{EB}$.
Therefore,
$\overrightarrow{EA}+\overrightarrow{EB}+\overrightarrow{EC}+\overrightarrow{ED}=\vec{0}$.

15. Multiple applications of the Triangle Law for adding vectors show that $\overrightarrow{RM}+\vec{b}=\vec{a}+\overrightarrow{TP}$ (since both are equal to the undrawn vector $\overrightarrow{TM}$), and that $\overrightarrow{RM}+\vec{a}=\vec{b}+\overrightarrow{SQ}$ (since both are equal to the undrawn vector $\overrightarrow{RQ}$).
Adding these two equations gives
$2\overrightarrow{RM}+\vec{a}+\vec{b}=\vec{a}+\vec{b}+\overrightarrow{TP}+\overrightarrow{SQ}$
$2\overrightarrow{RM}=\overrightarrow{TP}+\overrightarrow{SQ}$

16. $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ represent the diagonals of a parallelogram with sides $\vec{a}$ and $\vec{b}$.
Since $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$, and the only parallelogram with equal diagonals is a rectangle, the parallelogram must also be a rectangle.

17.



Let point M be defined as shown. Two applications of the triangle law for adding vectors show that

$\overrightarrow{GQ} + \overrightarrow{QM} + \overrightarrow{MG} = \vec{0}$

$\overrightarrow{GR} + \overrightarrow{RM} + \overrightarrow{MG} = \vec{0}$

Adding these two equations gives

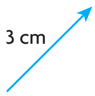
$\overrightarrow{GQ} + \overrightarrow{QM} + 2\,\overrightarrow{MG} + \overrightarrow{GR} + \overrightarrow{RM} = \vec{0}$

From the given information, $2\overrightarrow{MG} = \overrightarrow{GP}$ and $\overrightarrow{QM} + \overrightarrow{RM} = \vec{0}$ (since they are opposing vectors of equal length), so $\overrightarrow{GQ} + \overrightarrow{GP} + \overrightarrow{GR} = \vec{0}$, as desired.

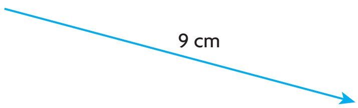
Section 6.3, pp. 298–301

1. A vector cannot equal a scalar.

2. **a.**



b.



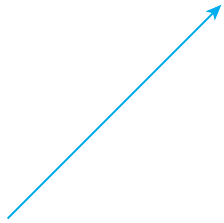
c.

d.

3. E25°N describes a direction that is 25° toward the north of due east. N65°E and “a bearing of 65°” both describe a direction that is 65° toward the east of due north.

4. Answers may vary. For example:

a.



$2\vec{v}$

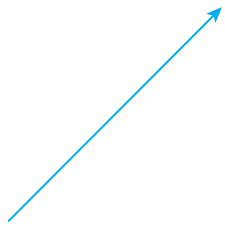
b.

$\frac{1}{2}\vec{v}$

c.

$-\frac{2}{3}\vec{v}$

d.

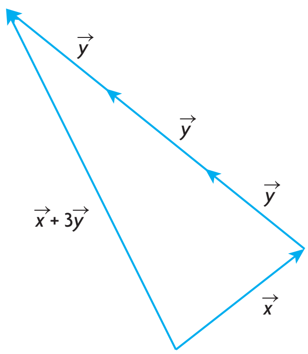


$-2\vec{v}$

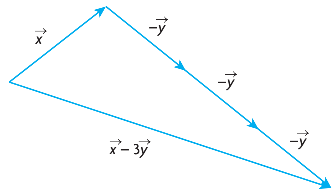
e.

$\frac{1}{|\vec{v}|}\vec{v}$

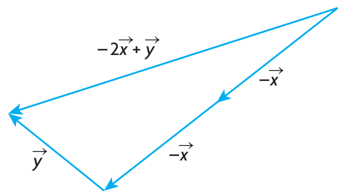
5. **a.**



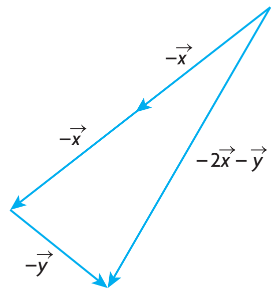
b.



c.

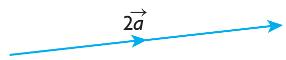


d.

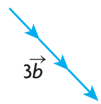


6. Answers may vary. For example:

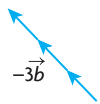
a.



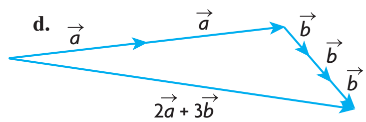
b.



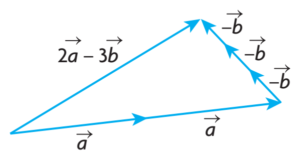
c.



d.



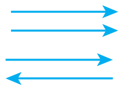
e.



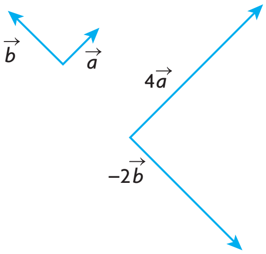
7. **a.** Answers may vary. For example: $m = 4$, $n = -3$, infinitely many

b. Answers may vary. For example: $d = 2$, $e = 0$, $f = -1$; not unique

8.



9. yes

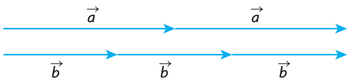


10. **a.** collinear
b. not collinear
c. not collinear
d. collinear

11. **a.** $\frac{1}{|\vec{x}|}\vec{x}$ is a vector with length 1 unit in the same direction as $\vec{x}$.

b. $-\frac{1}{|\vec{x}|}\vec{x}$ is a vector with length 1 unit in the opposite direction of $\vec{x}$.

12.



$m = \frac{2}{3}$

13. **a.** $-\frac{2}{3}\vec{a}$ **d.** $\frac{2}{3}|\vec{a}|$

b. $\frac{1}{3}\vec{a}$ **e.** $\frac{4}{3}\vec{a}$

c. $\frac{1}{3}|\vec{a}|$

14. $\sqrt{5}$ or 2.24, $\theta = 26.6°$ from $\vec{x}$ toward $2\vec{x} + \vec{y}$

15. 2.91, 9.9° from $\vec{x}$ toward $\vec{y}$

16. $\vec{b} = \frac{1}{|\vec{a}|}\vec{a}$

$|\vec{b}| = \left|\frac{1}{|\vec{a}|}\vec{a}\right|$

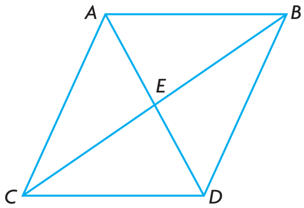
$|\vec{b}| = \frac{1}{|\vec{a}|}|\vec{a}|$

$|\vec{b}| = 1$

17. $\overrightarrow{AD} = \vec{c} + \overrightarrow{CD}$
$\overrightarrow{AD} = \vec{b} + \overrightarrow{BD}$
$2\overrightarrow{AD} = \vec{c} + \vec{b} + \overrightarrow{CD} + \overrightarrow{BD}$
But $\overrightarrow{CD} + \overrightarrow{BD} = 0$.
So, $2\overrightarrow{AD} = \vec{c} + \vec{b}$, or
$\overrightarrow{AD} = \frac{1}{2}\vec{c} + \frac{1}{2}\vec{b}$.

18. $\overrightarrow{MN} = \vec{b} - \vec{a}$
$\overrightarrow{QR} = 2\vec{b} - 2\vec{a}$
Notice that
$2\overrightarrow{MN} = 2\vec{b} - 2\vec{a}$
$= \overrightarrow{QR}$
We can conclude that $\overrightarrow{QR}$ is parallel to $\overrightarrow{MN}$ and $|\overrightarrow{QR}| = 2|\overrightarrow{MN}|$.

19.



Answers may vary. For example:
a. $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \overrightarrow{CD}$
b. $\vec{u} = \overrightarrow{AD}$ and $\vec{v} = \overrightarrow{AE}$
c. $\vec{u} = \overrightarrow{AC}$ and $\vec{v} = \overrightarrow{DB}$
d. $\vec{u} = \overrightarrow{ED}$ and $\vec{v} = \overrightarrow{AD}$

20. **a.** $|n| = 3|m|$
b. $m = 0$, $n = 0$

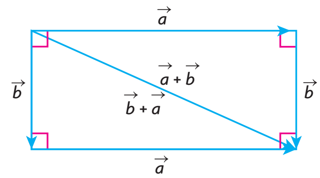
21. **a.** $\overrightarrow{CD} = \vec{b} - \vec{a}$
b. $\overrightarrow{BE} = 2\vec{b} - 2\vec{a}$
$= 2(\vec{b} - \vec{a})$
$= 2\overrightarrow{CD}$
The two are, therefore, parallel (collinear) and $\overrightarrow{BE} = 2|\overrightarrow{CD}|$.

22. Applying the triangle law for adding vectors shows that
$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC}$
The given information states that
$\overrightarrow{AB} = \frac{2}{3}\overrightarrow{DC}$
$\frac{3}{2}\overrightarrow{AB} = \overrightarrow{DC}$
By the properties of trapezoids, this gives
$\frac{3}{2}\overrightarrow{AE} = \overrightarrow{EC}$, and since
$\overrightarrow{AC} = \overrightarrow{AE} + \overrightarrow{EC}$, the original equation gives
$\overrightarrow{AE} + \frac{3}{2}\overrightarrow{AE} = \overrightarrow{AD} + \frac{3}{2}\overrightarrow{AB}$
$\frac{5}{2}\overrightarrow{AE} = \overrightarrow{AD} + \frac{3}{2}\overrightarrow{AB}$
$\overrightarrow{AE} = \frac{2}{5}\overrightarrow{AD} + \frac{3}{5}\overrightarrow{AB}$

Section 6.4, pp. 306–307

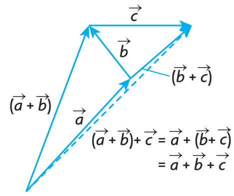
1. **a.** 0
b. 1
c. $\vec{0}$
d. 1

2.

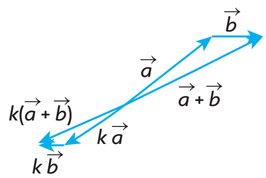


$\vec{a} + \vec{b} = \vec{b} + \vec{a}$

3.



4. Answers may vary. For example:



5. $\overrightarrow{PQ} = \overrightarrow{RQ} + \overrightarrow{SR} + \overrightarrow{TS} + \overrightarrow{PT}$
$= (\overrightarrow{RQ} + \overrightarrow{SR}) + (\overrightarrow{TS} + \overrightarrow{PT})$
$= (\overrightarrow{SR} + \overrightarrow{RQ}) + (\overrightarrow{PT} + \overrightarrow{TS})$
$= \overrightarrow{SQ} + \overrightarrow{PS}$
$= \overrightarrow{PS} + \overrightarrow{SQ}$
$= \overrightarrow{PQ}$

6. **a.** $\overrightarrow{EC}$
b. $\vec{0}$
c. Yes, the diagonals of a rectangular prism are of equal length.

7. $-4\vec{a} + 9\vec{b} - 24\vec{c}$

8. **a.** $12\vec{i} - 17\vec{j} + 5\vec{k}$
b. $-7\vec{i} + 11\vec{j} - 4\vec{k}$
c. $= -6\vec{i} + 13\vec{j} - 7\vec{k}$

9. $\vec{x} = \frac{5}{13}\vec{a} - \frac{18}{13}\vec{b}$,
$\vec{y} = \frac{1}{13}\vec{a} + \frac{12}{13}\vec{b}$

10. $\vec{a} = \vec{x} - \vec{y}$
$= \frac{2}{3}\vec{y} + \frac{1}{3}\vec{z} - (\vec{b} + \vec{z})$
$= \frac{2}{3}\vec{y} - \frac{2}{3}\vec{z} - \vec{b}$
$= \frac{2}{3}(\vec{y} - \vec{z}) - \vec{b}$
$= \frac{2}{3}\vec{b} - \vec{b}$
$= -\frac{1}{3}\vec{b}$

11. a. $\overrightarrow{AG} = \vec{a} + \vec{b} + \vec{c}$,
$\overrightarrow{BH} = -\vec{a} + \vec{b} + \vec{c}$,
$\overrightarrow{CE} = -\vec{a} - \vec{b} + \vec{c}$,
$\overrightarrow{DF} = \vec{a} - \vec{b} + \vec{c}$

b. $|\overrightarrow{AG}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2$
$= |-\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2$
$= |\overrightarrow{BH}|^2$
$\therefore |\overrightarrow{AG}| = |\overrightarrow{BH}|$

12. Applying the triangle law for adding vectors shows that
$\overrightarrow{TY} = \overrightarrow{TZ} + \overrightarrow{ZY}$
The given information states that
$\overrightarrow{TX} = 2\overrightarrow{ZY}$
$\frac{1}{2}\overrightarrow{TX} = \overrightarrow{ZY}$
By the properties of trapezoids, this gives
$\frac{1}{2}\overrightarrow{TO} = \overrightarrow{OY}$, and since
$\overrightarrow{TY} = \overrightarrow{TO} + \overrightarrow{OY}$, the original equation gives
$\overrightarrow{TO} + \frac{1}{2}\overrightarrow{TO} = \overrightarrow{TZ} + \frac{1}{2}\overrightarrow{TX}$
$\frac{3}{2}\overrightarrow{TO} = \overrightarrow{TZ} + \frac{1}{2}\overrightarrow{TX}$
$\overrightarrow{TO} = \frac{2}{3}\overrightarrow{TZ} + \frac{1}{3}\overrightarrow{TX}$

Mid-Chapter Review, pp. 308–309

1. a. $\overrightarrow{AB} = \overrightarrow{DC}$, $\overrightarrow{BA} = \overrightarrow{CD}$,
$\overrightarrow{AD} = \overrightarrow{BC}$, $\overrightarrow{CB} = \overrightarrow{DA}$
There is not enough information to determine if there is a vector equal to $\overrightarrow{AP}$.

b. $|\overrightarrow{PD}| = |\overrightarrow{DA}|$
$= |\overrightarrow{BC}|$ (parallelogram)

2. a. $\overrightarrow{RV}$ c. $\overrightarrow{PS}$ e. $\overrightarrow{PS}$
b. $\overrightarrow{RV}$ d. $\overrightarrow{RU}$ f. $\overrightarrow{PQ}$

3. a. $\sqrt{3}$
b. 53°

4. $t = 4$ or $t = -4$

5. In quadrilateral $PQRS$, look at $\triangle PQR$. Joining the midpoints B and C creates a vector $\overrightarrow{BC}$ that is parallel to $\overrightarrow{PR}$ and half the length of $\overrightarrow{PR}$. Look at $\triangle SPR$. Joining the midpoints A and D creates a vector $\overrightarrow{AD}$ that is parallel to $\overrightarrow{PR}$ and half the length of $\overrightarrow{PR}$. $\overrightarrow{BC}$ is parallel to $\overrightarrow{AD}$ and equal in length to $\overrightarrow{AD}$. Therefore, $ABCD$ is a parallelogram.

6. a. $2\sqrt{21}$
b. 71°
c. $\frac{1}{|\vec{u} + \vec{v}|}(\vec{u} + \vec{v}) = \frac{1}{2\sqrt{21}}(\vec{u} + \vec{v})$
d. $20\sqrt{7}$

7. 3

8. $|\vec{m} + \vec{n}| = \big||\vec{m}| - |\vec{n}|\big|$

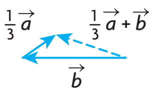
9. $\overrightarrow{BC} = -\vec{y}$, $\overrightarrow{DC} = \vec{x}$,
$\overrightarrow{BD} = -\vec{x} - \vec{y}$, $\overrightarrow{AC} = \vec{x} - \vec{y}$

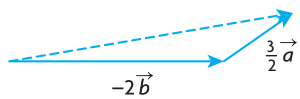
10. Construct a parallelogram with sides $\overrightarrow{OA}$ and $\overrightarrow{OC}$. Since the diagonals bisect each other, $2\overrightarrow{OB}$ is the diagonal equal to $\overrightarrow{OA} + \overrightarrow{OC}$. Or
$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$ and $\overrightarrow{AB} = \frac{1}{2}\overrightarrow{AC}$.
So, $\overrightarrow{OB} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC}$.
And $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$.
Now $\overrightarrow{OB} = \overrightarrow{OA} + \frac{1}{2}(\overrightarrow{OC} - \overrightarrow{OA})$,
Multiplying by 2 gives
$2\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC}$.

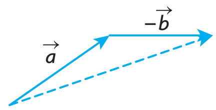
11. $\overrightarrow{BD} = 2\vec{x} + \vec{y}$
$\overrightarrow{BC} = 2\vec{x} - \vec{y}$

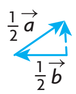
12. 460 km/h, south

13. a. $\overrightarrow{PT}$
b. $\overrightarrow{PT}$
c. $\overrightarrow{SR}$

14. a.


b.


c.


d.


15. $\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QS}$
$= 3\vec{b} - \vec{a}$
$\overrightarrow{RS} = \overrightarrow{QS} - \overrightarrow{QR}$
$= -3\vec{a}$

Section 6.5, pp. 316–318

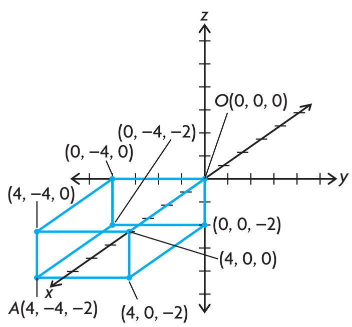
1. No, as the y-coordinate is not a real number.

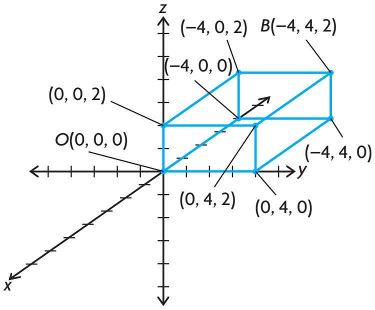
2. a. We first arrange the x-, y-, and z-axes (each a copy of the real line) in a way so that each pair of axes are perpendicular to each other (i.e., the x- and y-axes are arranged in their usual way to form the xy-plane, and the z-axis passes through the origin of the xy-plane and is perpendicular to this plane). This is easiest viewed as a "right-handed system," where, from the viewer's perspective, the positive z-axis points upward, the positive x-axis points out of the page, and the positive y-axis points rightward in the plane of the page. Then, given point $P(a, b, c)$, we locate this point's unique position by moving a units along the x-axis, then from there b units parallel to the y-axis, and finally c units parallel to the z-axis. It's associated unique position vector is determined by drawing a vector with tail at the origin $O(0, 0, 0)$ and head at P.

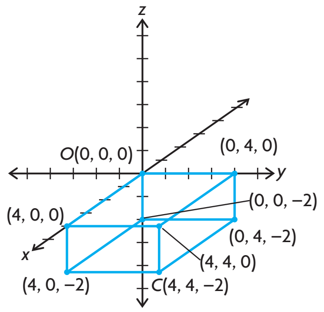
b. Since this position vector is unique, its coordinates are unique. Therefore, $a = -4$, $b = -3$, and $c = -8$.

3. a. $a = 5$, $b = -3$, and $c = 8$.
b. $(5, -3, 8)$

4. This is not an acceptable vector in I^3 as the z-coordinate is not an integer. However, since all of the coordinates are real numbers, this is acceptable as a vector in R^3.

5.






6. **a.** $A(0, -1, 0)$ is located on the y-axis. $B(0, -2, 0)$ $C(0, 2, 0)$, and $D(0, 10, 0)$ are three other points on this axis.

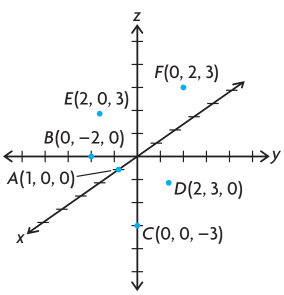
b. $\overrightarrow{OA} = (0, -1, 0)$, the vector with tail at the origin $O(0, 0, 0)$ and head at A.

7. **a.** Answers may vary. For example: $\overrightarrow{OA} = (0, 0, 1)$, $\overrightarrow{OB} = (0, 0, -1)$, $\overrightarrow{OC} = (0, 0, -5)$

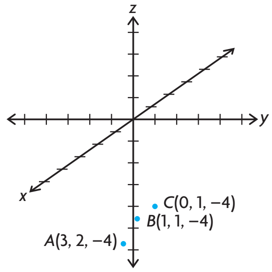
b. Yes, these vectors are collinear (parallel), as they all lie on the same line, in this case the z-axis.

c. A general vector lying on the z-axis would be of the form $\overrightarrow{OA} = (0, 0, a)$ for any real number a. Therefore, this vector would be represented by placing the tail at O and the head at the point (0, 0, a) on the z-axis.

8.

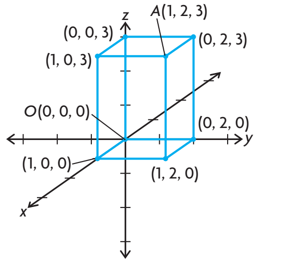


9. **a.**

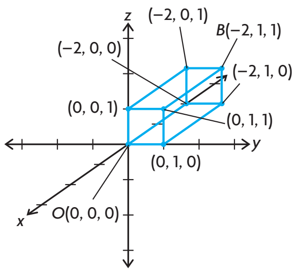


b. $z = -4$

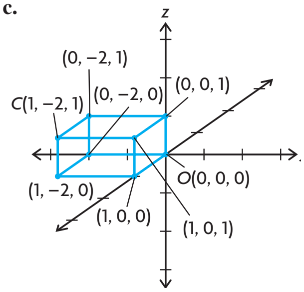
10. **a.**



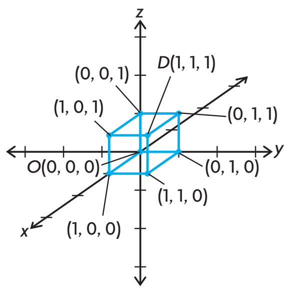
b.



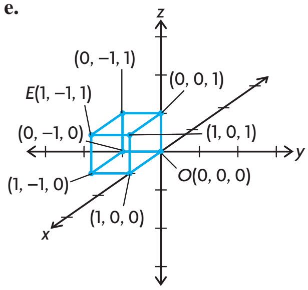
c.



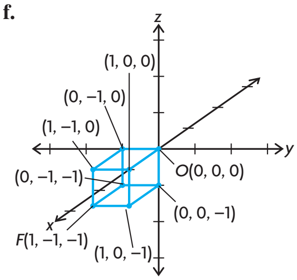
d.



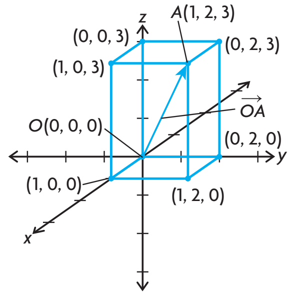
e.



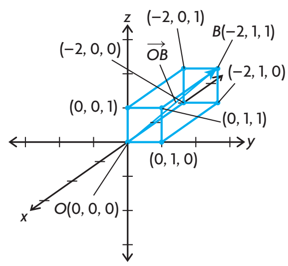
f.



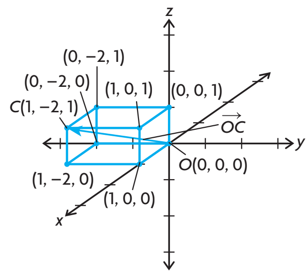
11. **a.** $\overrightarrow{OA} = (1, 2, 3)$



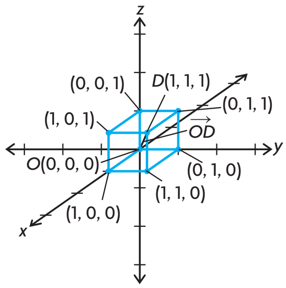
b. $\overrightarrow{OB} = (-2, 1, 1)$



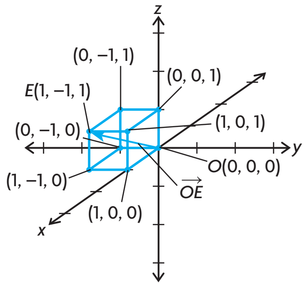
c. $\overrightarrow{OC} = (1, -2, 1)$



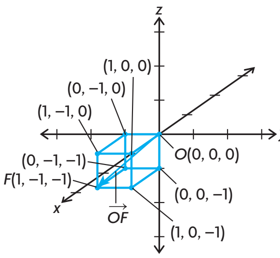
d. $\overrightarrow{OD} = (1, 1, 1)$



e. $\overrightarrow{OE} = (1, -1, 1)$



f. $\overrightarrow{OF} = (1, -1, -1)$



12. **a.** $a = 11\ c = 5$

b. Since P and Q represent the same point in R^3, they will have the same associated position vector, i.e., $|\overrightarrow{OP}| = |\overrightarrow{OQ}|$. So, since these vectors are equal, they will certainly have equal magnitudes, i.e., $|\overrightarrow{OP}| = |\overrightarrow{OQ}|$.

13. xy-plane, $z = 0$; xz-plane, $y = 0$; yz-plane, $x = 0$.

14. **a.** Every point on the plane containing points M, N, and P has y-coordinate equal to 0. Therefore, the equation of the plane containing these points is $y = 0$ (this is just the xz-plane).

b. The plane $y = 0$ contains the origin $O(0, 0, 0)$, and so since it also contains the points M, N, and P as well, it will contain the position vectors associated with these points joining O (tail) to the given point (head). That is, the plane $y = 0$ contains the vectors $\overrightarrow{OM}$, $\overrightarrow{ON}$, and $\overrightarrow{OP}$.

15. **a.** $A(-2, 0, 0)$, $B(-2, 4, 0)$, $C(0, 4, 0)$, $D(0, 0, -7)$, $E(0, 4, -7)$, $F(-2, 0, -7)$

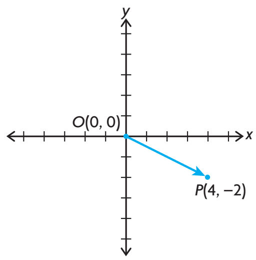
b. $\overrightarrow{OA} = (-2, 0, 0)$,
$\overrightarrow{OB} = (-2, 4, 0)$,
$\overrightarrow{OC} = (0, 4, 0)$, $\overrightarrow{OD} = (0, 0, -7)$,
$\overrightarrow{OE} = (0, 4, -7)$,
$\overrightarrow{OF} = (-2, 0, -7)$

c. 7 units

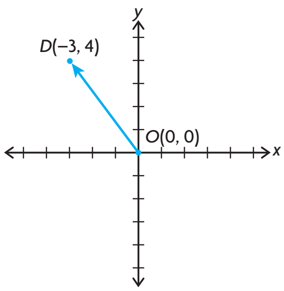
d. $y = 4$

e. Every point contained in rectangle $BCEP$ has y-coordinate equal to 4, and so is of the form $(x, 4, z)$, where x and z are real numbers such that $-2 \le x \le 0$ and $-7 \le z \le 0$.

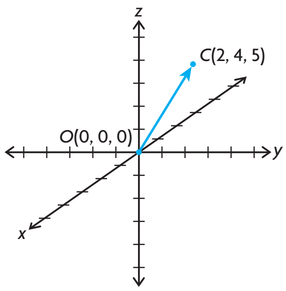
16. **a.**



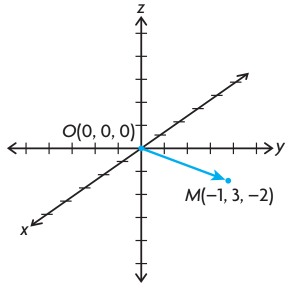
b.



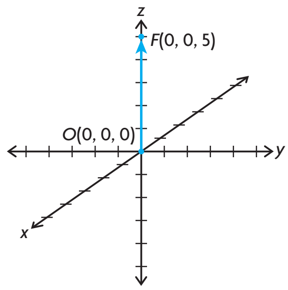
c.



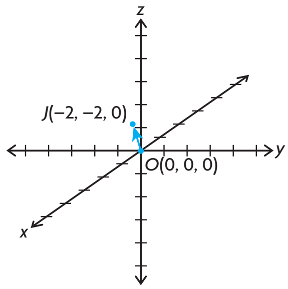
d.



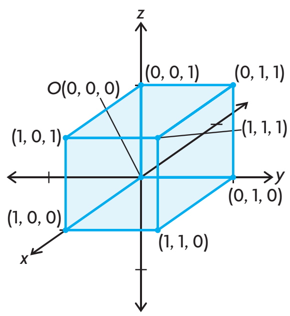
e.



f.



17.



18. First, $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB}$ by the triangle law of vector addition, where $\overrightarrow{OA} = (5, -10, 0)$, $\overrightarrow{OB} = (0, 0, -10)$, $\overrightarrow{OP}$ and $\overrightarrow{OA}$ are drawn in standard position (starting from the origin O $(0, 0, 0)$), and $\overrightarrow{OB}$ is drawn starting from the head of $\overrightarrow{OA}$. Notice that $\overrightarrow{OA}$

lies in the xy-plane, and $\overrightarrow{OB}$ is perpendicular to the xy-plane (so is perpendicular to $\overrightarrow{OA}$). So, $\overrightarrow{OP}$, $\overrightarrow{OA}$, and $\overrightarrow{OB}$ form a right triangle and, by the Pythagorean theorem,

$|\overrightarrow{OP}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2$

Similarly, $\overrightarrow{OA} = \vec{a} + \vec{b}$ by the triangle law of vector addition, where $\vec{a} = (5, 0, 0)$ and $\vec{b} = (0, -10, 0)$, and these three vectors form a right triangle as well. So,

$$\begin{aligned}|\overrightarrow{OA}|^2 &= |\vec{a}|^2 + |\vec{b}|^2\\ &= 25 + 100\\ &= 125\end{aligned}$$

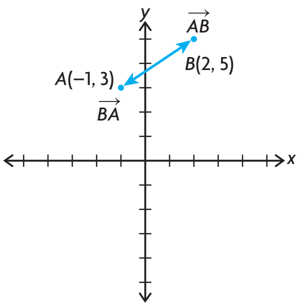
Obviously $|\overrightarrow{OB}|^2 = 100$, and so substituting gives

$$\begin{aligned}|\overrightarrow{OP}|^2 &= |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2\\ &= 125 + 100\\ &= 225\\ |\overrightarrow{OP}| &= \sqrt{225}\\ &= 15\end{aligned}$$

19. $(6, -5, 2)$

Section 6.6, pp. 324–326

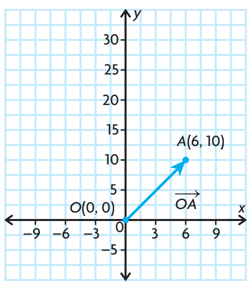
1. a. $\overrightarrow{AB} = (3, 2)$, $\overrightarrow{BA} = (-3, -2)$



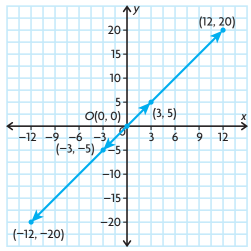
b. $|\overrightarrow{OA}| \doteq 3.16$, $|\overrightarrow{OB}| \doteq 5.39$

c. $|\overrightarrow{AB}| \doteq 3.61$, $|\overrightarrow{BA}| \doteq 3.61$

2.



a.



b. $\frac{1}{2}\overrightarrow{OA}$ and $-\frac{1}{2}\overrightarrow{OA}$, $2\overrightarrow{OA}$ and $-2\overrightarrow{OA}$

3. 5

4. a. $a = -3$, $b = 5$

b. 5.83

5. a. $|\vec{a}| = 61$, $|\vec{b}| = 41$

b. $|\vec{a} + \vec{b}| \doteq 100.02$, $|\vec{a} - \vec{b}| \doteq 28.28$

6. a. $(-2, 7)$

b. $(-30, 0)$

c. $(1, 11)$

7. a. $7\vec{i} - 8\vec{j}$

b. $3\vec{i} - 51\vec{j}$

c. $-29\vec{i} + 28\vec{j}$

8. a. about 4.12

b. about 6.71

c. about 18.38

d. about 18.38

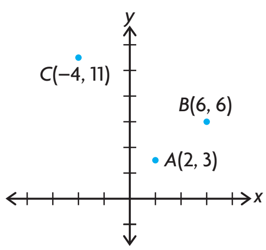
9. a. $\overrightarrow{AB} = (4, 2)$, $\overrightarrow{CD} = (2, 4)$, $\overrightarrow{EF} = (-6, 4)$, $\overrightarrow{GH} = (5, 0)$

b. $|\overrightarrow{AB}| \doteq 4.47$, $|\overrightarrow{CD}| \doteq 4.47$, $|\overrightarrow{EF}| \doteq 7.21$, $|\overrightarrow{GH}| = 5$

10. a. $\overrightarrow{OC} = (17, -3)$, $\overrightarrow{BA} = (-5, 9)$, $\overrightarrow{BC} = (6, 3)$

b. $\overrightarrow{OA} = (6, 3) = \overrightarrow{BC}$, so obviously we will have $|\overrightarrow{OA}| = |\overrightarrow{BC}|$.

11. a.

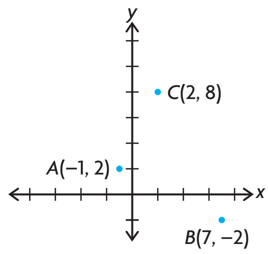


b. $\overrightarrow{AB}(4, 3)$, $|\overrightarrow{AB}| = 5$, $|\overrightarrow{AC}| = 10$, $\overrightarrow{CB} \doteq 11.18$

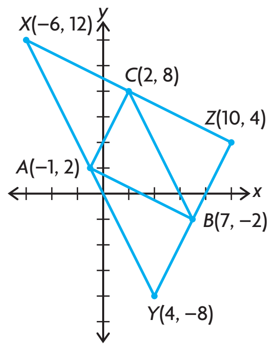
c. $|\overrightarrow{CB}|^2 = 125$, $|\overrightarrow{AC}|^2 = 100$, $|\overrightarrow{AB}|^2 = 25$

Since $|\overrightarrow{CB}|^2 = |\overrightarrow{AC}|^2 + |\overrightarrow{AB}|^2$, the triangle is a right triangle.

12. a.



b.

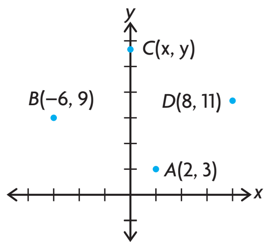


c. $X(-6, 12)$, $Y(4, -8)$, $Z(10, 4)$

13. a. $x = 7$, $y = -2$

b. $x = -12$, $y = 4$.

14. a.



b. Because $ABCD$ is a rectangle, we will have

$$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{AD}\\ (x, y) - (-6, 9) &= (8, 11) - (2, 3)\\ (x + 6, y - 9) &= (6, 8)\\ x + 6 &= 6\\ y - 9 &= 8\end{aligned}$$

So, $x = 0$ and $y = 17$, i.e., $C(0, 17)$.

15. a. $P\left(\frac{21}{10}, 0\right)$

b. $Q\left(0, -\frac{21}{4}\right)$

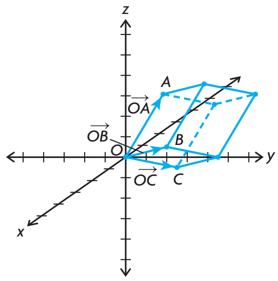
16. $\left(\frac{9}{41}, \frac{40}{41}\right)$

17. a. about 80.9°

b. about 35.4°

Section 6.7, pp. 332–333

1. **a.** $-1\vec{i} + 2\vec{j} + 4\vec{k}$
   b. about 4.58
2. $\vec{OB} = (3, 4, -4)$, $|\vec{OB}| = 6.40$
3. 3
4. **a.** $(-1, 6, 11)$
   b. $|\vec{OA}| = 13$, $|\vec{OB}| = 3$,
   $|\vec{OP}| \doteq 12.57$
   c. $\vec{AB} = (5, -2, -13)$,
   $|\vec{AB}| \doteq 14.07$,
   $\vec{AB}$ represents the vector from the tip of $\vec{OA}$ to the tip of $\vec{OB}$. It is the difference between the two vectors.
5. **a.** $(1, -3, 3)$
   b. $(-7, -16, 8)$
   c. $\left(-\frac{13}{2}, 2, \frac{3}{2}\right)$
   d. $(2, 30, -13)$
6. **a.** $\vec{i} - 2\vec{j} + 2\vec{k}$
   b. $3\vec{i} + 0\vec{j} + 0\vec{k}$
   c. $9\vec{i} + 3\vec{j} - 3\vec{k}$
   d. $-9\vec{i} - 3\vec{j} + 3\vec{k}$
7. **a.** about 5.10
   b. about 1.41
   c. about 5.39
   d. about 11.18
8. $\vec{x} = \vec{i} + 4\vec{j} - \vec{k}$,
   $\vec{y} = -2\vec{i} - 2\vec{j} + 6\vec{k}$
9. **a.** The vectors $\vec{OA}$, $\vec{OB}$, and $\vec{OC}$ represent the xy-plane, xz-plane, and yz-plane, respectively. They are also the vector from the origin to points $(a, b, 0)$, $(a, 0, c)$, and $(0, b, c)$, respectively.
   b. $\vec{OA} = a\vec{i} + b\vec{j} + 0\vec{k}$,
   $\vec{OB} = a\vec{i} + 0\vec{j} + c\vec{k}$,
   $\vec{OC} = 0\vec{i} + b\vec{j} + c\vec{k}$
   c. $|\vec{OA}| = \sqrt{a^2 + b^2}$,
   $|\vec{OB}| = \sqrt{a^2 + c^2}$,
   $|\vec{OB}| = \sqrt{b^2 + c^2}$
   d. $(0, -b, c)$, $\vec{AB}$ is a direction vector from A to B.
10. **a.** 7
    b. 13
    c. $(5, 2, 9)$
    d. 10.49
    e. $(-5, -2, -9)$
    f. 10.49
11. In order to show that $ABCD$ is a parallelogram, we must show that $\vec{AB} = \vec{DC}$ or $\vec{BC} = \vec{AD}$. This will show they have the same direction, thus the opposite sides are parallel.
    $\vec{AB} = (3, -4, 12)$
    $\vec{DC} = (3, -4, 12)$
    We have shown $\vec{AB} = \vec{DC}$ and $\vec{BC} = \vec{AD}$, so $ABCD$ is a parallelogram.
12. $a = \frac{2}{3}$, $b = 7$, $c = 0$
13. **a.**



    b. $V_1 = (0, 0, 0)$,
    $V_2 = (-2, 2, 5)$,
    $V_3 = (0, 4, 1)$,
    $V_4 = (0, 5, -1)$,
    $V_5 = (-2, 6, 6)$,
    $V_6 = (-2, 7, 4)$,
    $V_7 = (0, 9, 0)$,
    $V_8 = (-2, 11, 5)$
14. $(1, 0, 0)$
15. 4.36

Section 6.8, pp. 340–341

1. They are collinear, thus a linear combination is not applicable.
2. It is not possible to use $\vec{0}$ in a spanning set. Therefore, the remaining vectors only span R^2.
3. The set of vectors spanned by $(0, 1)$ is $m(0, 1)$. If we let $m = -1$, then $m(0, 1) = (0, -1)$.
4. $\vec{i}$ spans the set $m(1, 0, 0)$. This is any vector along the x-axis. Examples: $(2, 0, 0)$, $(-21, 0, 0)$.
5. As in question 2, it isn't possible to use $\vec{0}$ in a spanning set.
6. $\{(-1, 2), (-1, 1)\}$, $\{(2, -4), (-1, 1)\}$, $\{(-1, 1), (-3, 6)\}$ are all the possible spanning sets for R^2 with 2 vectors.
7. **a.** $14\vec{i} - 43\vec{j} + 40\vec{k}$
   b. $-7\vec{i} + 23\vec{j} - 14\vec{k}$
8. $\{(1, 0, 0), (0, 1, 0)\}$:
   $(-1, 2, 0) = -1(1, 0, 0) + 2(0, 1, 0)$
   $(3, 4, 0) = 3(1, 0, 0) + 4(0, 1, 0)$
   $\{(1, 1, 0), (0, 1, 0)\}$
   $(-1, 2, 0) = -1(1, 1, 0) + 3(0, 1, 0)$
   $(3, 4, 0) = 3(1, 1, 0) + (0, 1, 0)$
9. **a.** It is the set of vectors in the xy-plane.
   b. $-2(1, 0, 0) + 4(0, 1, 0)$
   c. By part a., the vector is not in the xy-plane. There is no combination that would produce a number other than 0 for the z-component.
   d. It would still only span the xy-plane. There would be no need for that vector.
10. $a = -2$, $b = 24$, $c = 3$
11. $(-10, -34) = 2(-1, 3) - 8(1, 5)$
12. **a.** $a = x + y$,
    $b = x + 2y$
    b. $(2, -3) = -1(2, -1) - 4(-1, 1)$
    $(124, -5) = 119(2, 1) + 114(-1, 1)$
    $(4, -11) = -7(2, 1) - 18(-1, 1)$
13. **a.** The statement $a(-1, 2, 3) + b(4, 1, -2) = (-14, -1, 16)$ does not have a consistent solution.
    b. $3(-1, 3, 4) - 5(0, -1, 1) = (-3, 14, 7)$
14. -7
15. $m = 2$, $n = 3$; Non-parallel vectors cannot be equal, unless their magnitudes equal 0.
16. Answers may vary. For example:
    $p = -6$ and $q = 1$,
    $p = 25$ and $q = 0$,
    $p = \frac{13}{3}$ and $q = \frac{2}{3}$
17. As in question 15, non-parallel vectors. Their magnitudes must be 0 again to make the equality true.
    $m^2 + 2m - 3 = (m - 1)(m + 3)$
    $m = 1, -3$
    $m^2 + m - 6 = (m - 2)(m + 3)$
    $m = 2, -3$
    So, when $m = -3$, their sum will be 0.

Review Exercise, pp. 344–347

1. **a.** false; Let $\vec{b} = -\vec{a} \neq 0$, then:
   $|\vec{a} + \vec{b}| = |\vec{a} + (-\vec{a})|$
   $= |0|$
   $= 0 < |\vec{a}|$
   b. true; $|\vec{a} + \vec{b}|$ and $|\vec{a} + \vec{c}|$ both represent the lengths of the diagonal of a parallelogram, the first with sides $\vec{a}$ and $\vec{b}$ and the second with sides $\vec{a}$ and $\vec{c}$; since both parallelograms have $\vec{a}$ as a side and diagonals of equal length $|\vec{b}| = |\vec{c}|$.
   c. true; Subtracting $\vec{a}$ from both sides shows that $\vec{b} = \vec{c}$.

d. true; Draw the parallelogram formed by $\overrightarrow{RF}$ and $\overrightarrow{SW}$. $\overrightarrow{FW}$ and $\overrightarrow{RS}$ are the opposite sides of a parallelogram and must be equal.

e. true; the distributive law for scalars

f. false; Let $\vec{b} = -\vec{a}$ and let $\vec{c} = \vec{d} \neq 0$. Then,
$|\vec{a}| = |-\vec{a}| = |\vec{b}|$ and $|\vec{c}| = |\vec{d}|$
but $|\vec{a} + \vec{b}| = |\vec{a} + (-\vec{a})| = 0$
$|\vec{c} + \vec{b}| = |\vec{c} + \vec{c}| = |2\vec{c}|$
so $|\vec{a} + \vec{b}| \neq |\vec{c} + \vec{d}|$

2. **a.** $20\vec{a} - 30\vec{b} + 8\vec{c}$
b. $\vec{a} - 3\vec{b} - 3\vec{c}$

3. **a.** $\overrightarrow{XY} = (-2, 3, 6)$,
$|\overrightarrow{XY}| = 7$
b. $\left(-\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right)$

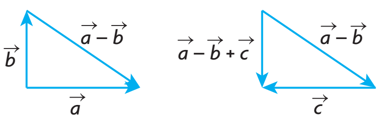
4. **a.** $(-6, -3, -6)$
b. $\left(-\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$

5. $\left(-\frac{6}{7}, \frac{2}{7}, \frac{3}{7}\right)$

6. **a.** $\overrightarrow{OA} + \overrightarrow{OB} = (-3, 8, -8)$,
$\overrightarrow{OA} - \overrightarrow{OB} = (9, -4, -4)$
b. $\theta \doteq 84.4°$

7. **a.** $|\overrightarrow{AB}| = \sqrt{14}$,
$|\overrightarrow{BC}| = \sqrt{59}$,
$|\overrightarrow{CA}| = \sqrt{45}$
b. 12.5
c. 18.13
d. $(6, 2, -2)$

8. **a.**



b. 5

9. $\frac{1}{2}(-11, 7) + \left(-\frac{3}{2}\right)(-3, 1) = (-1, 2)$,
$\frac{1}{3}(-11, 7) + \left(-\frac{2}{3}\right)(-1, 2) = (-3, 1)$,
$3(-3, 1) + 2(-1, 2) = (-11, 7)$

10. **a.** $x - 3y + 6z = 0$ where $P(x, y, z)$ is the point.
b. $(0, 0, 0)$ and $\left(1, \frac{1}{3}, 0\right)$

11. **a.** $a = -3, b = 26.5, c = 10$
b. $a = 8, b = \frac{7}{3}, c = -10$

12. **a.** yes
b. yes

13. **a.** $|\overrightarrow{AB}|^2 = 9$, $|\overrightarrow{AC}|^2 = 3$, $|\overrightarrow{BC}|^2 = 6$
Since $|\overrightarrow{AB}|^2 = |\overrightarrow{AC}|^2 + |\overrightarrow{BC}|^2$ the triangle is right-angled
b. $\frac{\sqrt{6}}{3}$

14. **a.** $\overrightarrow{DA}$, $\overrightarrow{BC}$ and $\overrightarrow{EB}$, $\overrightarrow{ED}$
b. $\overrightarrow{DC}$, $\overrightarrow{AB}$ and $\overrightarrow{CE}$, $\overrightarrow{EA}$
c. $|\overrightarrow{AD}|^2 + |\overrightarrow{DC}|^2 = |\overrightarrow{AC}|^2$
But $|\overrightarrow{AC}|^2 = |\overrightarrow{DB}|^2$
Therefore, $|\overrightarrow{AD}|^2 + |\overrightarrow{DC}|^2 = |\overrightarrow{DB}|^2$

15. **a.** $C(3, 0, 5)$, $P(3, 4, 5)$, $E(0, 4, 5)$, $F(0, 4, 0)$
b. $\overrightarrow{DB} = (3, 4, -5)$,
$\overrightarrow{CF} = (-3, 4, -5)$
c. 90°
d. 50.2°

16. **a.** 7.74
b. 2.83
c. 2.83

17. **a.** 1236.9 km
b. S14.0°W

18. **a.** Any pair of nonzero, noncollinear vectors will span R^2. To show that (2, 3) and (3, 5) are noncollinear, show that there does not exist any number k such that $k(2, 3) = (3, 5)$.
Solve the system of equations:
$2k = 3$
$3k = 5$
Solving both equations gives two different values for k, $\frac{3}{2}$ and $\frac{5}{3}$, so (2, 3) and (3, 5) are noncollinear and thus span R^2.
b. $m = -770, n = 621$

19. **a.** Find a and b such that

$$\begin{aligned}(5, 9, 14) &= a(-2, 3, 1) \\ &\quad + b(3, 1, 4) \\ (5, 9, 14) &= (-2a, 3a, a) \\ &\quad + (3b, b, 4b) \\ (5, 9, 14) &= (-2a + 3b, 3a \\ &\quad + b, a + 4b)\end{aligned}$$

i. $5 = -2a + 3b$
ii. $9 = 3a + b$
iii. $14 = a + 4b$
Use the method of elimination with i. and iii.

$$\begin{aligned}2(14) &= 2(a + 4b) \\ 28 &= 2a + 8b \\ +\quad 5 &= -2a + 3b \\ \hline 33 &= 11b \\ 3 &= b\end{aligned}$$

By substitution, $a = 2$.
$\vec{a}$ lies in the plane determined by $\vec{b}$ and $\vec{c}$ because it can be written as a linear combination of $\vec{b}$ and $\vec{c}$.

b. If vector $\vec{a}$ is in the span of $\vec{b}$ and $\vec{c}$, then $\vec{a}$ can be written as a linear combination of $\vec{b}$ and $\vec{c}$. Find m and n such that

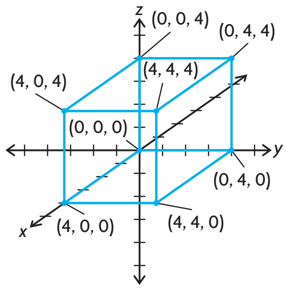
$$\begin{aligned}(-13, 36, 23) &= m(-2, 3, 1) \\ &\quad + n(3, 1, 4) \\ &= (-2m, 3m, m) \\ &\quad + (3n, n, 4n) \\ &= (-2m + 3n, \\ &\quad 3m + n, m + 4n)\end{aligned}$$

Solve the system of equations:
$-13 = -2m + 3n$
$36 = 3m + n$
$23 = m + 4n$
Use the method of elimination:

$$\begin{aligned}2(23) &= 2(m + 4n) \\ 46 &= 2m + 8n \\ +\ -13 &= -2m + 3n \\ \hline 33 &= 11n \\ 3 &= n\end{aligned}$$

By substitution, $m = 11$.
So, vector $\vec{a}$ is in the span of $\vec{b}$ and $\vec{c}$.

20. **a.**



b. $(-4, -4, -4)$
c. $(-4, 0, -4)$
d. $(4, 4, 0)$

21. 7

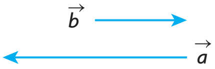
22. **a.** $|\overrightarrow{AB}| = 10$,
$|\overrightarrow{BC}| = 2\sqrt{5} = 4.47$,
$|\overrightarrow{CA}| = \sqrt{80} = 8.94$
b. If A, B, and C are vertices of a right triangle, then

$$\begin{aligned}|\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2 &= |\overrightarrow{AB}|^2 \\ |\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2 &= (2\sqrt{5})^2 + (\sqrt{80})^2 \\ &= 20 + 80 \\ &= 100 \\ |\overrightarrow{AB}|^2 &= 10^2 \\ &= 100\end{aligned}$$

So, triangle ABC is a right triangle.

23. **a.** $\vec{a} + \vec{b} + \vec{c}$
b. $\vec{a} - \vec{b}$
c. $-\vec{b} - \vec{a} + \vec{c}$
d. $\vec{0}$
e. $\vec{b} + \vec{c}$

24.

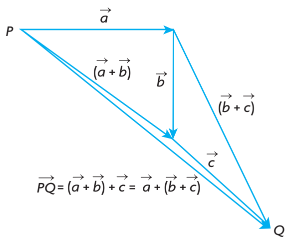


25. **a.** $\sqrt{|\vec{a}|^2 + |\vec{b}|^2}$
b. $\sqrt{|\vec{a}|^2 + |\vec{b}|^2}$
c. $\sqrt{4|\vec{a}|^2 + 9|\vec{b}|^2}$

26. **Case 1** If $\vec{b}$ and $\vec{c}$ are collinear, then $2\vec{b} + 4\vec{c}$ is also collinear with both $\vec{b}$ and $\vec{c}$. But $\vec{a}$ is perpendicular to $\vec{b}$ and $\vec{c}$, so $\vec{a}$ is perpendicular to $2\vec{b} + 4\vec{c}$.
Case 2 If $\vec{b}$ and $\vec{c}$ are not collinear, then by spanning sets, $\vec{b}$ and $\vec{c}$ span a plane in R^3, and $2\vec{b} + 4\vec{c}$ is in that plane. If $\vec{a}$ is perpendicular to $\vec{b}$ and $\vec{c}$, then it is perpendicular to the plane and all vectors in the plane. So, $\vec{a}$ is perpendicular to $2\vec{b} + 4\vec{c}$.

Chapter 6 Test, p. 348

1. Let P be the tail of $\vec{a}$ and let Q be the head of $\vec{c}$. The vector sums $[\vec{a} + (\vec{b} + \vec{c})]$ and $[(\vec{a} + \vec{b}) + \vec{c}]$ can be depicted as in the diagram below, using the triangle law of addition.
We see that $\overrightarrow{PQ} = \vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$. This is the associative property for vector addition.



2. **a.** (8, 4, 8)
b. 12
c. $\left(-\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$

3. $\sqrt{19}$

4. **a.** $\vec{x} = 2\vec{b} - 3\vec{a}, \vec{y} = 3\vec{b} - 5\vec{a}$
b. $a = 1, b = 5, c = -11$

5. **a.** $\vec{a}$ and $\vec{b}$ span R^2, because any vector (x, y) in R^2 can be written as a linear combination of $\vec{a}$ and $\vec{b}$. These two vectors are not multiples of each other.
b. $p = -2, q = 3$

6. **a.** $(1, 12, -29) = -2(3, 1, 4) + 7(1, 2, -3)$
b. $\vec{r}$ cannot be written as a linear combination of $\vec{p}$ and $\vec{q}$. In other words, $\vec{r}$ does not lie in the plane determined by $\vec{p}$ and $\vec{q}$.

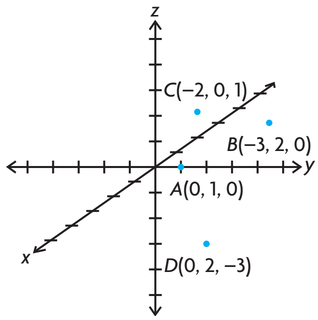
7. $\sqrt{13}$, $\theta \doteq 3.61$; 73.9° relative to x

8. $\overrightarrow{DE} = \overrightarrow{CE} - \overrightarrow{CD}$
$\overrightarrow{DE} = \vec{b} - \vec{a}$
Also,
$\overrightarrow{BA} = \overrightarrow{CA} - \overrightarrow{CB}$
$\overrightarrow{BA} = 2\vec{b} - 2\vec{a}$
Thus,
$\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BA}$

Chapter 7

Review of Prerequisite Skills, p. 350

1. $v \doteq 806$ km/h N 7.1° E
2. 15.93 units W 32.2° N
3.



4. **a.** $(3, -2, 7)$; $l \doteq 7.87$
b. $(-9, 3, 14)$; $l \doteq 16.91$
c. $(1, 1, 0)$; $l \doteq 1.41$
d. $(2, 0, -9)$; $l \doteq 9.22$

5. **a.** $(x, y, 0)$
b. $(x, 0, z)$
c. $(0, y, z)$

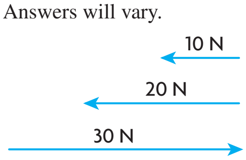
6. **a.** $\vec{i} - 7\vec{j}$
b. $6\vec{i} - 2\vec{j}$
c. $-8\vec{i} + 11\vec{j} + 3\vec{k}$
d. $4\vec{i} - 6\vec{j} + 8\vec{k}$

7. **a.** $\vec{i} + 3\vec{j} - \vec{k}$
b. $5\vec{i} + \vec{j} - \vec{k}$
c. $12\vec{i} + \vec{j} - 2\vec{k}$

Section 7.1, pp. 362–364

1. **a.** 10 N is a melon, 50 N is a chair, 100 N is a computer
b. Answers will vary.

2. **a.**



b. 180°

3. a line along the same direction

4. For three forces to be in equilibrium, they must form a triangle, which is a planar figure.

5. **a.** The resultant is 13 N at an angle of N 22.6° W. The equilibrant is 13 N at an angle of S 22.6° W.
b. The resultant is 15 N at an angle of S 36.9° W. The equilibrant is 15 N at N 36.9° E.

6. **a.** yes **b.** yes **c.** no **d.** yes

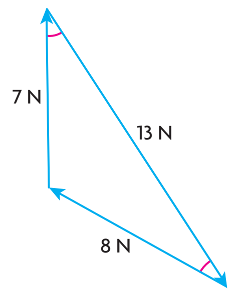
7. Arms 90 cm apart will yield a resultant with a smaller magnitude than at 30 cm apart. A resultant with a smaller magnitude means less force to counter your weight, hence a harder chin-up.

8. The resultant would be 12.17 N at 34.7° from the 6 N force toward the 8 N force. The equilibrant would be 12.17 N at 145.3° from the 6 N force away from the 8 N force.

9. 9.66 N 15° from given force, 2.95 N perpendicular to 9.66 N force

10. 49 N directed up the ramp

11. **a.**

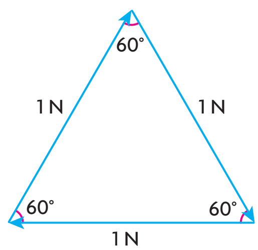


b. 60°

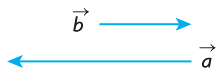
12. approximately 7.1 N 45° south of east

13. **a.** 7
b. The angle between f_1 and the resultant is 16.3°. The angle between $\vec{f}_1$ and the equilibrant is 163.7°.

14. **a.**



For these three equal forces to be in equilibrium, they must form an equilateral triangle. Since the resultant will lie along one of these lines, and since all angles of an equilateral triangle are 60°, the resultant will be at a 60° angle with the other two vectors.

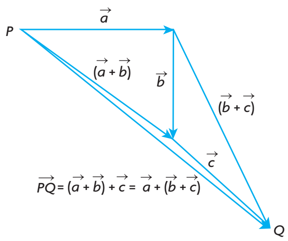
24. 

25. a. $\sqrt{|\vec{a}|^2 + |\vec{b}|^2}$
b. $\sqrt{|\vec{a}|^2 + |\vec{b}|^2}$
c. $\sqrt{4|\vec{a}|^2 + 9|\vec{b}|^2}$

26. **Case 1** If $\vec{b}$ and $\vec{c}$ are collinear, then $2\vec{b} + 4\vec{c}$ is also collinear with both $\vec{b}$ and $\vec{c}$. But $\vec{a}$ is perpendicular to $\vec{b}$ and $\vec{c}$, so $\vec{a}$ is perpendicular to $2\vec{b} + 4\vec{c}$.
Case 2 If $\vec{b}$ and $\vec{c}$ are not collinear, then by spanning sets, $\vec{b}$ and $\vec{c}$ span a plane in R^3, and $2\vec{b} + 4\vec{c}$ is in that plane. If $\vec{a}$ is perpendicular to $\vec{b}$ and $\vec{c}$, then it is perpendicular to the plane and all vectors in the plane. So, $\vec{a}$ is perpendicular to $2\vec{b} + 4\vec{c}$.

Chapter 6 Test, p. 348

1. Let P be the tail of $\vec{a}$ and let Q be the head of $\vec{c}$. The vector sums $[\vec{a} + (\vec{b} + \vec{c})]$ and $[(\vec{a} + \vec{b}) + \vec{c}]$ can be depicted as in the diagram below, using the triangle law of addition.
We see that $\overrightarrow{PQ} = \vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$. This is the associative property for vector addition.



2. a. (8, 4, 8)
b. 12
c. $\left(-\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$

3. $\sqrt{19}$

4. a. $\vec{x} = 2\vec{b} - 3\vec{a}, \vec{y} = 3\vec{b} - 5\vec{a}$
b. $a = 1, b = 5, c = -11$

5. a. $\vec{a}$ and $\vec{b}$ span R^2, because any vector (x, y) in R^2 can be written as a linear combination of $\vec{a}$ and $\vec{b}$. These two vectors are not multiples of each other.
b. $p = -2, q = 3$

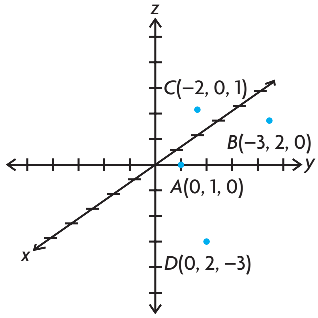
6. a. $(1, 12, -29) = -2(3, 1, 4) + 7(1, 2, -3)$
b. $\vec{r}$ cannot be written as a linear combination of $\vec{p}$ and $\vec{q}$. In other words, $\vec{r}$ does not lie in the plane determined by $\vec{p}$ and $\vec{q}$.

7. $\sqrt{13}$, $\theta \doteq 3.61$; 73.9° relative to x

8. $\overrightarrow{DE} = \overrightarrow{CE} - \overrightarrow{CD}$
$\overrightarrow{DE} = \vec{b} - \vec{a}$
Also,
$\overrightarrow{BA} = \overrightarrow{CA} - \overrightarrow{CB}$
$\overrightarrow{BA} = 2\vec{b} - 2\vec{a}$
Thus,
$\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BA}$

Chapter 7

Review of Prerequisite Skills, p. 350

1. $v \doteq 806$ km/h N 7.1° E
2. 15.93 units W 32.2° N
3. 

4. a. $(3, -2, 7)$; $l \doteq 7.87$
b. $(-9, 3, 14)$; $l \doteq 16.91$
c. $(1, 1, 0)$; $l \doteq 1.41$
d. $(2, 0, -9)$; $l \doteq 9.22$

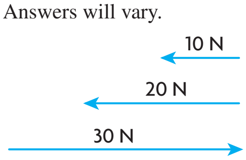
5. a. $(x, y, 0)$
b. $(x, 0, z)$
c. $(0, y, z)$

6. a. $\vec{i} - 7\vec{j}$
b. $6\vec{i} - 2\vec{j}$
c. $-8\vec{i} + 11\vec{j} + 3\vec{k}$
d. $4\vec{i} - 6\vec{j} + 8\vec{k}$

7. a. $\vec{i} + 3\vec{j} - \vec{k}$
b. $5\vec{i} + \vec{j} - \vec{k}$
c. $12\vec{i} + \vec{j} - 2\vec{k}$

Section 7.1, pp. 362–364

1. a. 10 N is a melon, 50 N is a chair, 100 N is a computer
b. Answers will vary.

2. a. 

b. 180°

3. a line along the same direction

4. For three forces to be in equilibrium, they must form a triangle, which is a planar figure.

5. a. The resultant is 13 N at an angle of N 22.6° W. The equilibrant is 13 N at an angle of S 22.6° W.
b. The resultant is 15 N at an angle of S 36.9° W. The equilibrant is 15 N at N 36.9° E.

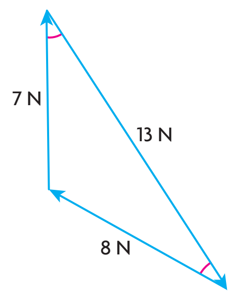
6. a. yes b. yes c. no d. yes

7. Arms 90 cm apart will yield a resultant with a smaller magnitude than at 30 cm apart. A resultant with a smaller magnitude means less force to counter your weight, hence a harder chin-up.

8. The resultant would be 12.17 N at 34.7° from the 6 N force toward the 8 N force. The equilibrant would be 12.17 N at 145.3° from the 6 N force away from the 8 N force.

9. 9.66 N 15° from given force, 2.95 N perpendicular to 9.66 N force

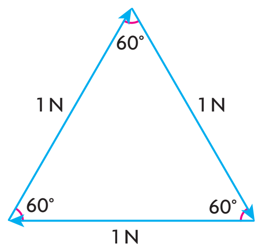
10. 49 N directed up the ramp

11. a. 

b. 60°

12. approximately 7.1 N 45° south of east

13. a. 7
b. The angle between f_1 and the resultant is 16.3°. The angle between $\vec{f_1}$ and the equilibrant is 163.7°.

14. a. 

For these three equal forces to be in equilibrium, they must form an equilateral triangle. Since the resultant will lie along one of these lines, and since all angles of an equilateral triangle are 60°, the resultant will be at a 60° angle with the other two vectors.

b. Since the equilibrant is directed opposite the resultant, the angle between the equilibrant and the other two vectors is $180° - 60° = 120°$.

15. 7.65 N, 67.5° from $\vec{f_2}$ toward $\vec{f_3}$

16. 45° rope: 175.73 N
30° rope: 143.48

17. 24 cm string: approximately 39.2 N,
32 cm string: approximately 29.4 N

18. 8.5° to the starboard side

19. **a.** magnitude for resultant and equilbrant $\doteq 13.75$ N

b. $\theta_{5N} \doteq 111.3°$, $\theta_{8N} \doteq 125.6°$, $\theta_{10N} \doteq 136.7°$

20. We know that the resultant of these two forces is equal in magnitude and angle to the diagonal line of the parallelogram formed with $\vec{f_1}$ and $\vec{f_2}$ as legs and has diagonal length $|\vec{f_1} + \vec{f_2}|$. We also know from the cosine rule that $|\vec{f_1} + \vec{f_2}|^2$
$= |\vec{f_1}|^2 + |\vec{f_2}|^2 - 2|\vec{f_1}||\vec{f_2}|\cos\phi$,
where ϕ is the supplement to θ in, our parallelogram. Since we know $\phi = 180 - \theta$, then $\cos\phi = \cos(180 - \theta) = -\cos\theta$. Thus, we have

$$|\vec{f_1} + \vec{f_2}|^2 = |\vec{f_1}|^2 + |\vec{f_2}|^2 - 2|\vec{f_1}||\vec{f_2}|\cos\phi$$

$$|\vec{f_1} + \vec{f_2}|^2 = |\vec{f_1}|^2 + |\vec{f_2}|^2 - 2|\vec{f_1}||\vec{f_2}|\cos\phi$$

$$|\vec{f_1} + \vec{f_2}| = \sqrt{|\vec{f_1}|^2 + |\vec{f_2}|^2 + 2|\vec{f_1}||\vec{f_2}|\cos\theta}$$

Section 7.2, pp. 369–370

1. **a.** 84 km/h in the direction of the train's movement
b. 76 km/h in the direction of the train's movement

2. **a.** 500 km/h north
b. 700 km/h north

3. 304.14, W 9.5° S

4. 60° upstream

5. **a.** 2 m/s forward
b. 22 m/s in the direction of the car

6. 13 m/s, N 37.6° W

7. **a.** 732.71 km/h, N 5.5° W
b. about 732.71 km

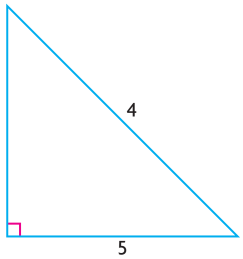
8. **a.** about 1383 km
b. about 12.5° east of north

9. **a.** about 10.4° south of west
b. 2 h, 53.1° downstream to the bank

10. **a.** 5 km/h
b. about 0.67 km
c. 20 min

11. **a.** about 18.4° west of north
b. about 108 km/h

12.



Since her swimming speed is a maximum of 4 km/h, this is her maximum resultant magnitude, which is also the hypotenuse of the triangle formed by her and the river's velocity vector. Since one of these legs is 5 km/h, we have a triangle with a leg larger than its hypotenuse, which is impossible.

13. **a.** about 68 m
b. 100 s

14. **a.** about 58.5°, upstream
b. about 58.6 s

15. 35 h

Section 7.3, pp. 377–378

1. To be guaranteed that the two vectors are perpendicular, the vectors must be nonzero.

2. $\vec{a} \cdot \vec{b}$ is a scalar, and a dot product is only defined for vectors.

3. Answers may vary. For example, let $\vec{a} = \hat{i}, \vec{b} = \vec{j}, \vec{c} = -\vec{i}\,\hat{i} \cdot \vec{a} \cdot \vec{b} \cdot \vec{b} = 0$, $\vec{a} \cdot \vec{c} \cdot \vec{c} = 0$, but $\vec{a} = -\vec{c}$.

4. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} = \vec{b} \cdot \vec{c}$ because $\vec{c} = \vec{a}$

5. -1

6. **a.** 16
b. -6.93
c. 0
d. -1
e. 0
f. -26.2

7. **a.** 30°
b. 80°
c. 53°
d. 127°
e. 60°
f. 120°

8. 22.5

9. **a.** $2|\vec{a}|^2 - 15|\vec{b}|^2 + 7\vec{a} \cdot \vec{b}$
b. $6|\vec{x}|^2 - 19\vec{x} \cdot \vec{y} + 3|\vec{y}|^2$

10. 0

11. 1

12. **a.** $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b})$
$= \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b}$
$= |\vec{a}|^2 + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{b} + |\vec{b}|^2$
$= |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2$

b. $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b})$
$= \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{b}$
$= |\vec{a}|^2 - \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{b} - |\vec{b}|^2$
$= |\vec{a}|^2 - |\vec{b}|^2$

13. **a.** $|\vec{a}|^2 = \vec{a} \cdot \vec{a}$
$= (\vec{b} + \vec{c}) \cdot (\vec{b} + \vec{c})$
$= |\vec{b}|^2 + 2\vec{b} \cdot \vec{c} + |\vec{c}|^2$

b. $\vec{b} \cdot \vec{c} = 0$, vectors are perpendicular. Therefore $|\vec{a}|^2 = |\vec{b}|^2 + |\vec{c}|^2$, which is the Pythagorean theory.

14. 14

15. $|\vec{u} + \vec{v}|^2 + |\vec{u} - \vec{v}|^2$
$= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$
$+ (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})$
$= |\vec{u}|^2 + 2\vec{u} \cdot \vec{v} + |\vec{v}|^2 + |\vec{u}|^2$
$- 2\vec{u} \cdot \vec{v} + |\vec{v}|^2$
$= 2|\vec{u}|^2 + 2|\vec{v}|^2$

16. 3

17. -7

18. $\vec{d} = \vec{b} - \vec{c}$
$\vec{b} = \vec{d} + \vec{c}$
$\vec{c} \cdot \vec{a} = ((\vec{b} \cdot \vec{a})\vec{a}) \cdot \vec{a}$
$\vec{c} \cdot \vec{a} = (\vec{b} \cdot \vec{a})(\vec{a} \cdot \vec{a})$ because
$\vec{b} \cdot \vec{a}$ is a scalar
$\vec{c} \cdot \vec{a} = (\vec{b} \cdot \vec{a})|\vec{a}|^2$
$\vec{c} \cdot \vec{a} = (\vec{d} + \vec{c}) \cdot \vec{a}$ because $|\vec{a}| = 1$
$\vec{c} \cdot \vec{a} = \vec{d} \cdot \vec{a} + \vec{c} \cdot \vec{a}$
$\vec{d} \cdot \vec{a} = 0$

Section 7.4, pp. 385–387

1. Any vector of the form (c, c) is perpendicular to $\vec{a}$. Therefore, there are infinitely many vectors perpendicular to $\vec{a}$. Answers may vary. For example: $(1, 1), (2, 2), (3, 3)$.

2. **a.** 0; 90°
b. $34 > 0$; acute
c. $-3 < 0$; obtuse

3. Answer may vary. For example:
a. $(0, 0, 1)$ is perpendicular to every vector in the *xy*-plane.
b. $(0, 1, 0)$ is perpendicular to every vector in the *xz*-plane.
c. $(1, 0, 0)$ is perpendicular to every vector in the *yz*-plane.

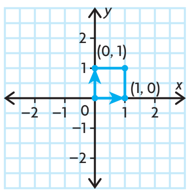
4. **a.** $(1, 2, -1)$ and $(4, 3, 10)$;
$(-4, -5, -6)$ and $\left(5, -3, -\frac{5}{6}\right)$
b. no

5. **a.** The vectors must be in R^3, which is impossible
b. This is not possible since R^3 does not exist in R^2.

6. **a.** about 148°
b. about 123°
c. about 64°
d. about 154°

7. **a.** $k = \frac{2}{3}$
b. $k \geq 0$

8. **a.**



b. $(1, 1)$ and $(1, -1)$; $(1, 1)$ and $(-1, 1)$
c. $(1, 1) \cdot (1, -1)$
$= 1 - 1$
$= 0$
or
$(1, 1) \cdot (-1, 1)$
$= -1 + 1$
$= 0$

9. **a.** 90°
b. 30°

10. **a. i.** $p = \frac{8}{3}$; $q = 3$
ii. Answers may vary. For example, $p = 1, q = -50$.
b. Unique for collinear vectors; not unique for perpendicular vectors

11. $\theta_A = 90°$; $\theta_B \doteq 26.6°$; $\theta_C \doteq 63.4°$

12. **a.** $O = (0, 0, 0)$, $A = (7, 0, 0)$, $B = (7, 4, 0)$, $C = (0, 4, 0)$, $D = (7, 0, 5)$, $E = (0, 4, 5)$, $F = (0, 0, 5)$
b. 50°

13. **a.** Answers may vary. For example, $(3, 1, 1)$.
b. Answers may vary. For example, $(1, 1, 1)$.

14. 3 or −1

15. **a.** $3 + 4p + q = 0$
b. 0

16. Answers may vary. For example, $(1, 0, 1)$ and $(1, 1, 3)$.
$(x, y, z)(1, 2, -1) = 0$
$x + 2y - z = 0$
Let $x = z = 1$.
$(1, 0, 1)$ is perpendicular to $(1, 2, -1)$ and $(-2, -4, 2)$.
Let $x = y = 1$.
$(1, 1, 3)$ is perpendicular to $(1, 2, -1)$ and $(-2, -4, 2)$.

17. 4 or $-\frac{44}{65}$

18. **a.** $\vec{a} \cdot \vec{b} = 0$
Therefore, since the two diagonals are perpendicular, all the sides must be the same length.
b. $\overrightarrow{AB} = (1, 2, -1)$,
$\overrightarrow{BC} = (2, 1, 1)$,
$|\overrightarrow{AB}| = |\overrightarrow{BC}| = \sqrt{6}$
c. $\theta_1 = 60°$; $\theta_2 = 120°$

19. **a.** $(6, 18, -4)$
b. 87.4°

20. $\alpha \doteq 109.5°$ or $\theta \doteq 70.5°$

Mid-Chapter Review, pp. 388–389

1. **a.** 3
b. 81

2. 15 cm cord: 117.60 N;
20 cm cord: 88.20 N

3. 0

4. **a.** about 575.1 km/h at S 7.06° E
b. about 1.74 h

5. **a.** about 112.61 N
b. about 94.49 N

6. 4.5

7. **a.** 34
b. $\frac{34}{63}$

8. **a.** 0
b. 5
c. $5\vec{i} - 4\vec{j} + 3\vec{k}$
d. 0
e. 34
f. 9

9. **a.** $x = -3$ or $x = -\frac{1}{3}$
b. no value

10. **a.** $\vec{i} - 4\vec{j} - \vec{k}$
b. 24
c. $\sqrt{2}$ or 1.41
d. −4
e. −12

11. about 126.9°

12. $\vec{F} \doteq 6.08$ N, 25.3° from the 4 N force towards the 3 N force. $\vec{E} \doteq 6.08$ N, $180° - 25.3° = 154.7°$ from the 4 N force away from the 3 N force.

13. **a.** about 109.1°
b. about 87.9°

14. **a.** about N 2.6° E
b. about 2.17 h

15. $\vec{x} = \left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ or $\left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$

16. **a.** about 6.12 m
b. about 84.9 s

17. **a.** when $\vec{x}$ and $\vec{y}$ have the same length
b. Vectors $\vec{a}$ and $\vec{b}$ determine a parallelogram. Their sum $\vec{a} + \vec{b}$ is one diagonal of the parallelogram formed, with its tail in the same location as the tails of $\vec{a}$ and $\vec{b}$. Their difference $\vec{a} - \vec{b}$ is the other diagonal of the parallelogram.

18. about 268.12 N

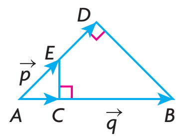
Section 7.5, pp. 398–400

1. **a.** scalar projection $= 2$,
vector projection $= 2\vec{i}$
b. scalar projection $= 3$,
vector projection $= 3\vec{j}$

2. Using the formula would cause a division by 0. Generally the $\vec{0}$ has any direction and 0 magnitude. You cannot project onto nothing.

3. You are projecting $\vec{a}$ onto the tail of $\vec{b}$, which is a point with magnitude 0. Therefore, it is $\vec{0}$; the projections of $\vec{b}$ onto the tail of $\vec{a}$ are also 0 and $\vec{0}$.

4. Answers may vary. For example, $\vec{p} = \overrightarrow{AE}$, $\vec{q} = \overrightarrow{AB}$



scalar projection $\vec{p}$ on $\vec{q} = |\overrightarrow{AC}|$,
vector projection $\vec{p}$ on $\vec{q} = \overrightarrow{AC}$,
scalar projection $\vec{q}$ on $\vec{p} = |\overrightarrow{AD}|$,
vector projection $\vec{q}$ on $\vec{p} = \overrightarrow{AD}$

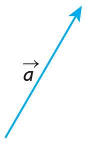
5. scalar projection of $\vec{a}$ on $\vec{i} = -1$,
vector projection of $\vec{a}$ on $\vec{i} = -\vec{i}$,
scalar projection of $\vec{a}$ on $\vec{j} = 2$,
vector projection of $\vec{a}$ on $\vec{j} = 2\vec{j}$,
scalar projection of $\vec{a}$ on $\vec{k} = -5$,
vector projection of $\vec{a}$ on $\vec{k} = -5\vec{k}$;
Without having to use formulae, a projection of $(-1, 2, 5)$ on $\vec{i}$, $\vec{j}$, or $\vec{k}$ is the same as a projection of $(-1, 0, 0)$ on $\vec{i}$, $(0, 2, 0)$ on $\vec{j}$, and $(0, 0, 5)$ on $\vec{k}$, which intuitively yields the same result.

6. **a.** scalar projection: $\frac{\vec{p} \cdot \vec{q}}{|\vec{q}|} = \frac{458}{21}$,
vector projection: $\frac{458}{441}(-4, 5, -20)$
b. about 82.5°, about 74.9°, about 163.0°

7. **a.** scalar projection: 0,
vector projection: $\vec{0}$
b. scalar projection: 2,
vector projection: $2\vec{i}$
c. scalar projection: $\frac{50}{13}$,
vector projection: $\frac{50}{169}(-5, 12)$

8. **a.** The scalar projection of $\vec{a}$ on the x-axis $(X, 0, 0)$ is -1; The vector projection of $\vec{a}$ on the x-axis is $-\vec{i}$; The scalar projection of $\vec{a}$ on the y-axis $(0, Y, 0)$ is 2; The vector projection of $\vec{a}$ on the y-axis is $2\vec{j}$; The scalar projection of $\vec{a}$ on the z-axis $(0, 0, Z)$ is 4; The vector projection of $\vec{a}$ on the z-axis is $4\vec{k}$.
b. The scalar projection of $m\,\vec{a}$ on the x-axis $(X, 0, 0)$ is $-m$; The vector projection of $m\,\vec{a}$ on the x-axis is $-m\vec{i}$; The scalar projection of $m\,\vec{a}$ on the y-axis $(0, Y, 0)$ is $2m$; The vector projection $m\,\vec{a}$ on the y-axis $(0, Y, 0)$ is $2m\vec{j}$; The scalar projection of $m\,\vec{a}$ on the z-axis $(0, 0, Z)$ is $4m$; The vector projection of $m\,\vec{a}$ on the z-axis is $4m\vec{k}$.

9. **a.**



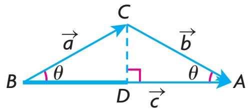
vector projection: $\vec{a}$
scalar projection: $|\vec{a}|$
b. $|\vec{a}|\cos\theta = |\vec{a}|\cos 0$
$= |\vec{a}|(1)$
$= |\vec{a}|.$
The vector projection is the scalar projection multiplied by $\frac{\vec{a}}{|\vec{a}|}$,
$|\vec{a}| \times \frac{\vec{a}}{|\vec{a}|} = \vec{a}.$

10. **a.**

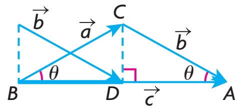
b. $\frac{(-\vec{a}) \cdot \vec{a}}{|\vec{a}|} = \frac{-|\vec{a}|^2}{|\vec{a}|}$
$= -|\vec{a}|$
So, the vector projection is
$-|\vec{a}|\left(\frac{\vec{a}}{|\vec{a}|}\right) = -\vec{a}.$

11. **a.** scalar projection of $\overrightarrow{AB}$ on the x-axis is -2; vector projection of $\overrightarrow{AB}$ on the x-axis is $-2\vec{i}$; scalar projection of $\overrightarrow{AB}$ on the y-axis is 1; vector projection of $\overrightarrow{AB}$ on the y-axis is $\vec{j}$; scalar projection of $\overrightarrow{AB}$ on the z-axis is 2; vector projection of $\overrightarrow{AB}$ on the z-axis is $2\vec{k}$.
b. 70.5°

12. **a.** $|\overrightarrow{BD}|$

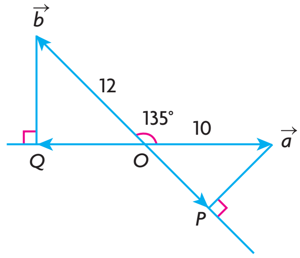


b. $|\overrightarrow{BD}|$



c. In an isosceles triangle, CD is a median and a right bisector of BA.
d. Yes

13. **a.** -7.07, -8.49
b.



$\overrightarrow{OQ}$ is the vector projection of $\vec{b}$ on $\vec{a}$
$\overrightarrow{OP}$ is the vector projection of $\vec{a}$ on $\vec{b}$

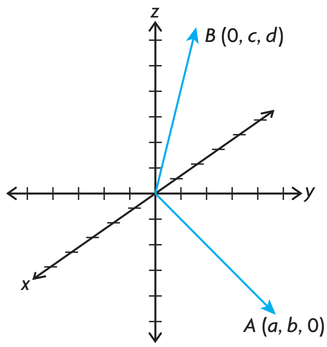
14. **a.** $-\frac{1}{3}$
b. The scalar projection of $\overrightarrow{BC}$ on $\overrightarrow{OD}$ is $\frac{19}{3} \cdot -\frac{1}{3} + \frac{19}{13} = 6$
The scalar projection of $\overrightarrow{AC}$ on $\overrightarrow{OD}$ is 6.
c. Same lengths and both are in the direction of $\overrightarrow{OD}$. Add to get one vector.

15. **a.** $1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$
$$= \left(\frac{a}{\sqrt{a^2 + b^2 + c^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2 + b^2 + c^2}}\right)^2 + \left(\frac{c}{\sqrt{a^2 + b^2 + c^2}}\right)^2$$
$$= \frac{a^2}{a^2 + b^2 + c^2} + \frac{b^2}{a^2 + b^2 + c^2} + \frac{c^2}{a^2 + b^2 + c^2}$$
$$= \frac{a^2 + b^2 + c^2}{a^2 + b^2 + c^2}$$
$$= 1$$
b. Answers may vary. For example:
$\left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $(0, \sqrt{3}, 1)$
c. If two angles add to 90°, then all three will add to 180°.

16. **a.** about 54.7°
b. about 125.3°

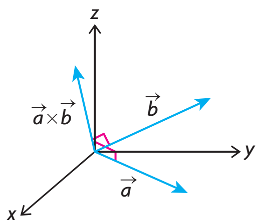
17. $\cos^2 x + \sin^2 x = 1$
$\cos^2 x = 1 - \sin^2 x$
$1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$
$1 = (1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma)$
$1 = 3 - (\sin^2\alpha + \sin^2\beta + \sin^2\gamma)$
$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

18. Answers may vary. For example:

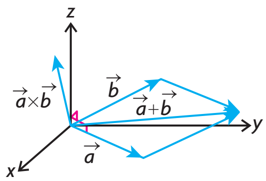


Section 7.6, pp. 407–408

1. **a.**



$\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$. Thus, their dot product must equal 0. The same applies to the second case.



b. $\vec{a} + \vec{b}$ is still in the same plane formed by $\vec{a}$ and $\vec{b}$, thus $\vec{a} + \vec{b}$ is perpendicular to $\vec{a} \times \vec{b}$ making the dot product 0 again.
c. Once again, $\vec{a} - \vec{b}$ is still in the same plane formed by $\vec{a}$ and $\vec{b}$, thus $\vec{a} - \vec{b}$ is perpendicular to $\vec{a} \times \vec{b}$ making the dot product 0 again.

2. $\vec{a} \times \vec{b}$ produces a vector, not a scalar. Thus, the equality is meaningless.

3. **a.** It's possible because there is a vector crossed with a vector, then dotted with another vector, producing a scalar.
   b. This is meaningless because $\vec{a} \cdot \vec{b}$ produces a scalar. This results in a scalar crossed with a vector, which is meaningless.
   c. This is possible. $\vec{a} \times \vec{b}$ produces a vector, and $\vec{c} + \vec{d}$ also produces a vector. The result is a vector dotted with a vector producing a scalar.
   d. This is possible. $\vec{a} \times \vec{b}$ produces a scalar, and $\vec{c} \times \vec{d}$ produces a vector. The product of a scalar and vector produces a vector.
   e. This is possible. $\vec{a} \times \vec{b}$ produces a vector, and $\vec{c} \times \vec{d}$ produces a vector. The cross product of a vector and vector produces a vector.
   f. This is possible. $\vec{a} \times \vec{b}$ produces a vector. When added to another vector, it produces another vector.

4. **a.** $(-7, -8, -2)$
   b. $(1, 5, 1)$
   c. $(-11, -33, 22)$
   d. $(-19, -22, 7)$
   e. $(3, 3, -1)$
   f. $(-8, -26, 11)$

5. 1

6. **a.** $(-4, 0, 0)$
   b. Vectors of the form $(0, b, c)$ are in the yz-plane. Thus, the only vectors perpendicular to the yz-plane are those of the form $(a, 0, 0)$ because they are parallel to the x-axis.

7. **a.** $(1, 2, 1) \times (2, 4, 2)$
   $= (2(2) - 1(4), 1(2) - 1(2), 1(4) - 2(2))$
   $= (0, 0, 0)$
   b. $(a, b, c) \times (ka, kb, kc)$
   $= (b(kc) - c(kb), c(ka) - a(kc), a(kb) - b(ka))$
   Using the associative law of multiplication, we can rearrange this:
   $= (bck - bck, ack - ack, abk - abk)$
   $= (0, 0, 0)$

8. **a.** $\vec{p} \times (\vec{q} + \vec{r}) = (-26, -7, 3)$
   $\vec{p} \times \vec{q} + \vec{p} \times \vec{r} = (-26, -7, 3)$
   b. $\vec{p} \times (\vec{q} + \vec{r}) = (-3, 2, 5)$
   $\vec{p} \times \vec{q} + \vec{p} \times \vec{r} = (-3, 2, 5)$

9. **a.** $\vec{i} \times \vec{j} = (1, 0, 0) \times (0, 1, 0) = \vec{k}$
   $-\vec{j} \times \vec{i} = (0, -1, 0) \times (1, 0, 0) = \vec{k}$
   b. $\vec{j} \times \vec{k} = (0, 1, 0) \times (0, 0, 1) = \vec{i}$
   $-\vec{k} \times \vec{j} = (0, 0, -1) \times (0, 1, 0) = \vec{i}$
   c. $\vec{k} \times \vec{i} = (0, 0, 1) \times (1, 0, 0) = \vec{j}$
   $-\vec{i} \times \vec{k} = (-1, 0, 0) \times (0, 0, 1) = \vec{j}$

10. $k(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)(a_1, a_2, a_3)$
    $= k(a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_2a_1b_3 + a_3a_1b_2 - a_3a_2b_1)$
    $= k(0)$
    $= 0$
    $\vec{a}$ is perpendicular to $k(\vec{a} \times \vec{b})$.

11. **a.** $(0, 0, 6), (0, 0, -6)$
    b. $(0, 0, 0)$
    c. All the vectors are in the xy-plane. Thus, their cross product in part b. is between vectors parallel to the z-axis and so parallel to each other. The cross product of parallel vectors is $\vec{0}$.

12. Let $\vec{x} = (1, 0, 1)$, $\vec{y} = (1, 1, 1)$, and $\vec{z} = (1, 2, 3)$
    Then, $\vec{x} \times \vec{y} = (0 - 1, 1 - 1, 1 - 0)$
    $= (-1, 0, 1)$
    $(\vec{x} \times \vec{y}) \times \vec{z}$
    $= (0 - 2, 1 - (-3), -3 - 0)$
    $= (-2, 4, -3)$
    $\vec{y} \times \vec{z} = (3 - 2, 1 - 3, 2 - 1)$
    $= (1, -2, 1)$
    $\vec{x} \times (\vec{y} \times \vec{z}) = (0 + 2, 1 - 1, -2 - 0)$
    $= (2, 0, -2)$
    Thus, $(\vec{x} \times \vec{y}) \times \vec{z} \neq \vec{x} \times (\vec{y} \times \vec{z})$.

13. By the distributive property of cross product:
    $= (\vec{a} - \vec{b}) \times \vec{a} + (\vec{a} - \vec{b}) \times \vec{b}$
    By the distributive property again:
    $= \vec{a} \times \vec{a} - \vec{b} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{b}$
    A vector crossed with itself equals $\rightarrow 0$, thus:
    $= -\vec{b} \times \vec{a} + \vec{a} \times \vec{b}$
    $= \vec{a} \times \vec{b} - \vec{b} \times \vec{a}$
    $= \vec{a} \times \vec{b} - (-\vec{a} \times \vec{b})$
    $= 2\vec{a} \times \vec{b}$

Section 7.7, pp. 414–415

1. By pushing as far away from the hinge as possible, $|\vec{r}|$ is increased, making the cross product bigger. By pushing at right angles, sine is its largest value, 1, making the cross product larger.

2. **a.** 0
   b. This makes sense because the vectors lie on the same line. Thus, the parallelogram would just be a line making its area 0.

3. **a.** 450 J
   b. about 10 078.91 J
   c. about 32 889.24 J
   d. 35 355.34 J

4. **a.** $\vec{k}$
   b. $-\vec{k}$
   c. $-\vec{j}$
   d. $\vec{j}$

5. **a.** $\sqrt{3}$ square units
   b. $\sqrt{213}$ square units

6. $2, \frac{-12}{5}$

7. **a.** $\frac{5\sqrt{6}}{2}$ square units
   b. $\frac{5\sqrt{6}}{2}$ square units.
   c. Any two sides of a triangle can be used to calculate its area.

8. about 0.99 J

9. $\frac{6}{\sqrt{26}}$ or about 1.18

10. **a.** $\vec{p} \times \vec{q} = (-6 - 3, 6 - 3, 1 + 4)$
    $= (-9, 3, 5)$
    $(\vec{p} \times \vec{q}) \times \vec{r}$
    $= (0 - 5, 5 + 0, -9 - 3)$
    $= (-5, 5, -12)$
    $a(1, -2, 3) + b(2, 1, 3)$
    $= (-5, 5, -12)$
    Looking at x-components:
    $a + 2b = -5$
    $a = -5 - 2b$
    y-components:
    $-2a + b = 5$
    Substitute a:
    $10 + 4b + b = 5$
    $5b = -5$
    $b = -1$
    Substitute b back into the x-components:
    $a = -5 + 2$
    $a = -3$
    Check in z-components:
    $3a + 3b = -12$
    $-9 - 3 = -12$
    b. $\vec{p} \cdot \vec{r} = 1 - 2 + 0 = -1$
    $\vec{q} \cdot \vec{r} = 2 + 1 + 0 = 3$
    $(\vec{p} \cdot \vec{r})\vec{q} - (\vec{q} \cdot \vec{r})\vec{p}$
    $= -1(2, 1, 3) - 3(1, -2, 3)$
    $= (-2, -1, -3) - (3, -6, 9)$
    $= (-2 - 3, -1 + 6, -3 - 9)$
    $= (-5, 5, -12)$

Review Exercise, pp. 418–421

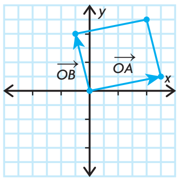
1. **a.** $(2, 0, 2)$
   b. $(-4, 0, -4)$
   c. 16
   d. The cross products are parallel, so the original vectors are in the same plane.

2. a. 3
   b. 7
   c. $4\sqrt{3}$
   d. $2\sqrt{17}$
   e. 5
   f. -1

3. a. 6
   b. $-\frac{54}{5}$

4. about 18.52°

5. a.

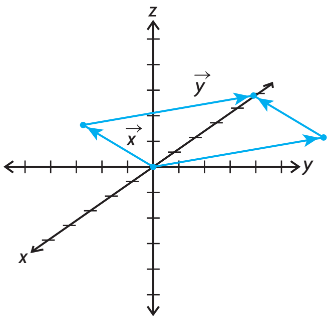


   b. about 77.9°

6. rope at 45°: about 87.86 N,
   rope at 30°: about 71.74 N

7. 304.14 km/h, W 9.46° N

8. a.



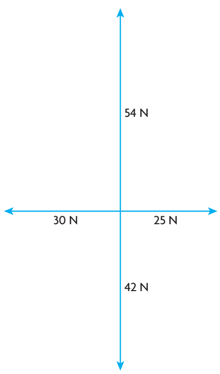
   b. approximately 56.78

9. $\left(\frac{9}{\sqrt{115}}, -\frac{5}{\sqrt{115}}, -\frac{3}{\sqrt{115}}\right)$

10. a. about 77.64° is the largest angle
    b. 36.50

11. 30 cm string: 78.4 N;
    40 cm string: 58.8 N

12. a.



    b. The resultant is 13 N in a direction N22.6°W. The equilibrant is 13 N in a direction S22.6°E.

13. a. Let D be the origin, then:
    $A = (2, 0, 0), B = (2, 4, 0),$
    $C = (0, 4, 0), D = (0, 0, 0),$
    $E = (2, 0, 3), F = (2, 4, 3),$
    $G = (0, 4, 3), H = (0, 0, 3)$
    b. about 44.31°
    c. about 3.58

14. 7.5

15. a. about 48.2°
    b. about 8 min 3 s
    c. Such a situation would have resulted in a right triangle where one of the legs is longer than the hypotenuse, which is impossible.

16. a. $\overrightarrow{OA} + \overrightarrow{OB} = (-3, 8, -8),$
    $\overrightarrow{OA} - \overrightarrow{OB} = (9, -4, -4)$
    b. about 84.36°

17. a. $a = 4$ and $b = -4$
    b. $\vec{p} \cdot \vec{q} = 2a - 2b - 18 = 0$
    c. $\left(\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right)$

18. a. about 74.62°
    b. about 0.75
    c. $(0.1875)(\sqrt{3}, -2, -3)$
    d. about 138.59°

19. a. special
    b. not special

20. a. $(-1, 1, 3)$
    b. $(-2, 2, 6)$
    c. 0
    d. $(-1, 1, 3)$

21. about 11.55 N

22. $(2, -8, -10)$

23. -141

24. 5 or -7

25. about 103.34°

26. a. $C = (3, 0, 5), F = (0, 4, 0)$
    b. $(-3, 4, -5)$
    c. about 111.1°

27. a. about 7.30
    b. about 3.84
    c. about 3.84

28. a. scalar: 1,
    vector: $\vec{i}$
    b. scalar: 1,
    vector: $\vec{j}$
    c. scalar: $\frac{1}{\sqrt{2}}$,
    vector: $\frac{1}{2}(\vec{k} + \vec{j})$

29. a. $|\vec{b}|, |\vec{c}|$
    b. $\vec{a}$; When dotted with $\vec{d}$, it equals 0.

30. 7.50 J

31. a. $\vec{a} \cdot \vec{b} = 6 - 5 - 1 = 0$
    b. $\vec{a}$ with the x-axis:
    $|\vec{a}| = \sqrt{4 + 25 + 1} = \sqrt{30}$
    $\cos(\alpha) = \frac{2}{\sqrt{30}}$
    $\vec{a}$ with the y-axis:
    $\cos(\beta) = \frac{5}{\sqrt{30}}$
    $\vec{a}$ with the z-axis:
    $\cos(\gamma) = \frac{-1}{\sqrt{30}}$
    $|\vec{b}| = \sqrt{9 + 1 + 1} = \sqrt{11}$
    $\vec{b}$ with the x-axis:
    $\cos(\alpha) = \frac{3}{\sqrt{11}}$
    $\vec{b}$ with the y-axis:
    $\cos(\beta) = \frac{-1}{\sqrt{11}}$
    $\vec{b}$ with the z-axis:
    $\cos(\gamma) = \frac{1}{\sqrt{11}}$
    c. $\vec{m}_1 \times \vec{m}_2 = \frac{6}{\sqrt{330}} - \frac{5}{\sqrt{330}} - \frac{1}{\sqrt{330}} = 0$

32. $|3\vec{i} + 3\vec{j} + 10\vec{k}| = \sqrt{118}$
    $|-\vec{i} + 9\vec{j} - 6\vec{k}| = \sqrt{118}$

33. a. $\cos\alpha = \frac{\sqrt{3}}{2},$
    $\cos\beta = \cos\gamma = \pm\frac{1}{2\sqrt{2}}$
    b. acute case: 69.3°,
    obtuse case: 110.7°

34. -5

35. $|\vec{a} + \vec{b}| = \sqrt{1 + 1 + 64} = \sqrt{66}$
    $|\vec{a} - \vec{b}| = \sqrt{1 + 81 + 16} = \sqrt{98}$
    $\frac{1}{4}|\vec{a} + \vec{b}|^2 - \frac{1}{4}|\vec{a} - \vec{b}|^2$
    $= \frac{66}{4} - \frac{98}{4} = -8$

36. $\vec{c} = \vec{b} - \vec{a}$
    $|\vec{c}|^2 = |\vec{b} - \vec{a}|^2$
    $= (\vec{b} - \vec{a})(\vec{b} - \vec{a})$
    $= |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b}$
    $= |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$

37. $\overrightarrow{AB} = (2, 0, 4)$
    $|\overrightarrow{AB}| = 2\sqrt{5}$
    $\overrightarrow{AC} = (1, 0, 2)$
    $|\overrightarrow{AC}| = \sqrt{5}$
    $\overrightarrow{BC} = (-1, 0, -2)$
    $|\overrightarrow{BC}| = \sqrt{5}$

$\cos A = 1$
$\cos B = 1$
$\cos C = -1$
area of triangle $ABC = 0$

Chapter 7 Test, p. 422

1. **a.** $(-4, -1, -3)$
b. $(-4, -1, -3)$
c. 0
d. 0

2. **a.** scalar projection: $\frac{1}{3}$,
vector projection: $\frac{1}{9}(2, -1, -2)$.
b. x-axis: 48.2°; y-axis: 109.5°; z-axis: 131.8°
c. $\sqrt{26}$ or 5.10

3. Both forces have a magnitude of 78.10 N. The resultant makes an angle 33.7° to the 40 N force and 26.3° to the 50 N force. The equilibrant makes an angle 146.3° to the 40 N force and 153.7° to the 50 N force.

4. 1004.99 km/h, N 5.7° W

5. **a.** 96 m downstream
b. 28.7° upstream

6. 3.50 square units.

7. cord at 45°: about 254.0 N; cord at 70°: about 191.1 N

8. **a.** 0
33
$$\frac{1}{4}|\vec{x} + \vec{y}|^2 - \frac{1}{4}|\vec{x} - \vec{y}|^2$$
$$= \frac{1}{4}(33) - \frac{1}{4}(33) = 0$$
So, the equation holds for these vectors.

b.
$$\begin{aligned}|\vec{x} + \vec{y}|^2 &= (\vec{x} + \vec{y})(\vec{x} + \vec{y})\\ &= (\vec{x}\cdot\vec{x}) + (\vec{x}\cdot\vec{y})\\ &\quad + (\vec{y}\cdot\vec{x}) + (\vec{y}\cdot\vec{y})\\ &= (\vec{x}\cdot\vec{x}) + 2(\vec{x}\cdot\vec{y})\\ &\quad + (\vec{y}\cdot\vec{y})\end{aligned}$$
$$\begin{aligned}|\vec{x} - \vec{y}|^2 &= (\vec{x} - \vec{y})(\vec{x} - \vec{y})\\ &= (\vec{x}\cdot\vec{x}) + (\vec{x}\cdot -\vec{y})\\ &\quad + (-\vec{y}\cdot\vec{x})\\ &\quad + (-\vec{y}\cdot -\vec{y})\\ &= (\vec{x}\cdot\vec{x}) - 2(\vec{x}\cdot\vec{y})\\ &\quad + (\vec{y}\cdot\vec{y})\end{aligned}$$
So, the right side of the equation is
$$\frac{1}{4}|\vec{x} + \vec{y}|^2 - \frac{1}{4}|\vec{x} - \vec{y}|^2$$
$$= \frac{1}{4}(4(\vec{x}\cdot\vec{y}))$$
$$= \vec{x}\cdot\vec{y}$$

Chapter 8

Review of Prerequisite Skills, pp. 424–425

1. **a.** $(2, -9, 6)$
b. $(13, -12, -41)$

2. **a.** yes **c.** yes
b. yes **d.** no

3. yes

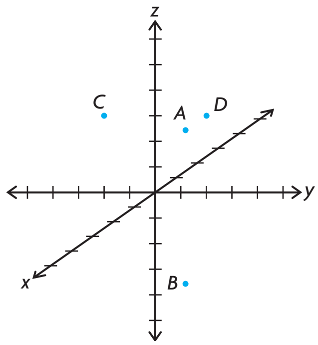
4. $t = 18$

5. **a.** $(3, 1)$
b. $(5, 6)$
c. $(-4, 7, 0)$

6. $\sqrt{2802}$

7. **a.** $(-22, -8, -13)$
b. $(0, 0, -3)$

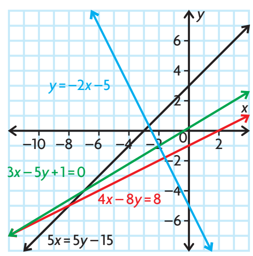
8.



9. **a.** $(-7, -3)$
b. $(10, 14)$
c. $(2, -8, 5)$
d. $(-4, 5, 4)$

10. **a.** $(7, 3)$
b. $(-10, -14)$
c. $(-2, 8, -5)$
d. $(4, -5, -4)$

11. **a.** slope: -2; y-intercept: -5
b. slope: $\frac{1}{2}$; y-intercept: -1
c. slope: $\frac{3}{5}$; y-intercept: $\frac{1}{5}$
d. slope: 1; y-intercept: 3



12. Answers may vary. For example:
a. $(8, 14)$
b. $(-15, 12, 9)$
c. $\vec{i} + 3\vec{j} - 2\vec{k}$
d. $-20\vec{i} + 32\vec{j} + 8\vec{k}$

13. **a.** 33
b. -33
c. 77
d. $(-11, -8, 28)$
e. $(11, 8, -28)$
f. $(55, 40, -140)$

14. The dot product of two vectors yields a real number, while the cross product of two vectors gives another vector.

Section 8.1, pp. 433–434

1. Direction vectors for a line are unique only up to scalar multiplication. So, since each of the given vectors is just a scalar multiple of $\left(\frac{1}{3}, \frac{1}{6}\right)$, each is an acceptable direction vector for the line.

2. **a.** Answers may vary. For example, $(-2, 7)$, $(1, 5)$, and $(4, 3)$.
b. $t = -5$
If $t = -5$, then $x = -14$ and $y = 15$. So $P(-14, 15)$ is a point on the line.

3. Answers may vary. For example:
a. direction vector: $(2, 1)$; point: $(3, 4)$
b. direction vector: $(2, -7)$; point: $(1, 3)$
c. direction vector: $(0, 2)$; point: $(4, 1)$
d. direction vector: $(-5, 0)$; point: $(0, 6)$

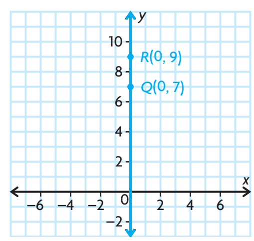
4. Answers may vary. For example:
$\vec{r} = (2, 1) + t(-5, 4), t \in \mathbf{R}$
$\vec{q} = (-3, 5) + s(5, -4), s \in \mathbf{R}$

5. **a.** $R(-9, 18)$ is a point on the line. When $t = 7$, $x = -9$ and $y = 18$.
b. Answers may vary. For example: $\vec{r} = (-9, 18) + t(-1, 2), t \in \mathbf{R}$
c. Answers may vary. For example: $\vec{r} = (-2, 4) + t(-1, 2), t \in \mathbf{R}$

6. Answers may vary. For example:
a. $(-3, -4)$, $(0, 0)$, and $(3, 4)$
b. $\vec{r} = t(1, 1), t \in \mathbf{R}$
c. This describes the same line as part a.

7. One can multiply a direction vector by a constant to keep the same line, but multiplying the point yields a different line.

8. **a.**



b. $\vec{r} = (0, 7) + t(0, 2), t \in \mathbf{R}$; $x = 0$, $y = 7 + 2t, t \in \mathbf{R}$

$\cos A = 1$
$\cos B = 1$
$\cos C = -1$
area of triangle $ABC = 0$

Chapter 7 Test, p. 422

1. **a.** $(-4, -1, -3)$
b. $(-4, -1, -3)$
c. 0
d. 0

2. **a.** scalar projection: $\frac{1}{3}$,
vector projection: $\frac{1}{9}(2, -1, -2)$.
b. x-axis: 48.2°; y-axis: 109.5°; z-axis: 131.8°
c. $\sqrt{26}$ or 5.10

3. Both forces have a magnitude of 78.10 N. The resultant makes an angle 33.7° to the 40 N force and 26.3° to the 50 N force. The equilibrant makes an angle 146.3° to the 40 N force and 153.7° to the 50 N force.

4. 1004.99 km/h, N 5.7° W

5. **a.** 96 m downstream
b. 28.7° upstream

6. 3.50 square units.

7. cord at 45°: about 254.0 N;
cord at 70°: about 191.1 N

8. **a.** 0
33
$$\frac{1}{4}|\vec{x} + \vec{y}|^2 - \frac{1}{4}|\vec{x} - \vec{y}|^2$$
$$= \frac{1}{4}(33) - \frac{1}{4}(33) = 0$$
So, the equation holds for these vectors.

b.
$$\begin{aligned}|\vec{x} + \vec{y}|^2 &= (\vec{x} + \vec{y})(\vec{x} + \vec{y})\\ &= (\vec{x}\cdot\vec{x}) + (\vec{x}\cdot\vec{y})\\ &\quad + (\vec{y}\cdot\vec{x}) + (\vec{y}\cdot\vec{y})\\ &= (\vec{x}\cdot\vec{x}) + 2(\vec{x}\cdot\vec{y})\\ &\quad + (\vec{y}\cdot\vec{y})\end{aligned}$$
$$\begin{aligned}|\vec{x} - \vec{y}|^2 &= (\vec{x} - \vec{y})(\vec{x} - \vec{y})\\ &= (\vec{x}\cdot\vec{x}) + (\vec{x}\cdot -\vec{y})\\ &\quad + (-\vec{y}\cdot\vec{x})\\ &\quad + (-\vec{y}\cdot -\vec{y})\\ &= (\vec{x}\cdot\vec{x}) - 2(\vec{x}\cdot\vec{y})\\ &\quad + (\vec{y}\cdot\vec{y})\end{aligned}$$
So, the right side of the equation is
$$\frac{1}{4}|\vec{x} + \vec{y}|^2 - \frac{1}{4}|\vec{x} - \vec{y}|^2$$
$$= \frac{1}{4}(4(\vec{x}\cdot\vec{y}))$$
$$= \vec{x}\cdot\vec{y}$$

Chapter 8

Review of Prerequisite Skills, pp. 424–425

1. **a.** $(2, -9, 6)$
b. $(13, -12, -41)$

2. **a.** yes **c.** yes
b. yes **d.** no

3. yes

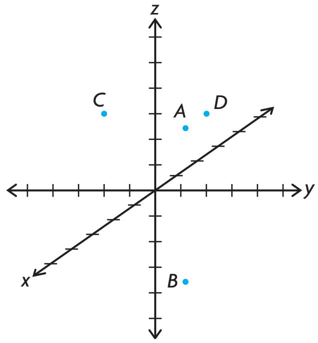
4. $t = 18$

5. **a.** $(3, 1)$
b. $(5, 6)$
c. $(-4, 7, 0)$

6. $\sqrt{2802}$

7. **a.** $(-22, -8, -13)$
b. $(0, 0, -3)$

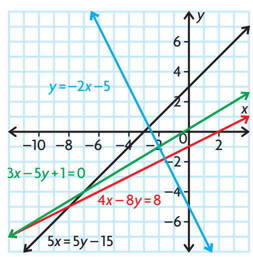
8.



9. **a.** $(-7, -3)$
b. $(10, 14)$
c. $(2, -8, 5)$
d. $(-4, 5, 4)$

10. **a.** $(7, 3)$
b. $(-10, -14)$
c. $(-2, 8, -5)$
d. $(4, -5, -4)$

11. **a.** slope: -2; y-intercept: -5
b. slope: $\frac{1}{2}$; y-intercept: -1
c. slope: $\frac{3}{5}$; y-intercept: $\frac{1}{5}$
d. slope: 1; y-intercept: 3



12. Answers may vary. For example:
a. $(8, 14)$
b. $(-15, 12, 9)$
c. $\vec{i} + 3\vec{j} - 2\vec{k}$
d. $-20\vec{i} + 32\vec{j} + 8\vec{k}$

13. **a.** 33
b. -33
c. 77
d. $(-11, -8, 28)$
e. $(11, 8, -28)$
f. $(55, 40, -140)$

14. The dot product of two vectors yields a real number, while the cross product of two vectors gives another vector.

Section 8.1, pp. 433–434

1. Direction vectors for a line are unique only up to scalar multiplication. So, since each of the given vectors is just a scalar multiple of $\left(\frac{1}{3}, \frac{1}{6}\right)$, each is an acceptable direction vector for the line.

2. **a.** Answers may vary. For example, $(-2, 7)$, $(1, 5)$, and $(4, 3)$.
b. $t = -5$
If $t = -5$, then $x = -14$ and $y = 15$. So $P(-14, 15)$ is a point on the line.

3. Answers may vary. For example:
a. direction vector: $(2, 1)$; point: $(3, 4)$
b. direction vector: $(2, -7)$; point: $(1, 3)$
c. direction vector: $(0, 2)$; point: $(4, 1)$
d. direction vector: $(-5, 0)$; point: $(0, 6)$

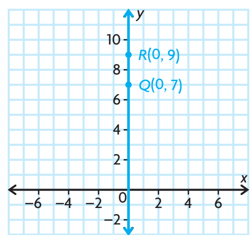
4. Answers may vary. For example:
$\vec{r} = (2, 1) + t(-5, 4), t \in \mathbf{R}$
$\vec{q} = (-3, 5) + s(5, -4), s \in \mathbf{R}$

5. **a.** $R(-9, 18)$ is a point on the line. When $t = 7$, $x = -9$ and $y = 18$.
b. Answers may vary. For example:
$\vec{r} = (-9, 18) + t(-1, 2), t \in \mathbf{R}$
c. Answers may vary. For example:
$\vec{r} = (-2, 4) + t(-1, 2), t \in \mathbf{R}$

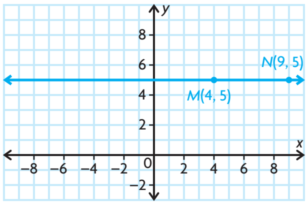
6. Answers may vary. For example:
a. $(-3, -4)$, $(0, 0)$, and $(3, 4)$
b. $\vec{r} = t(1, 1), t \in \mathbf{R}$
c. This describes the same line as part a.

7. One can multiply a direction vector by a constant to keep the same line, but multiplying the point yields a different line.

8. **a.**



b. $\vec{r} = (0, 7) + t(0, 2), t \in \mathbf{R}$; $x = 0$, $y = 7 + 2t, t \in \mathbf{R}$

9. **a.** 

b. $\vec{r} = (4, 5) + t(5, 0), t \in \mathbf{R}$;
$x = 4 + 5t, y = 5, t \in \mathbf{R}$

10. **a.** $\vec{r} = (2, 0) + t(5, -3), t \in \mathbf{R}$
b. $(0, -1.2)$

11. 9

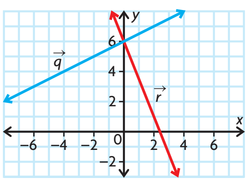
12. First, all the relevant vectors are found.
$\overrightarrow{AB} = (-2, 3)$
$\overrightarrow{AC} = (-4, 6)$
$\overrightarrow{AD} = (-6, 9)$
a. $\overrightarrow{AC} = (-4, 6) = 2(-2, 3) = 2\overrightarrow{AB}$
b. $\overrightarrow{AD} = (-6, 9) = 3(-2, 3) = 3\overrightarrow{AB}$
c. $\overrightarrow{AC} = (-4, 6) = \frac{2}{3}(-6, 9) = \frac{2}{3}\overrightarrow{AD}$

13. **a.** $A(5, 12)$; $B(-12, -5)$
b. $\sqrt{578}$ or about 24.04

14. In the parametric form, the second equation becomes $x = 1 + 6t$, $y = 6 + 4t, t \in \mathbf{R}$. If t is solved for in this equation, we obtain $t = \frac{x - 1}{6}$ and $t = \frac{y - 6}{4}$. Setting these two expressions equal to each other, the line is described by $\frac{x - 1}{6} = \frac{y - 6}{4}$, or by simplifying, $y - 6 = \frac{2}{3}x - \frac{2}{3}$. So, the second equation describes a line with a slope of $\frac{2}{3}$. If y is solved for in the first expression, we see that $y = \frac{2}{3}x + 5$. $(1, 6)$ is on the second line but not the first. Hence, both equations are lines with slope of $\frac{2}{3}$ and must be parallel.

Section 8.2, pp. 443–444

1. **a.** $\vec{m} = (6, -5)$
b. $\vec{n} = (5, 6)$
c. $(0, 9)$
d. $\vec{r} = (7, 9) + t(6, -5), t \in \mathbf{R}$;
$x = 7 + 6t, y = 9 - 5t, t \in \mathbf{R}$
e. $\vec{r} = (-2, 1) + t(5, 6), t \in \mathbf{R}$;
$x = -2 + 5t, y = 1 + 6t, t \in \mathbf{R}$

2. **a.–b.** 

c. It produces a different line.

3. **a.** $\vec{r} = (0, -6) + t(8, 7), t \in \mathbf{R}$;
$x = 8t, y = -6 + 7t, t \in \mathbf{R}$
b. $\vec{r} = (0, 5) + t(2, 3), t \in \mathbf{R}$;
$x = 2t, y = 5 + 3t, t \in \mathbf{R}$
c. $\vec{r} = (0, -1) + t(1, 0), t \in \mathbf{R}$;
$x = t, y = -1, t \in \mathbf{R}$
d. $\vec{r} = (4, 0) + t(0, 1), t \in \mathbf{R}$;
$x = 4, y = t, t \in \mathbf{R}$

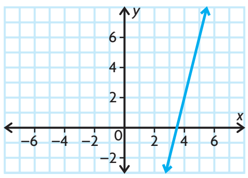
4. If the two lines have direction vectors that are collinear and share a point in common, then the two lines are coincident. In this example, both have $(3, 2)$ as a parallel direction vector and both have $(-4, 0)$ as a point on the line. Hence, the two lines are coincident.

5. **a.** The normal vectors for the lines are $(2, -3)$ and $(4, -6)$, which are collinear. Since in two dimensions, any two direction vectors perpendicular to $(2, -3)$ are collinear, the lines have collinear direction vectors. Hence, the lines are parallel.
b. $k = 12$

6. $4x + 5y - 21 = 0$

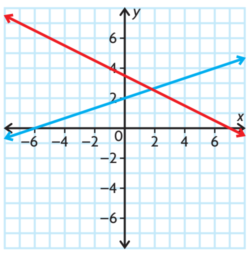
7. $x + y - 2 = 0$

8. $2x + y - 16 = 0$

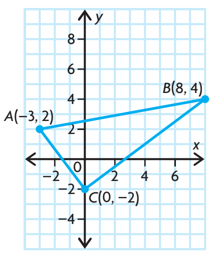
9. **a.** 

b. $4x - y - 14 = 0$

10. **a.** 82° **c.** 63° **e.** 54°
b. 42° **d.** 37° **f.** 63°

11. **a.** 

b. acute: 45°, obtuse: 135°

12. **a.** $(0, -2)$
b. 

c. $\overrightarrow{CA} = (-3 - 0, 2 - (-2))$
$= (-3, 4)$
$\overrightarrow{CB} = (8 - 0, 4 - (-2))$
$= (8, 6)$
$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-3)(8) + (4)(6)$
$= -24 + 24$
$= 0$
Since the dot product of the vectors is 0, the vectors are perpendicular, and $\angle ACB = 90°$.

13. The sum of the interior angles of a quadrilateral is 360°. The normals make 90° angles with their respective lines at A and C. The angle of the quadrilateral at B is $180° - \theta$. Let x represent the measure of the interior angle of the quadrilateral at O.
$90° + 90° + 180° - \theta + x = 360°$
$360° - \theta + x = 360°$
$x = \theta$
Therefore, the angle between the normals is the same as the angle between the lines.

14. $2 \pm \sqrt{3}$

Section 8.3, pp. 449–450

1. **a.** $(-3, 1, 8)$
b. $(1, -1, 3)$
c. $(-2, 1, 3)$
d. $(-2, -3, 1)$
e. $(3, -2, -1)$
f. $\left(\frac{1}{3}, -\frac{3}{4}, \frac{2}{5}\right)$

2. **a.** $(-1, 1, 9)$
b. $(2, 1, -1)$
c. $(3, -4, -1)$
d. $(-1, 0, 2)$
e. $(0, 0, 2)$
f. $(2, -1, 2)$

3. **a.** $\vec{r} = (-1, 2, 4) + t(4, -5, 1), t \in \mathbf{R}$;
$\vec{q} = (3, -3, 5) + s(-4, 5, -1), s \in \mathbf{R}$
b. $x = -1 + 4t, y = 2 - 5t,$
$z = 4 + t, t \in \mathbf{R}$; $x = 3 - 4s,$
$y = -3 + 5s, z = 5 - s, s \in \mathbf{R}$

4. **a.** $\vec{r} = (-1, 5, -4) + t(1, 0, 0), t \in \mathbf{R}$
b. $x = -1 + t, y = 5, z = -4, t \in \mathbf{R}$
c. Since two of the coordinates in the direction vector are zero, a symmetric equation cannot exist.

5. **a.** $\vec{r} = (-1, 2, 1) + t(3, -2, 1), t \in \mathbf{R}$;
$x = -1 + 3t, y = 2 - 2t,$
$z = 1 + t, t \in \mathbf{R}$;
$\frac{x + 1}{3} = \frac{y - 2}{-2} = \frac{z - 1}{1}$

b. $\vec{r} = (-1, 1, 0) + t(0, 1, 1), t \in \mathbf{R}$;
$x = -1, y = 1 + t, z = t, t \in \mathbf{R}$;
$\frac{y-1}{1} = \frac{z}{1}, x = -1$

c. $\vec{r} = (-2, 3, 0) + t(0, 1, 1), t \in \mathbf{R}$;
$x = -2, y = 3 + t, z = t, t \in \mathbf{R}$;
$\frac{y-3}{1} = \frac{z}{1}, x = -2$

d. $\vec{r} = (-1, 0, 0) + t(0, 1, 0), t \in \mathbf{R}$;
$x = -1, y = t, z = 0, t \in \mathbf{R}$;
Since two of the coordinates in the direction vector are zero, there is no symmetric equation for this line.

e. $\vec{r} = t(-4, 3, 0), t \in \mathbf{R}$;
$x = -4t, y = 3t, z = 0, t \in \mathbf{R}$;
$\frac{x}{-4} = \frac{y}{3}, z = 0$

f. $\vec{r} = (1, 2, 4) + t(0, 0, 1), t \in \mathbf{R}$;
$x = 1, y = 2, z = 4 + t, t \in \mathbf{R}$;
Since two of the coordinates in the direction vector are zero, there is no symmetric equation for this line.

6. **a.** $x = -6 + t, y = 10 - t$,
$z = 7 + t, t \in \mathbf{R}$;
$x = -7 + s, y = 11 - s$,
$z = 5, s \in \mathbf{R}$

b. about 35.3°

7. The directional vector of the first line is $(8, 2, -2) = -2(-4, -1, 1)$. So, $(-4, -1, 1)$ is a directional vector for the first line as well. Since $(-4, -1, 1)$ is also the directional vector of the second line, the lines are the same if the lines share a point. (1, 1, 3) is a point on the second line. Since
$1 = \frac{1+7}{8} = \frac{1+1}{2} = \frac{3-5}{-2}$, (1, 1, 3) is a point on the first line as well.
Hence, the lines are the same.

8. **a.** The line that passes through (0, 0, 3) with a directional vector of $(-3, 1, -6)$ is given by the parametric equation is $x = 3t, y = t, z = 3 - 6t, t \in \mathbf{R}$. So, the y-coordinate is equal to -2 only when $t = -2$. At $t = -2, x = -3(-2) = 6$ and $z = 3 - 6(-2) = 15$. So, $A(6, -2, 15)$ is a point on the line. So, the y-coordinate is equal to 5 only when $t = 5$. At $t = 5$, $x = -3(5) = -15$ and $z = 3 - 6(5) = -27$. So, $B(-15, 5, -27)$ is a point on the line.

b. $x = -3t, y = t, z = 3 - 6t$,
$-2 \le t \le 5$

9. -1

10. **a.** $(8, 4, -3), (0, -8, 13), (4, -2, 5)$
b. $(-9, 3, 15), (1, 1, 3), (-4, 2, 9)$
c. $(-4, 3, -4), (2, 1, 4), (-1, 2, 0)$
d. $(-4, -1, -2), (-4, 5, 8)\ (-4, 2, 3)$

11. **a.** $x = 4 - 4t, y = -2 - 6t$,
$z = 5 + 8t, t \in \mathbf{R}$;
$\frac{x-4}{-4} = \frac{y+2}{-6} = \frac{z-5}{8}$

b. $\vec{r} = (-4, 2, 9) + s(5, -1, -6)$,
$s \in \mathbf{R}$; $\frac{x+4}{5} = \frac{y-2}{-1} = \frac{z-9}{-6}$

c. $\vec{r} = (-1, 2, 0) + t(3, -1, 4), t \in \mathbf{R}$;
$x = -1 + 3t, y = 2 - t, z = 4t$,
$t \in \mathbf{R}$

d. $\vec{r} = (-4, 2, 3) + t(0, 3, 5), t \in \mathbf{R}$;
$x = -4, y = 2 + 3t, z = 3 + 5t$,
$t \in \mathbf{R}$

12. $x = 2 - 34t, y = -5 + 25t, z = 13t$,
$t \in \mathbf{R}$

13. $(-2, -1, 2), (2, 1, 2)$.

14. $P_1(2, 3, -2)$ and $P_2(4, -3, -4)$

15. about 17°

Mid-Chapter Review, pp. 451–452

1. **a.** $(-7, -2), (-5, 1), (-3, 4)$
b. $(-1, 5), (2, 3), (5, 1)$
c. $\left(-1, \frac{11}{5}\right), \left(0, \frac{8}{5}\right), (1, 1)$
d. $(-2, -4, 4), (4, 0, 6), (1, -2, 5)$

2. **a.** $\left(\frac{18}{5}, 0\right)$; $(0, 6)$
b. $\left(-\frac{14}{3}, 0\right)$; $(0, -3)$

3. approximately 86.8°

4. x-axis: about 51°; y-axis: about 39°

5. $5x - 7y - 41 = 0$

6. $\frac{x}{3} = \frac{y}{-4} = \frac{z-2}{4}$

7. $x = 1 + t, y = 2 - 9t, z = 5 + t, t \in \mathbf{R}$

8. approximately 79.3°, 137.7°, and 49.7°

9. $y = -4, \frac{x-3}{1} = \frac{z-6}{\sqrt{3}}$

10. x-axis: $x = t, y = 0, z = 0, t \in \mathbf{R}$;
y-axis: $x = 0, y = t, z = 0, t \in \mathbf{R}$;
z-axis: $x = 0, y = 0, z = t, t \in \mathbf{R}$

11. **a.** -7
b. $\frac{1}{19}$

12. 17.2 units, 12

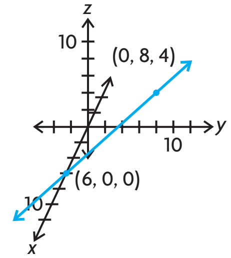
13. **a.** $\vec{r} = (0, 6) + t(4, -3), t \in \mathbf{R}$
b. $x = 4t, y = 6 - 3t, t \in \mathbf{R}$
c. about 36.9°
d. $\vec{r} = t(3, 4), t \in \mathbf{R}$

14. $x + 6y - 32 = 0$;
$\vec{r} = (-4, 6) + t(12, -2), t \in \mathbf{R}$;
$x = -4 + 12t, y = 6 - 2t, t \in \mathbf{R}$

15. $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

16. **a.** $x = -5 + 3t, y = 10 - 2t, t \in \mathbf{R}$
b. $x = 1 + t, y = -1 + t, t \in \mathbf{R}$
c. $x = 0, y = t, t \in \mathbf{R}$

17. **a.** yz-plane at (0, 8, 4); xz-plane at (6, 0, 0); xy-plane at (6, 0, 0)
b. x-axis at (6, 0, 0)
c.



18. **a.** $\vec{r} = (1, -2, 8) + t(-5, -2, 1)$,
$t \in \mathbf{R}$; $x = 1 - 5t, y = -2 - 2t$,
$z = 8 + t, t \in \mathbf{R}$;
$\frac{x-1}{-5} = \frac{y+2}{-2} = \frac{z-8}{1}$

b. $\vec{r} = (3, 6, 9) + t(2, 4, 6), t \in \mathbf{R}$;
$x = 3 + 2t, y = 6 + 4t$,
$z = 9 + 6t, t \in \mathbf{R}$;
$\frac{x-3}{2} = \frac{y-6}{4} = \frac{z-9}{6}$

c. $\vec{r} = (0, 0, 6) + t(-1, 5, 1), t \in \mathbf{R}$;
$x = -t, y = 5t, z = 6 + t, t \in \mathbf{R}$;
$\frac{x}{-1} = \frac{y}{5} = \frac{z-6}{1}$

d. $\vec{r} = (2, 0, 0) + t(0, 0, -2), t \in \mathbf{R}$;
$x = 2, y = 0, z = -2t, t \in \mathbf{R}$; There is no symmetric equation for this line.

19. $\vec{r} = t(5, -5, -1), t \in \mathbf{R}$

20. $x = t, y = -8 - 13t, z = 1, t \in \mathbf{R}$

21. $(1, 3, -5), -3(1, 3, -5)$.

22. Since $\frac{7-4}{3} = \frac{-1+2}{1} = \frac{8-6}{2} = 1$, the point $(7, -1, 8)$ lies on the line.

Section 8.4 pp. 459–460

1. **a.** plane; **b.** line;
c. line; **d.** plane;

2. **a.** $(4, -24, 9)$
b. $(1, -2, 5)$
c. $\vec{r} = (2, 1, 3) + s(4, -24, 9)$
$+ t(1, -2, 5), t, s \in \mathbf{R}$

3. **a.** $(0, 0, -1)$
b. $(2, -3, -3)$ and $(0, 5, -2)$
c. $(-2, -17, 10)$
d. $m = 0$ and $n = 3$
e. For the point $B(0, 15, -8)$, the first two parametric equations are the same, yielding $m = 0$ and $n = 3$; however, the third equation would then give:
$-8 = -1 - 3m - 2n$
$-8 = -1 - 3(0) - 2(3)$
$-8 = -7$
which is not true. So, there can be no solution.

4. a. $\vec{r} = (-2, 3, 1) + t(0, 0, 1) + s(3, -3, 0), t, s \in \mathbf{R}$
b. $\vec{r} = (-2, 3, -2) + t(0, 0, 1) + s(3, -3, -1), t, s \in \mathbf{R}$

5. a. $\vec{r} = (1, 0, -1) + s(2, 3, -4) + t(4, 6, -8), t, s \in \mathbf{R}$, does not represent a plane because the direction vectors are the same. We can rewrite the second direction vector as $(2)(2, 3, -4)$. And so we can rewrite the equation as:
$\vec{r} = (1, 0, -1) + s(2, 3, -4) + 2t(2, 3, -4)$
$= (1, 0, -1) + (s + 2t)(2, 3, -4)$
$= (1, 0, -1) + n(2, 3, 4), n \in \mathbf{R}$
This is an equation of a line in $\mathbf{R}^3$.

6. a. $\vec{r} = (-1, 2, 7) + t(4, 1, 0) + s(3, 4, -1), t, s \in \mathbf{R}$;
$x = -1 + 4t + 3s,$
$y = 2 + t + 4s,$
$z = 7 - s, t, s \in \mathbf{R}$
b. $\vec{r} = (1, 0, 0) + t(-1, 1, 0) + s(-1, 0, 1), t, s \in \mathbf{R}$;
$x = 1 - t - s,$
$y = t,$
$z = s, t, s \in \mathbf{R}$
c. $\vec{r} = (1, 1, 0) + t(3, 4, -6) + s(7, 1, 2), t, s \in \mathbf{R}$;
$x = 1 + 3t + 7s,$
$y = 1 + 4t + s,$
$z = -6t + 2s, t, s \in \mathbf{R}$

7. a. $s = 1$ and $t = 1$
b. $(0, 5, -4) = (2, 0, 1) + s(4, 2, -1) + t(-1, 1, 2)$ gives the following parametric equations:
$0 = 2 + 4s + t \Rightarrow t = 2 + 4s$
$5 = 2s + t$
$5 = 2s + (2 + 4s)$
$3 = 6s$
$\frac{1}{2} = s$
$t = 2 + 4\left(\frac{1}{4}\right)$
$t = 2 + 2 = 4$
The third equation then says:
$-4 = 1 - s + 2t$
$-4 = 1 - \frac{1}{2} + 2(4)$
$-4 = \frac{17}{2}$, which is a false statement. So, the point $A(0, 5, -4)$ is not on the plane.

8. a. $\vec{l} = (-3, 5, 6) + s(-1, 1, 2), s \in \mathbf{R}$;
$\vec{p} = (-3, 5, 6) + t(2, 1, -3), t \in \mathbf{R}$
b. $(-3, 5, 6)$

9. $(0, 0, 5)$

10. $\vec{r} = (2, 1, 3) + s(4, 1, 5) + t(3, -1, 2), t, s \in \mathbf{R}$

11. $\vec{r} = m(2, -1, 7) + n(-2, 2, 3), m, n \in \mathbf{R}$

12. a. $(1, 0, 0)$, $(0, 1, 0)$ and $(1, 1, 0)$, $(-1, 1, 0)$
b. $\vec{r} = s(1, 0, 0) + t(0, 1, 0), t, s \in \mathbf{R}$;
$x = s,$
$y = t,$
$z = 0, t, s \in \mathbf{R}$

13. a. $\vec{r} = s(-1, 2, 5) + t(3, -1, 7), t, s \in \mathbf{R}$
b. $\vec{r} = (-2, 2, 3) + s(-1, 2, 5) + t(3, -1, 7), t, s \in \mathbf{R}$
c. The planes are parallel since they have the same direction vectors.

14. $(-4, 7, 1) - (-3, 2, 4) = (-1, 5, -3)$,
$\frac{27}{13}(-3, 2, 4) - \frac{17}{13}(-4, 7, 1)$
$= (-1, -5, 7)$

15. $\vec{r} = (0, 3, 0) + t(0, 3, 2), t \in \mathbf{R}$

16. The fact that the plane $\vec{r} = \overrightarrow{OP_0} + s\vec{a} + t\vec{b}$ contains both of the given lines is easily seen when letting $s = 0$ and $t = 0$, respectively.

Section 8.5 pp. 468–469

1. a. $\vec{n} = (A, B, C) = (1, -7, -18)$
b. In the Cartesian equation:
$Ax + By + Cz + D = 0$
If $D = 0$, the plane passes through the origin.
c. $(0, 0, 0)$, $(11, -1, 1)$, $(-11, 1, -1)$

2. a. $\vec{n} = (A, B, C) = (2, -5, 0)$
b. In the Cartesian equation: $D = 0$. So, the plane passes through the origin.
c. $(0, 0, 0)$, $(5, 2, 0)$ $(5, 2, 1)$

3. a. $\vec{n} = (A, B, C) = (1, 0, 0)$
b. In the Cartesian equation: $D = 0$. So, the plane passes through the origin.
c. $(0, 0, 0)$, $(0, 1, 0)$ $(0, 0, 1)$

4. a. $x + 5y - 7z = 0$
b. $-8x + 12y + 7z = 0$

5. *Method 1:* Let $A(x, y, z)$ be a point on the plane. Then, $\overrightarrow{PA} = (x + 3, y - 3, z - 5)$ is a vector on the plane.
$\vec{n} \cdot \overrightarrow{PA} = 0$
$(x + 3) + 7(y - 3) + 5(z - 5) = 0$
$x + 7y + 5z - 43 = 0.$
Method 2: $\vec{n} = (1, 7, 5)$ so the Cartesian equation is
$x + 7y + 5z + D = 0$
We know the point $(-3, 3, 5)$ is on the plane and must satisfy the equation, so
$(-3) + 7(3) + 5(5) + D = 0$
$43 + D = 0$
$D = -43$
This also gives the equation:
$x + 7y + 5z - 43 = 0.$

6. a. $7x + 19y - 3z - 28 = 0$
b. $7x + 19y - 3z - 28 = 0$
c. There is only one simplified Cartesian equation that satisfies the given information, so the equations must be the same.

7. $7x + 17y - 13z - 24 = 0$

8. $20x + 9y + 7z - 47 = 0$

9. a. $\left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$
b. $\left(\frac{4}{\sqrt{26}}, -\frac{3}{\sqrt{26}}, \frac{1}{\sqrt{26}}\right)$
c. $\left(\frac{3}{13}, -\frac{4}{13}, \frac{12}{13}\right)$

10. $21x - 15y - z - 1 = 0$

11. $2x - 4y - z + 6 = 0$

12. a. First determine their normal vectors, $\vec{n_1}$ and $\vec{n_2}$. Then the angle between the two planes can be determined from the formula:
$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$
b. 30°

13. a. 53.3°
b. $2x - 3y - z + 5 = 0$

14. a. 8
b. $-\frac{5}{2}$
c. No, the planes cannot ever be coincident. If they were, then they would also be parallel, so $k = 8$, and we would have the two equations:
$4x + 8y - 2z + 1 = 0.$
$2x + 4y - z + 4 = 0 \Rightarrow$
$4x + 8y - 2z + 8 = 0$. Here all of the coefficients are equal except for the D-values, which means that they don't coincide.

15. $3x + 5y - z - 18 = 0$

16. $-\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y + \sqrt{3}z = 0$

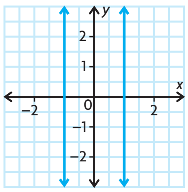
17. $8x - 2y - 16z - 5 = 0$

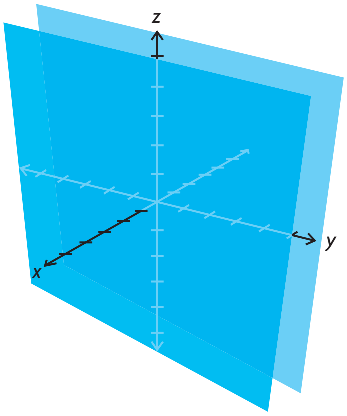
Section 8.6, pp. 476–477

1. a. A plane parallel to the yz-axis, but two units away, in the negative x direction.
b. A plane parallel to the xz-axis, but three units away, in the positive y direction.
c. A plane parallel to the xy-axis, but 4 units away, in the positive z direction.

2. $(-2, 3, 4)$

3. P must lie on plane π_1 since the point has an x-coordinate of 5, and doesn't have a y-coordinate of 6.

4. 



5. a. i. x-intercept $= 9$,
y-intercept $= 6$,
no z-intercept

ii. x-intercept $= 40$,
y-intercept $= -30$,
z-intercept $= 24$

iii. no x-intercept,
y-intercept $= 3$,
z-intercept $= -39$

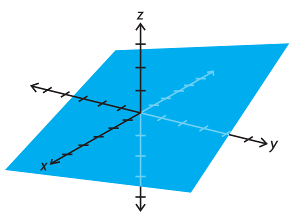
b. i. $(0, 0, 1)$, $(3, -2, 0)$

ii. $(4, 3, 0)$, $(5, 0, -3)$

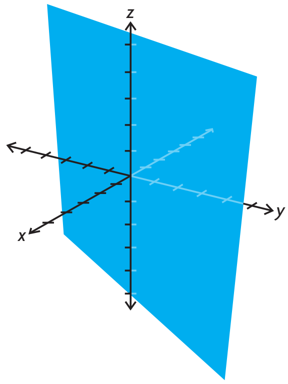
iii. $(1, 0, 0)$, $(0, 1, 13)$

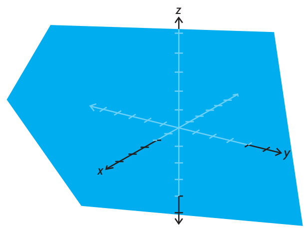
6. a. i. $(0, 0, 0)$, $(1, 2, 0)$, $(0, 5, 1)$

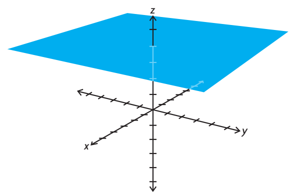
ii. $2x - y = 0$

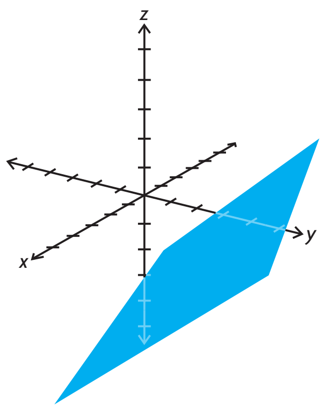
b. 

7. yz-plane, xz-plane, xy-plane

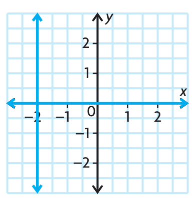
8. a. 

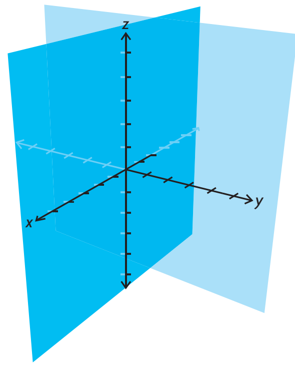
b. 

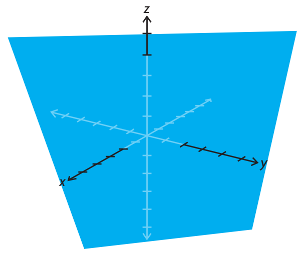
c. 

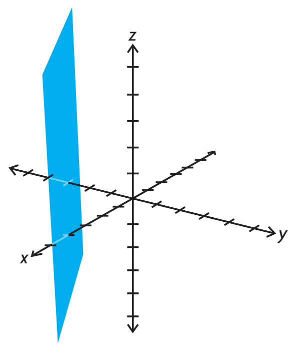
d. 

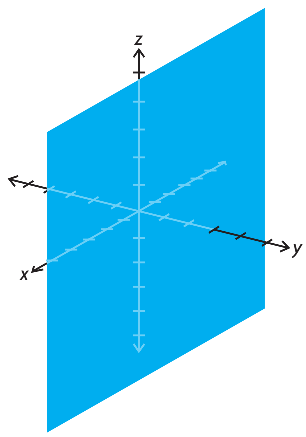
9. a. $y(x + 2) = 0$

b. 

c. 

10. a. 

b. 

c. 

11. a. $\frac{x}{3} + \frac{y}{4} + \frac{z}{6} = 1$

b. $\frac{x}{5} - \frac{z}{7} = 1$

c. $\frac{z}{8} = 1$

Review Exercise, pp. 480–483

1. Answers may vary. For example,
$x = 1 + s + t,$
$y = 2 - s,$
$z = -1 + 2s + 3t$

2. $3x + y - z - 6 = 0$
$\vec{AC} = (2, -1, 5) = \vec{c}$
$\vec{BC} = (1, 0, 3) = \vec{b}$
$\vec{r} = (1, 2, -1) + s(2, -1, 5) + t(1, 0, 3), s, t \in \mathbf{R}$
$\vec{b} \times \vec{c} = (1, 0, 3) \times (2, -1, 5) = (3, 1, -1)$
$Ax + By + Cz + D = 0$
$(3)x + (1)y + (-1)z + D = 0$
$3(1) + (2) - 1(-1) + D = 0$
$D = -6$
$3x + y - z - 6 = 0$
Both Cartesian equations are the same regardless of which vectors are used.

3. **a.** Answers may vary. For example,
$\vec{r} = (4, 3, 9) + t(7, 1, 1), t \in \mathbf{R};$
$x = 4 + 7t, y = 3 + t, z = 9 + t, t \in \mathbf{R};$
$$\frac{x - 4}{7} = \frac{y - 3}{1} = \frac{z - 9}{1}$$
b. Answers may vary. For example,
$\vec{r} = (4, 3, 9) + t(7, 1, 1) + s(3, 2, 3), t, s \in \mathbf{R};$
$x = 4 + 7t + 3s, y = 3 + t + 2s, z = 9 + t + 3s, t, s \in \mathbf{R}$
c. There are two parameters.

4. $\vec{r} = (7, 1, -2) + t(2, -3, 1), t \in \mathbf{R};$
$x = 7 + 2t, y = 1 - 3t, z = -2 + t;$
$$\frac{x - 7}{2} = \frac{y - 1}{-3} = \frac{z + 2}{1}$$

5. **a.** $x - 3y - 3z - 3 = 0$
b. $3x + 5y - 2z - 7 = 0$
c. $3y + z - 7 = 0$

6. $19x - 7y - 8z = 0$

7. $\vec{r} = (-1, 2, 1) + t(0, 1, 0) + s(0, 0, 1)\ t, s \in \mathbf{R};$
$x = -1, y = 2 + t, z = 1 + s$

8. $3x + y - z - 7 = 0$

9. $34x + 32y - 7z - 229 = 0$

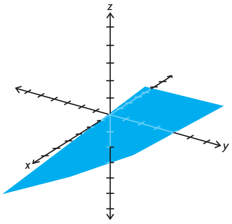
10. Answers may vary. For example,
$\vec{r} = (2, 3, -3) + s(3, -2, 1), s \in \mathbf{R};$
$x = 2 + 3s, y = 3 - 2s, z = -3 + s;$
$$\frac{x - 2}{3} = \frac{y - 3}{-2} = \frac{z + 3}{1}$$

11. Answers may vary. For example,
$\vec{r} = (0, 0, 6) + s(1, 0, 3) + t(3, -5, -1), s, t \in \mathbf{R};$
$x = s + 3t, y = -5t, z = 6 + 3s - t$

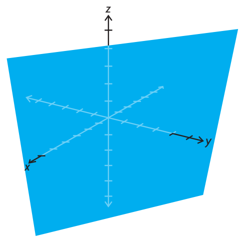
12. Answers may vary. For example,
$\vec{r} = (0, 0, 7) + t(1, 0, 2), t \in \mathbf{R};$
$x = t, y = 0, z = 7 + 2t;$

13. $\vec{r} = (3, -4, 1) + s(1, -3, -5) + t(4, 3, -1), s, t \in \mathbf{R};$
$x = 3 + s + 4t,$
$y = -4 - 3s + 3t$
$z = 1 - 5s - t, s, t \in \mathbf{R};$
$18x - 19y + 15z - 145 = 0$

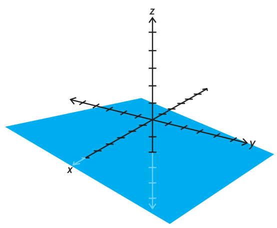
14. **a.**



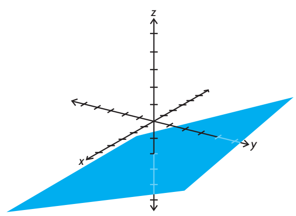
b.



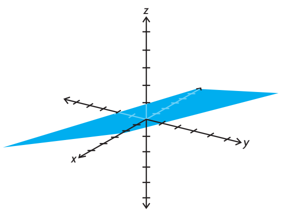
c.



d.



e.



15. **a.** Answers may vary. For example,
$\vec{r} = (3, 1, 2) + t(2, 4, 1) + s(2, 3, -3), t, s \in \mathbf{R};$
$x = 3 + 2t + 2s,$
$y = 1 + 4t + 3s, z = 2 + t - 3s;$
$15x - 8y + 2z - 41 = 0$
b. Answers may vary. For example,
$\vec{BC} = (-4, 0, 11)$
$D = -18$
$-4x + 11z - 18 = 0$
c. Answers may vary. For example,
$\vec{r} = (4, 1, -1) + t(1, -3, 5) + s(0, 0, 1), t, s \in \mathbf{R};$
$x = 4 + t, y = 1 - 3t,$
$z = -1 + 5t + s;$
$3x + y - 13 = 0$
d. Answers may vary. For example,
$\vec{r} = (1, 3, -5) + t(1, 3, 9) + s(1, -9, -1), t, s \in \mathbf{R};$
$x = 1 + t + s, y = 3 + 3t - 9s,$
$z = -5 + 9t - s;$
$78x + 10y - 12z - 168 = 0$

16. They are in the same plane because both planes have the same normal vectors and Cartesian equations.
$L_1: \vec{r} = (1, 2, 3) + s(-3, 5, 21) + t(0, 1, 3), s, t \in \mathbf{R}$
$L_2: \vec{r} = (1, -1, -6) + u(1, 1, 1) + v(2, 5, 11), u, v \in \mathbf{R}$
$(-3, 5, 21) \times (0, 1, 3) = (-6, 9, -3) = (2, -3, 1)$
$(1, 1, 1) \times (2, 5, 11) = (6, -9, 3) = (2, -3, 1)$
$Ax + By + Cz + D = 0$
$2x - 3y + z + D = 0$
$2(1) - 3(2) + (3) + D = 0$
$D = 1$
$2(1) - 3(-1) + (-6) + D = 0$
$D = 1$
$2x - 3y + z + 1 = 0$

17. $\left(\frac{20}{3}, \frac{10}{3}, -\frac{1}{3}\right)$

18. **a.** The plane is parallel to the z-axis through the points (3, 0, 0) and (0, −2, 0).
b. The plane is parallel to the y-axis through the points (6, 0, 0) and (0, 0, −2).
c. The plane is parallel to the x-axis through the points (0, 3, 0) and (0, 0, −6).

19. **a.** A
b. $a = -8, b = -1$

20. **a.** 45.0° **b.** 59.0° **c.** 37.4° **d.** 90°

21. **a.** 44.2°
b. 90°

22. **a.** **i.** no **ii.** yes **iii.** no
b. **i.** yes **ii.** no **iii.** no

23. $(x, y, z) = (4, 1, 6) + p(3, -2, 1) + q(-6, 6, -1)$
$(x, y, z) = (4, 1, 6) + 4(3, -2, 1) + 2(-6, 6, -1)$
$(x, y, z) = (4, 5, 8) \neq (4, 5, 6)$

24. $x = 1 + s + 3t, y = 4 - t,$
$z = 4 - 3s + t, s, t \in \mathbf{R}$

25. A plane has two parameters, because a plane goes in two different directions, unlike a line that goes only in one direction.

26. This equation will always pass through the origin, because you can always set $s = 0$ and $t = -1$ to obtain $(0, 0, 0)$.

27. a. They do not form a plane, because these three points are collinear.
$\vec{r} = (-1, 2, 1) + t(3, 1, -2)$

b. They do not form a plane, because the point lies on the line.
$\vec{r} = (4, 9, -3) + t(1, -4, 2)$
$\vec{r} = (4, 9, -3) + 4(1, -4, 2)$
$= (8, -7, 5)$

28. $bcx + acy + abz - abc = 0$

29. $6x - 5y + 12z + 46 = 0$

30. a., b. $\vec{r} = (1, -3, 2) + t(-3, 7, -4) + s(5, -2, 3)\ t, s \in \mathbf{R}$;
$x = 1 - 3t + 5s$,
$y = -3 + 7t - 2s$,
$z = 2 - 4t + 3s$

c. $13x - 11y - 29z + 12 = 0$

d. no

31. a. $4x - 2y + 5z = 0$

b. $4x - 2y + 5z + 19 = 0$

c. $4x - 2y + 5z - 22 = 0$

32. a. These lines are coincident. The angle between them is 0°.

b. $\left(\frac{3}{2}, 5\right)$, 86.82°

33. a. $\vec{r} = (1, 3, 5) + t(-2, -4, -10)$, $t \in \mathbf{R}$;
$x = 1 - 2t, y = 3 - 4t$,
$z = 5 - 10t$;
$\frac{x - 1}{-2} = \frac{y - 3}{-4} = \frac{z - 5}{-10}$

b. $\vec{r} = (1, 3, 5) + t(-8, 6, -2), t \in \mathbf{R}$;
$x = 1 - 8t, y = 3 + 6t$,
$z = 5 - 2t$;
$\frac{x - 1}{-8} = \frac{x - 3}{6} = \frac{x - 5}{-2}$

c. $\vec{r} = (1, 3, 5) + t(-6, -13, 14)$, $t \in \mathbf{R}$;
$x = 1 - 6t, y = 3 - 13t$,
$z = 5 + 14t$;
$\frac{x - 1}{-6} = \frac{x - 3}{-13} = \frac{x - 5}{14}$

d. $\vec{r} = (1, 3, 5) + t(1, 0, 0), t \in \mathbf{R}$;
$x = 1 + t, y = 3, z = 5$

e. $a = 0, b = 6, c = 4$;
$\vec{r} = (1, 3, 5) + t(0, 6, 4), t \in \mathbf{R}$

f. $\vec{r} = (1, 3, 5) + t(0, 1, 6)$;
$x = 1, y = 3 + t, z = 5 + 6t$

34. a. $2x - 4y + 5z + 23 = 0$

b. $29x + 27y + 24z - 86 = 0$

c. $z - 3 = 0$

d. $3x + y - 4z + 26 = 0$

e. $y - 2z - 4 = 0$

f. $-5x + y + 7z + 18 = 0$

Chapter 8 Test, p. 484

1. a. i. $\vec{r} = (1, 2, 4) + s(1, -2, -1) + t(3, 2, 0), s, t \in \mathbf{R}$;
$x = 1 + s + 3t$,
$y = 2 - 2s + 2t, z = 4 - s$,
$s, t \in \mathbf{R}$

ii. $2x - 3y + 8z - 28 = 0$

b. no

2. a. $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$

b. $(6, 4, 3)$

3. a. $\vec{r} = s(2, 1, 3) + t(1, 2, 5), s, t \in \mathbf{R}$

b. $-x - 7y + 3z = 0$

4. a. $\vec{r} = (4, -3, 5) + s(2, 0, -3) + t(5, 1, -1), s, t \in \mathbf{R}$

b. $3x - 13y + 2z - 61 = 0$

5. a. $\left(0, 5, -\frac{1}{2}\right)$

b. $\frac{x}{4} = \frac{y - 5}{-2} = \frac{1}{2}$

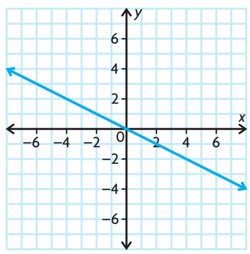
6. a. about 70.5°

b. i. 4

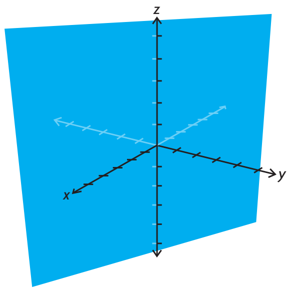
ii. $-\frac{1}{5}$

c. The y-intercepts are different and the planes are parallel.

7. a.



b.



c. The equation for the plane can be written as $Ax + By + 0z = 0$. For any real number t,
$A(0) + B(0) + 0(t) = 0$, so $(0, 0, t)$ is on the plane. Since this is true for all real numbers, the z-axis is on the plane.

Chapter 9

Review of Prerequisite Skills, p. 487

1. a. yes **c.** yes
b. no **d.** no

2. Answers may vary. For example:

a. $\vec{r} = (2, 5) + t(5, -2), t \in \mathbf{R}$;
$x = 2 + 5t, y = 5 - 2t, t \in \mathbf{R}$

b. $\vec{r} = (-3, 7) + t(7, -14), t \in \mathbf{R}$;
$x = -3 + 7t, y = 7 - 14t, t \in \mathbf{R}$

c. $\vec{r} = (-1, 0) + t(-2, -11), t \in \mathbf{R}$;
$x = -1 + -2t, y = -11t, t \in \mathbf{R}$

d. $\vec{r} = (1, 3, 5) + t(5, -10, -5), t \in \mathbf{R}$;
$x = 1 + 5t, y = 3 - 10t, z = 5 - 5t$, $t \in \mathbf{R}$

e. $\vec{r} = (2, 0, -1) + t(-3, 5, 3), t \in \mathbf{R}$;
$x = 2 - 3t, y = 5t, z = -1 + 3t$, $t \in \mathbf{R}$

f. $\vec{r} = (2, 5, -1) + t(10, -10, -6)$, $t \in \mathbf{R}$;
$x = 2 + 10t, y = 5 - 10t, z = -1 - 6t, t \in \mathbf{R}$

3. a. $2x + 6y - z - 17 = 0$

b. $y = 0$

c. $4x - 3y - 15 = 0$

d. $6x - 5y + 3z = 0$

e. $11x - 6y - 38 = 0$

f. $x + y - z - 6 = 0$

4. $5x + 11y + 2z - 21 = 0$

5. L_1 is not parallel to the plane. L_1 is on the plane.
L_2 is parallel to the plane.
L_3 is not parallel to the plane.

6. a. $x - y - z - 2 = 0$

b. $x + 6y - 10z - 30 = 0$

7. $\vec{r} = (1, -4, 3) + t(1, 3, 3) + s(0, 1, 0), s, t \in \mathbf{R}$

8. $3y + z = 13$

Section 9.1, pp. 496–498

1. a. $\pi: x - 2y - 3z = 6$,
$\vec{r} = (1, 2, -3) + s(5, 1, 1)\ s \in \mathbf{R}$

b. This line lies on the plane.

2. a. A line and a plane can intersect in three ways: (1) The line and the plane have zero points of intersection. This occurs when the lines are not incidental, meaning they do not intersect.
(2) The line and the plane have only one point of intersection. This occurs when the line crosses the plane at a single point.
(3) The line and the plane have an infinite number of intersections. This occurs when the line is

25. A plane has two parameters, because a plane goes in two different directions, unlike a line that goes only in one direction.

26. This equation will always pass through the origin, because you can always set $s = 0$ and $t = -1$ to obtain $(0, 0, 0)$.

27. **a.** They do not form a plane, because these three points are collinear.
$\vec{r} = (-1, 2, 1) + t(3, 1, -2)$
b. They do not form a plane, because the point lies on the line.
$\vec{r} = (4, 9, -3) + t(1, -4, 2)$
$\vec{r} = (4, 9, -3) + 4(1, -4, 2)$
$= (8, -7, 5)$

28. $bcx + acy + abz - abc = 0$

29. $6x - 5y + 12z + 46 = 0$

30. **a., b.** $\vec{r} = (1, -3, 2) + t(-3, 7, -4) + s(5, -2, 3)$ $t, s \in \mathbf{R}$;
$x = 1 - 3t + 5s$,
$y = -3 + 7t - 2s$,
$z = 2 - 4t + 3s$
c. $13x - 11y - 29z + 12 = 0$
d. no

31. **a.** $4x - 2y + 5z = 0$
b. $4x - 2y + 5z + 19 = 0$
c. $4x - 2y + 5z - 22 = 0$

32. **a.** These lines are coincident. The angle between them is 0°.
b. $\left(\frac{3}{2}, 5\right)$, 86.82°

33. **a.** $\vec{r} = (1, 3, 5) + t(-2, -4, -10)$, $t \in \mathbf{R}$;
$x = 1 - 2t$, $y = 3 - 4t$,
$z = 5 - 10t$;
$\frac{x - 1}{-2} = \frac{y - 3}{-4} = \frac{z - 5}{-10}$
b. $\vec{r} = (1, 3, 5) + t(-8, 6, -2)$, $t \in \mathbf{R}$;
$x = 1 - 8t$, $y = 3 + 6t$,
$z = 5 - 2t$;
$\frac{x - 1}{-8} = \frac{x - 3}{6} = \frac{x - 5}{-2}$
c. $\vec{r} = (1, 3, 5) + t(-6, -13, 14)$, $t \in \mathbf{R}$;
$x = 1 - 6t$, $y = 3 - 13t$,
$z = 5 + 14t$;
$\frac{x - 1}{-6} = \frac{x - 3}{-13} = \frac{x - 5}{14}$
d. $\vec{r} = (1, 3, 5) + t(1, 0, 0)$, $t \in \mathbf{R}$;
$x = 1 + t$, $y = 3$, $z = 5$
e. $a = 0$, $b = 6$, $c = 4$;
$\vec{r} = (1, 3, 5) + t(0, 6, 4)$, $t \in \mathbf{R}$
f. $\vec{r} = (1, 3, 5) + t(0, 1, 6)$;
$x = 1$, $y = 3 + t$, $z = 5 + 6t$

34. **a.** $2x - 4y + 5z + 23 = 0$
b. $29x + 27y + 24z - 86 = 0$
c. $z - 3 = 0$
d. $3x + y - 4z + 26 = 0$

e. $y - 2z - 4 = 0$
f. $-5x + y + 7z + 18 = 0$

Chapter 8 Test, p. 484

1. **a. i.** $\vec{r} = (1, 2, 4) + s(1, -2, -1) + t(3, 2, 0)$, $s, t \in \mathbf{R}$;
$x = 1 + s + 3t$,
$y = 2 - 2s + 2t$, $z = 4 - s$,
$s, t \in \mathbf{R}$
ii. $2x - 3y + 8z - 28 = 0$
b. no

2. **a.** $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$
b. $(6, 4, 3)$

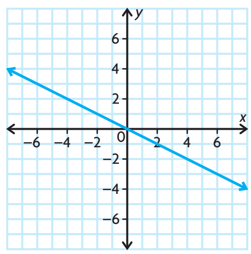
3. **a.** $\vec{r} = s(2, 1, 3) + t(1, 2, 5)$, $s, t \in \mathbf{R}$
b. $-x - 7y + 3z = 0$

4. **a.** $\vec{r} = (4, -3, 5) + s(2, 0, -3) + t(5, 1, -1)$, $s, t \in \mathbf{R}$
b. $3x - 13y + 2z - 61 = 0$

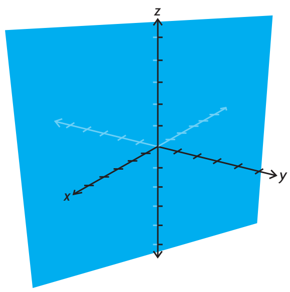
5. **a.** $\left(0, 5, -\frac{1}{2}\right)$
b. $\frac{x}{4} = \frac{y - 5}{-2} = \frac{1}{2}$

6. **a.** about 70.5°
b. i. 4
ii. $-\frac{1}{5}$
c. The y-intercepts are different and the planes are parallel.

7. **a.**



b.



c. The equation for the plane can be written as $Ax + By + 0z = 0$. For any real number t, $A(0) + B(0) + 0(t) = 0$, so $(0, 0, t)$ is on the plane. Since this is true for all real numbers, the z-axis is on the plane.

Chapter 9

Review of Prerequisite Skills, p. 487

1. **a.** yes **c.** yes
b. no **d.** no

2. Answers may vary. For example:
a. $\vec{r} = (2, 5) + t(5, -2)$, $t \in \mathbf{R}$;
$x = 2 + 5t$, $y = 5 - 2t$, $t \in \mathbf{R}$
b. $\vec{r} = (-3, 7) + t(7, -14)$, $t \in \mathbf{R}$;
$x = -3 + 7t$, $y = 7 - 14t$, $t \in \mathbf{R}$
c. $\vec{r} = (-1, 0) + t(-2, -11)$, $t \in \mathbf{R}$;
$x = -1 + -2t$, $y = -11t$, $t \in \mathbf{R}$
d. $\vec{r} = (1, 3, 5) + t(5, -10, -5)$, $t \in \mathbf{R}$;
$x = 1 + 5t$, $y = 3 - 10t$, $z = 5 - 5t$,
$t \in \mathbf{R}$
e. $\vec{r} = (2, 0, -1) + t(-3, 5, 3)$, $t \in \mathbf{R}$;
$x = 2 - 3t$, $y = 5t$, $z = -1 + 3t$,
$t \in \mathbf{R}$
f. $\vec{r} = (2, 5, -1) + t(10, -10, -6)$,
$t \in \mathbf{R}$;
$x = 2 + 10t$, $y = 5 - 10t$, $z = -1 - 6t$, $t \in \mathbf{R}$

3. **a.** $2x + 6y - z - 17 = 0$
b. $y = 0$
c. $4x - 3y - 15 = 0$
d. $6x - 5y + 3z = 0$
e. $11x - 6y - 38 = 0$
f. $x + y - z - 6 = 0$

4. $5x + 11y + 2z - 21 = 0$

5. L_1 is not parallel to the plane. L_1 is on the plane.
L_2 is parallel to the plane.
L_3 is not parallel to the plane.

6. **a.** $x - y - z - 2 = 0$
b. $x + 6y - 10z - 30 = 0$

7. $\vec{r} = (1, -4, 3) + t(1, 3, 3) + s(0, 1, 0)$, $s, t \in \mathbf{R}$

8. $3y + z = 13$

Section 9.1, pp. 496–498

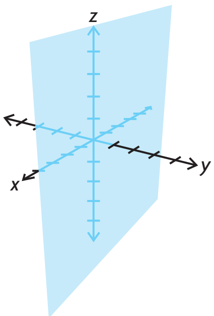
1. **a.** π: $x - 2y - 3z = 6$,
$\vec{r} = (1, 2, -3) + s(5, 1, 1)$ $s \in \mathbf{R}$
b. This line lies on the plane.

2. **a.** A line and a plane can intersect in three ways: (1) The line and the plane have zero points of intersection. This occurs when the lines are not incidental, meaning they do not intersect.
(2) The line and the plane have only one point of intersection. This occurs when the line crosses the plane at a single point.
(3) The line and the plane have an infinite number of intersections. This occurs when the line is

coincident with the plane, meaning the line lies on the plane.

b. Assume that the line and the plane have more than one intersection, but not an infinite number. For simplicity, assume two intersections. At the first intersection, the line crosses the plane. In order for the line to continue on, it must have the same direction vector. If the line has already crossed the plane, then it continues to move away from the plane, and can not intersect again. So, the line and the plane can only intersect zero, one, or infinitely many times.

3. **a.** The line $\vec{r} = s(1, 0, 0)$ is the x-axis.
b. The plane is parallel to the xz-plane, but just one unit away to the right.
c.



d. There are no intersections between the line and the plane.

4. **a.** For $x + 4y + z - 4 = 0$, if we substitute our parametric equations, we have $(-2 + t) + 4(1 - t) + (2 + 3t) + 4 = 0$
All values of t give a solution to the equation, so all points on the line are also on the plane.
b. For the plane $2x - 3y + 2x - 3y + 4z - 11 = 0$, we can substitute the parametric equations derived from $\vec{r} = (1, 5, 6) + t(1, -2, -2)$:
$2(1 + t) - 3(5 - 2t) + 4(6 - 2t) - 11 = 0$
All values of t give a solution to this equation, so all points on the line are also on the plane.

5. **a.** $2(-1 - s) - 2(1 + 2s) + 3(2s) - 1 = -5$
Since there are no values of s such that $-5 = 0$, this line and plane do not intersect.
b. $2(1 + 2t) - 4(-2 + 5t) + 4(1 + 4t) - 13 = 1$
Since there are no values of t such that $1 = 0$, there are no solutions, and the plane and the line do not intersect.

6. **a.** The direction vector is $\vec{m} = (-1, 2, 2)$ and the normal is $\vec{n} = (2, -2, 3)$, $\vec{m} \cdot \vec{n} = 0$. So the line is parallel to the plane, but $2(-1) - 2(1) + 3(0) - 1 = -5 \neq 0$. So, the point on the line is not on the plane.
b. The direction vector is $\vec{m} = (2, 5, 4)$ and the normal is $\vec{n} = (2, -4, 4)$, $\vec{m} \cdot \vec{n}$, $= 0$, so the line is parallel to the plane. and
$2(1) - 4(-2) + 4(1) - 13 = 1 \neq 0$
So, the point on the line is not on the plane.

7. **a.** $(-19, 0, 10)$
b. $(-11, 1, 0)$

8. **a.** There is no intersection and the lines are skew.
b. $(4, 1, 2)$

9. **a.** not skew
b. not skew
c. not skew
d. skew

10. 8

11. **a.** Comparing components results in the equation $s - t = -4$ for each component.
b. From L_1, we see that at $(-2, 3, 4)$, $s = 0$. When this occurs, $t = 4$. Substituting this into L_2, we get $(-30, 11, -4) + 4(7, -2, 2) = (-2, 3, 4)$. Since both of these lines have the same direction vector and a common point, the lines are coincidental.

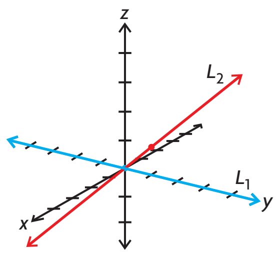
12. **a.** 3
b. $\left(\frac{2}{11}, \frac{53}{11}, \frac{46}{11}\right)$

13. 3

14. **a.** $(-6, 1, 3)$
b. 3

15. **a.** $(4, 1, 12)$
b. $\vec{r} = (4, 1, 12) + t(42, 55, -10)$, $t \in \mathbf{R}$

16. **a.**



b. (0, 0, 0)
c. If $p = 0$ and $q = 0$, the intersection occurs at (0, 0, 0).

17. **a.** Represent the lines parametrically, and then substitute into the equation for the plane. For the first equation, $x = t, y = 7 - 8t, z = 1 + 2t$.
Substituting into the plane equation, $2t + 7 - 8t + 3 + 6t - 10 = 0$.
Simplifying, $0t = 0$. So, the line lies on the plane.
For the second line, $x = 4 + 3s, y = -1, z = 1 - 2s$.
Substituting into the plane equation, $8 + 6s - 1 + 3 - 6s - 10 = 0$.
Simplifying, $0s = 0$. This line also lies on the plane.
b. $(1, -1, 3)$

18. Answers may vary. For example, $\vec{r} = (2, 0, 0) + p(2, 0, 1)$, $p \in \mathbf{R}$.

Section 9.2, pp. 507–509

1. **a.** linear
b. not linear
c. linear
d. not linear

2. Answers may vary. For example:
a. $x + y + 2z = -15$
$x + 2y + z = -3$
$2x + y + z = -10$
b. $(-3, 4, -8)$

3. **a.** yes
b. no

4. **a.** $(-2, -3)$
b. $(-2, -3)$
The two systems are equivalent because they have the same solution.

5. **a.** $(6, 1)$
b. $(-3, 5)$
c. $(-4, 3)$

6. **a.** These two lines are parallel, and therefore cannot have an intersection.
b. The second equation is five times the first; therefore, the lines are coincident.

7. **a.** $x = t, y = 2t - 3, t \in \mathbf{R}$
b. $x = t, y = s, z = 2s - t, t \in \mathbf{R}$

8. **a.** $2x + y = -11$
b. $2x + y = -11$
$2(3t + 3) + (-6t - 17) =$
$6t - 6t + 6 - 17 = -11$

9. **a.** $k \neq 12$
b. not possible
c. $k = 12$

10. **a** infinitely many

b. $x = t$,

$y = \dfrac{11}{4} - \dfrac{1}{2}t, t \in \mathbf{R}$

c. This equation will not have any integer solutions because the left side is an even function and the right side is an odd function.

11. **a.** $x = -a + b, y = -\frac{1}{3}b + \frac{2}{3}a$

b. Since they have different direction vectors, these two equations are not parallel or coincident and will intersect somewhere.

12. **a.** $(-1, -2, 3)$

b. $(3, 4, 12)$

c. $(4, 6, -8)$

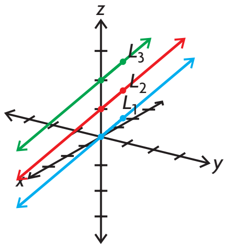
d. $(60, 120, -180)$

e. $(2, 4, 1)$

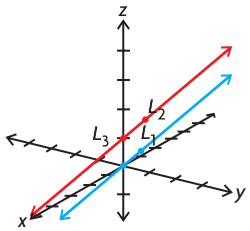
f. $(-2, 3, 6)$

13. Answers may vary. For example:

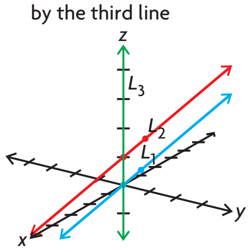
a. Three lines parallel



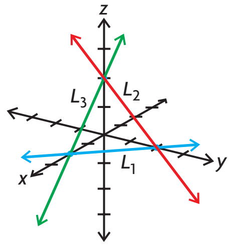
Two lines coincident and the third parallel



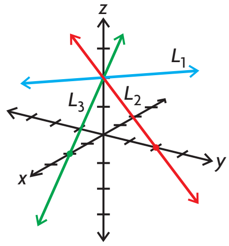
Two parallel lines cut by the third line



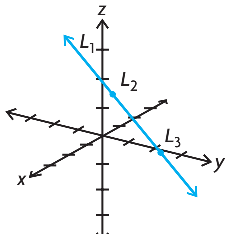
The lines form a triangle



b. Lines meet in a point



c. Three coincident lines



14. $(a - c, -a + b + c, a - b)$

15. **a.** $k = 2$

b. $k = -2$

c. $k \neq \pm 2$

Section 9.3, pp. 516–517

1. **a.** The two equations represent planes that are parallel and not coincident.

b. Answers may vary. For example: $x - y + z = 1, x - y + z = -2$

2. **a.** $x = \dfrac{1}{2} + \dfrac{1}{2}s - t, y = s, z = t;$ $s, t \in \mathbf{R}$; the two planes are coincident.

b. Answers may vary. For example: $x - y + z = -1,$ $2x - 2y + 2z = -2$

3. **a.** $x = 1 + s, y = s, z = -2, s \in \mathbf{R}$; the two planes intersect in a line.

b. Answers may vary. For example: $x - y + z = -1, x - y - z = 3$

4. **a.** $m = \frac{1}{2}, p = 2q, q = 1$, and $p = 2$; The value for m is unique, but p just has to be twice q and arbitrary values can be chosen.

b. $m = \frac{1}{2}, q = 1$, and $p = 3$; The value for m is unique, but p and q can be arbitrarily chosen as long as $p \neq 2q$.

c. $m = -20$; This value is unique, since only one value was found to satisfy the given conditions.

d. $m = -20, p = 1, q = 1$; The value for m is unique from the solution to **c.**, but the values for p and q can be arbitrary since the only value which can change the angle between the planes is m.

5. **a.** $x = 9s, y = -3s, z = s, s \in \mathbf{R}$

b. $x = -3t, y = t, z = -\dfrac{1}{3}t, t \in \mathbf{R}$

c. Since t is an arbitrary real number, we can express t as part b. $t = -3s$, $s \in \mathbf{R}$.

6. **a.** yes; plane

b. no

c. yes; line

d. yes; line

e. yes; line

f. yes; line

7. **a.** $x = 1 - s - t, y = s, z = t, s, t \in \mathbf{R}$

b. no solution

c. $x = -2s, y = -2, z = s, s \in \mathbf{R}$

d. $x = -s + 5, y = -s - 1, z = s,$ $s \in \mathbf{R}$

e. $x = \dfrac{5}{4}s, y = s, z = 1 - \dfrac{3}{4}s, s \in \mathbf{R}$

f. $x = s - 8, y = s, z = 4, s \in \mathbf{R}$

8. **a.** The system will have an infinite number of solutions for any value of k.

b. No, there is no value of k for which the system will not have a solution.

9. $\vec{r}_2 = (-2, 3, 6) + s(-5, -8, 2),$ $s \in \mathbf{R}$

10. The line of intersection of the two planes,

$x = 1 - 2s, y = 2 - 2s, z = s; s \in \mathbf{R};$

$5x + 3y + 16z - 11 = 0$

$5(1 - 2s) + 3(2 - 2s) +$

$16(s) - 11 = 0$

$5 + 6 - 11 - 10s - 6s + 16s = 0$

$0 = 0$

Since this is true, the line is contained in the plane.

11. **a.** $x = 1 + s, y = 1 + s, z = s, s \in \mathbf{R}$

b. about 1.73

12. $8x + 14y - 3z - 8 = 0$

Mid-Chapter Review, pp. 518–519

1. **a.** $(-2, 6, 0)$
b. $(2, 0, 10)$
c. $(0, 3, 5)$

2. **a, c.** Answers may vary. For example:
$x = 2 + 3t, y = 1 + 3t,$
$z = 3 - 2t, t \in \mathbf{R};$
$x = 3 + t, y = -2, z = 5\ t \in \mathbf{R};$
$x = -8 + 7t, y = -5 + 3t,$
$z = 7 - 2t, t \in \mathbf{R}$
b. $(-1, -2, 5)$
d. C: $x = -8 + 7t, y = -5 + 3t,$
$z = 7 - 2t, t \in \mathbf{R}$
$t = 1$
$x = -8 + 7(1), y = -5 + 3(1),$
$z = 7 - 2(1)$
$x = -1, y = -2, z = 5$
$(-1, -2, 5)$
e. $(-1, -2, 5)$

3. **a.** $\vec{r} = (-7, 20, 0) + t(0, -2, 1),$
$t \in \mathbf{R}$
b. $\vec{r} = \left(-\frac{19}{7}, \frac{30}{7}, 0\right) + t(3, 3, -7),$
$t \in \mathbf{R}$
c. $(-7, 0, 10)$

4. **a.** $x = -\frac{11t}{5} - \frac{1}{40}, y = -\frac{2t}{5} - \frac{117}{40},$
$z = t, t \in \mathbf{R}$
b. $x = -\frac{11}{5}s + \frac{227}{5}, y = -\frac{2}{5}s + \frac{94}{5},$
$z = s, t \in \mathbf{R}$
c. The lines found in 4.a. and 4.b. do not intersect, because they are in parallel and distinct planes.

5. **a.** $a = 3$
b. $a = -3$
c. $a \neq \pm 3, a \in \mathbf{R}$

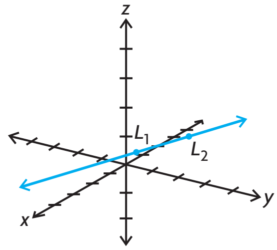
6. Since there is no t-value that satisfies the equations, there is no intersection, and these lines are skew.

7. **a.** no intersection
b. The lines are skew.

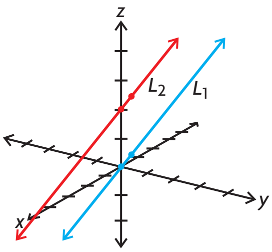
8. $(-3, 6, 6)$

9. **a.** $(3, 1, 2)$
b. $t \in \mathbf{R}$
These lines are the same, so either one of these lines can be used as their intersection.

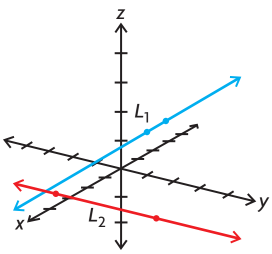
10. **a.** Answers may vary. For example:
i. coincident



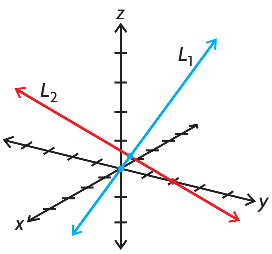
ii. parallel and distinct



iii. skew



iv. intersect in a point



b. **i.** When lines are the same, they are a multiple of each other.
ii. When lines are parallel, one equation is a multiple of the other equation, except for the constant term.
iii. When lines are skew, there are no common solutions to make each equation consistent.
iv. When the solution meets in a point, there is only one unique solution for the system.

11. **a.** when the line lies in the plane
b. Answers may vary. For example:
$\vec{r} = t(3, -5, -3), t \in \mathbf{R};$
$\vec{r} = t(3, -5, -3) + s(1, 1, 1),$
$t, s \in \mathbf{R}$

12. **a.** $(3, 8)$
b. no solution
c. $(2, 1, 4)$

13. **a.** The two lines intersect at a point.
b. The two planes are parallel and do not meet.
c. The three planes intersect at a point.

14. **a.** $\left(-\frac{1}{2}, -\frac{3}{2}, -\frac{3}{2}\right)$
b. $\theta = 90°$
c. $2x - y + z + 1 = 0$

Section 9.4, pp. 530–533

1. **a.** $(-9, -5, -4)$
b. This solution is the point at which all three planes meet.

2. **a.** Answers may vary. For example,
$3x - 3y + 3z = 12$ and
$2x - 2y + 2z = 8.$
b. These three planes are intersecting in one single plane because all three equations can be changed into one equivalent equation. They are coincident planes.
c. $x = t, y = s, z = s - t + 4, s, t \in \mathbf{R}$
d. $y = t, z = s, x = t - s + 4, s, t \in \mathbf{R}$

3. **a.** Answers may vary. For example,
$2x - y + 3z = -2, x - y + 4z = 3,$
and $3x - 2y + 7z = 2;$
$2x - y + 3z = -2, x - y + 4z = 3,$
and $2x - 2y + 8z = 5.$
b. no solutions

4. **a.** $\left(-3, \frac{11}{4}, -\frac{3}{2}\right)$
b. This solution is the point at which all three planes meet.

5. **a.** Since equation ③ = equation ②, equation ② and equation ③ are consistent or lie in the same plane. Equation ① meets this plane in a line.
b. $x = 0, y = t,$ and $z = 1 + t, t \in \mathbf{R}$

6. If you multiply equation ② by 5, you obtain a new equation, $5x - 5y + 15z = -1005$, which is inconsistent with equation ③.

7. **a.** Yes, when this equation is alone, this is true.
b. Answers may vary. For example:
$x + y + z = 2$
$2x + 2y + 2z = 4$
$3x + 3y + 3z = 12$

8. a. $(-1, -1, 0)$ is the point at which the three planes meet.
   b. $\left(-6, \frac{1}{2}, 3\right)$ is the point at which the three planes meet.
   c. $(-99, 100, -101)$ is the point at which the three planes meet.
   d. $(4, 2, 3)$ is the point at which the three planes meet.
9. a. $x = -\frac{1}{7}t - \frac{9}{7}, y = -\frac{15}{7} + \frac{3}{7}t$, and $z = t, t \in \mathbf{R}$; the planes intersect in a line.
   b. no solution
   c. $x = -t, y = 2$, and $z = t, t \in \mathbf{R}$; the planes intersect in a line.
10. a. $x = 0, y = t - 2$, and $z = t, t \in \mathbf{R}$
    b. $x = \frac{t - 3s}{2}, y = t$, and $z = s$, $s, t \in \mathbf{R}$
11. a. ① $x + y + z = 1$
    ② $x - 2y + z = 0$
    ③ $x - y + z = 0$
    Equation ① − equation ③ =
    Equation ④ = $2y = 1$ or $y = \frac{1}{2}$
    Equation ② − equation ③ =
    Equation ⑤ = $-y = 0$ or $y = 0$
    Since the y-variable is different in Equation ④ and Equation ⑤, the system is inconsistent and has no solution.
    b. Answers may vary. For example:
    $\vec{n_1} = (1, 1, 1)$
    $\vec{n_2} = (1, -2, 1)$
    $\vec{n_3} = (1, -1, 1)$
    $m_1 = \vec{n_1} \times \vec{n_2} = (3, 0, -3)$
    $m_2 = \vec{n_1} \times \vec{n_3} = (2, 0, -2)$
    $m_3 = \vec{n_2} \times \vec{n_3} = (-1, 0, 1)$
    c. The three lines of intersection are parallel and coplanar, so they form a triangular prism.
    d. Since $(\vec{n_1} \times \vec{n_2}) \cdot \vec{n_3} = 0$, a triangular prism forms.
12. a. Equation ① and equation ② have the same set of coefficients and variables; however, equations ① equals 3, while equation ② equals 6, which means there is no possible solution.
    b. All three equations equal different numbers, so there is no possible solution.
    c. Equation ② equals 18, while equation ③ equals 17, which means there is no possible solution.
    d. The coefficients of equation ① are half the coefficients of equation ③, but the constant term is not half the other constant term.
13. a. $(4, 3, -5)$
    b. $x = \frac{t - 2}{3}, y = \frac{5t + 5}{3}, z = t, t \in \mathbf{R}$
    c. $x = 0, y = t, z = t, t \in \mathbf{R}$
    d. no solution
    e. $x = -t, y = 2, z = t, t \in \mathbf{R}$
    f. $(0, 0, 0)$
14. a. $p = q = 5$
    b. $x = -\frac{2}{3}t + 3, y = \frac{1}{3}t - 2, z = t$, $t \in \mathbf{R}$
15. a. $m = 2$
    b. $m \neq \pm 2, m \in \mathbf{R}$
    c. $m = -2$
16. $(3, 6, 2)$

Section 9.5, pp. 540–541

1. a. $\frac{3}{5}$
   b. $\frac{56}{13}$ or 4.31
   c. $\frac{236}{\sqrt{1681}}$ or 5.76
2. a. $\frac{5}{\sqrt{5}}$ or 2.24
   b. $\frac{504}{25}$ or 20.16
3. a. 1.4
   b. about 3.92
   c. about 2.88
4. a. $d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$
   If you substitute the coordinates $(0, 0)$, the formula changes to $d = \frac{|A(0) + B(0) + C|}{\sqrt{A^2 + B^2}}$, which reduces to $d = \frac{|C|}{\sqrt{A^2 + B^2}}$.
   b. $\frac{24}{5}$
   c. $\frac{24}{5}$; the answers are the same
5. a. 3
   b. $\frac{7}{5}$ or 1.4
   c. $\frac{4}{\sqrt{13}}$ or 1.11
   d. $\frac{240}{13}$ or 18.46
6. a. about 1.80
   b. about 2.83
   c. about 3.44
7. a. about 2.83
   b. about 3.28
8. a. $\left(\frac{17}{11}, \frac{7}{11}, \frac{16}{11}\right)$
   b. about 1.65
9. about 3.06; $\left(-\frac{11}{14}, \frac{5}{14}, \frac{22}{14}\right)$
10. $\left(\frac{38}{21}, -\frac{44}{21}, \frac{167}{21}\right)$
11. a. about 1.75
    b. D and G
    c. about 3.61 units^2

Section 9.6, pp. 549–550

1. a. Yes, the calculations are correct. Point A lies in the plane.
   b. The answer 0 means that the point lies in the plane.
2. a. 3 c. 2 e. $\frac{11}{27}$ or 0.41
   b. 3 d. $\frac{5}{13}$ or 0.38
3. a. 5
   b. $6x + 8y - 24z + 13 = 0$
   c. Answers may vary. For example: $\left(-\frac{1}{6}, 0, \frac{1}{2}\right)$
4. a. 4 b. 4 c. 2
5. $\frac{2}{3}$ or 0.67
6. 3
7. about 1.51
8. a. about 3.46
   b. $U(1, 1, 2)$ is the point on the first line that produces the minimal distance to the second line at point $V(-1, -1, 0)$.

Review Exercise, pp. 552–555

1. $-\frac{4}{99}$
2. no solution
3. a. no solution
   b. $(99, 100, 101)$
4. a. All four points lie on the plane $3x + 4y - 2z + 1 = 0$
   b. about 0.19
5. a. 3
   b. $\frac{1}{12}$ or 0.08
6. $\vec{r} = (3, 1, 1) + t(2, -1, 2), t \in \mathbf{R}$
7. a. no solution
   b. no solution
   c. no solution

8. **a.** $x = -\frac{5}{7}t, y = 1 + \frac{2}{7}t, z = t, t \in \mathbf{R}$

b. $x = 3, y = \frac{1}{4}, z = -\frac{1}{2}$

c. $x = 3t - 3s + 7, y = t, z = s,$ $s, t \in \mathbf{R}$

9. **a.** $x = \frac{1}{2} + \frac{1}{36}t, y = -\frac{1}{2} + \frac{5}{12}t,$ $z = t, t \in \mathbf{R}$

b. $x = \frac{9}{8} - \frac{31}{24}t, y = \frac{1}{4} + \frac{1}{12}t, z = t,$ $t \in \mathbf{R}$

10. **a.** These three planes meet at the point $(-1, 5, 3)$.

b. The planes do not intersect. Geometrically, the planes form a triangular prism.

c. The planes meet in a line through the origin, with equation $x = t,$ $y = -7t, z = -5t, t \in \mathbf{R}$

11. 4.90

12. **a.** $x - 2y + z + 4 = 0$

$\vec{r} = (3, 1, -5) + s(2, 1, 0), s \in \mathbf{R}$

$\vec{m} \times \vec{n} = (2, 1, 0)(1, -2, 1) = 0$

Since the line's direction vector is perpendicular to the normal of the plane and the point $(3, 1, -5)$ lies on both the line and the plane, the line is in the plane.

b. $(-1, -1, -5)$

c. $x - 2y + z + 4 = 0$

$-1 - 2(-1) + (-5) + 4 = 0$

The point $(-1, -1, -5)$ is on the plane since it satisfies the equation of the plane.

d. $7x - 2y - 11z - 50 = 0$

13. **a.** 5.48

b. $(3, 0, -1)$

14. **a.** $(-2, -3, 0)$.

b. $\vec{r} = (-2, -3, 0) + t(1, -2, 1),$ $t \in \mathbf{R}$

15. **a.** $-10x + 9y + 8z + 16 = 0$

b. about 0.45

16. **a.** 1

b. $\vec{r} = (0, 0, -1) + t(4, 3, 7), t \in \mathbf{R}$

17. **a.** $x = 2, y = -1, z = 1$

b. $x = 7 - 3t, y = 3 - t, z = t, t \in \mathbf{R}$

18. $a = \frac{2}{3}, b = \frac{3}{4}, c = \frac{1}{2}$

19. $\left(4, -\frac{7}{4}, \frac{7}{2}\right)$

20. $\left(-\frac{5}{3}, \frac{8}{3}, \frac{4}{3}\right)$

21. **a.** $\vec{r} = \left(\frac{45}{4}, 0, -\frac{21}{4}\right) + t(11, 2, -5), t \in \mathbf{R};$

$\vec{r} = \left(-\frac{37}{2}, 0, \frac{15}{2}\right) + t(11, 2, -5), t \in \mathbf{R};$

$\vec{r} = (7, 0, -1) + t(11, 2, -5),$ $t \in \mathbf{R}$; $z = -1 - 5t, t \in \mathbf{R}$

b. All three lines of intersection found in part a. have direction vector $(11, 2, -5)$, and so they are all parallel. Since no pair of normal vectors for these three planes is parallel, no pair of these planes is coincident.

22. $\left(\frac{1}{2}, 1, \frac{1}{3}\right), \left(\frac{1}{2}, 1, -\frac{1}{3}\right), \left(\frac{1}{2}, -1, \frac{1}{3}\right),$ $\left(\frac{1}{2}, -1, -\frac{1}{3}\right), \left(-\frac{1}{2}, 1, \frac{1}{3}\right),$ $\left(\frac{1}{2}, -1, -\frac{1}{3}\right) \left(-\frac{1}{2}, 1, -\frac{1}{3}\right)$, and $\left(-\frac{1}{2}, -1, \frac{1}{3}\right)$

23. $y = \frac{7}{6}x^2 - \frac{3}{2}x - \frac{2}{3}$

24. $\left(\frac{29}{7}, \frac{4}{7}, -\frac{33}{7}\right)$

25. $A = 5, B = 2, C = -4$

26. **a.** $\vec{r} = (-1, -4, -6) + t(-5, -4, -3), t \in \mathbf{R}$

b. $\left(\frac{13}{2}, 2, -\frac{3}{2}\right)$

c. about 33.26 units^2

27. $6x - 8y + 9z - 115 = 0$

Chapter 9 Test, p. 556

1. **a.** $(3, -1, -5)$

b. $3 - (-1) + (-5) + 1 = 0$

$3 + 1 - 5 + 1 = 0$

$0 = 0$

2. **a.** $\frac{13}{12}$ or 1.08

b. $\frac{40}{3}$ or 13.33

3. **a.** $x = \frac{4t}{5}, y = 1 - \frac{t}{5}, z = t, t \in \mathbf{R}$

b. $(4, 0, 5)$

4. **a.** $(1, -5, 4)$

b. The three planes intersect at the point $(1, -5, 4)$.

5. **a.** $x = -\frac{1}{2} - \frac{t}{4}, y = \frac{3t}{4} + \frac{1}{2}, z = t,$ $t \in \mathbf{R}$

b. The three planes intersect at this line.

6. **a.** $m = -1, n = -3$

b. $x = -1, y = 1 - t, z = t, t \in \mathbf{R}$

7. 10.20

Cumulative Review of Vectors, pp. 557–560

1. **a.** about 111.0°

b. scalar projection: $-\frac{14}{13}$, vector projection: $\left(-\frac{52}{169}, \frac{56}{169}, -\frac{168}{169}\right)$

c. scalar projection: $-\frac{14}{3}$, vector projection: $\left(-\frac{28}{9}, \frac{14}{9}, \frac{28}{9}\right)$

2. **a.** $x = 8 + 4t, y = t, z = -3 - 3t,$ $t \in \mathbf{R}$

b. about 51.9°

3. **a.** $\frac{1}{2}$

b. 3

c. $\frac{3}{2}$

4. **a.** $-7\vec{i} - 19\vec{j} - 14\vec{k}$

b. 18

5. x-axis: about 42.0°, y-axis: about 111.8°, z-axis: about 123.9°

6. **a.** $(-7, -5, -1)$

b. $(-42, -30, -6)$

c. about 8.66 square units

d. 0

7. $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

8. **a.** vector equation: Answers may vary. $\vec{r} = (2, -3, 1) + t(-1, 5, 2), t \in \mathbf{R}$; parametric equation: $x = 2 - t, y = -3 + 5t,$ $z = 1 + 2t, t \in \mathbf{R}$

b. If the x-coordinate of a point on the line is 4, then $2 - t = 4$, or $t = -2$. At $t = -2$, the point on the line is $(2, -3, 1) - 2(-1, 5, 2)$ $= (4, -13, -3)$. Hence, $C(4, -13, -3)$ is a point on the line.

9. The direction vector of the first line is $(-1, 5, 2)$ and of the second line is $(1, -5, -2) = -(-1, 5, 2)$. So they are collinear and hence parallel. The lines coincide if and only if for any point on the first line and second line, the vector connecting the two points is a multiple of the direction vector for the lines. $(2, 0, 9)$ is a point on the first line and $(3, -5, 10)$ is a point on the second line. $(2, 0, 9) - (3, -5, 10) = (-1, 5, -1)$ $\neq k(-1, 5, 2)$ for $k \in \mathbf{R}$. Hence, the lines are parallel and distinct.

8. **a.** $x = -\frac{5}{7}t, y = 1 + \frac{2}{7}t, z = t, t \in \mathbf{R}$
b. $x = 3, y = \frac{1}{4}, z = -\frac{1}{2}$
c. $x = 3t - 3s + 7, y = t, z = s,$ $s, t \in \mathbf{R}$

9. **a.** $x = \frac{1}{2} + \frac{1}{36}t, y = -\frac{1}{2} + \frac{5}{12}t,$ $z = t, t \in \mathbf{R}$
b. $x = \frac{9}{8} - \frac{31}{24}t, y = \frac{1}{4} + \frac{1}{12}t, z = t,$ $t \in \mathbf{R}$

10. **a.** These three planes meet at the point $(-1, 5, 3)$.
b. The planes do not intersect. Geometrically, the planes form a triangular prism.
c. The planes meet in a line through the origin, with equation $x = t,$ $y = -7t, z = -5t, t \in \mathbf{R}$

11. 4.90

12. **a.** $x - 2y + z + 4 = 0$
$\vec{r} = (3, 1, -5) + s(2, 1, 0), s \in \mathbf{R}$
$\vec{m} \times \vec{n} = (2, 1, 0)(1, -2, 1) = 0$
Since the line's direction vector is perpendicular to the normal of the plane and the point $(3, 1, -5)$ lies on both the line and the plane, the line is in the plane.
b. $(-1, -1, -5)$
c. $x - 2y + z + 4 = 0$
$-1 - 2(-1) + (-5) + 4 = 0$
The point $(-1, -1, -5)$ is on the plane since it satisfies the equation of the plane.
d. $7x - 2y - 11z - 50 = 0$

13. **a.** 5.48
b. $(3, 0, -1)$

14. **a.** $(-2, -3, 0)$.
b. $\vec{r} = (-2, -3, 0) + t(1, -2, 1),$ $t \in \mathbf{R}$

15. **a.** $-10x + 9y + 8z + 16 = 0$
b. about 0.45

16. **a.** 1
b. $\vec{r} = (0, 0, -1) + t(4, 3, 7), t \in \mathbf{R}$

17. **a.** $x = 2, y = -1, z = 1$
b. $x = 7 - 3t, y = 3 - t, z = t, t \in \mathbf{R}$

18. $a = \frac{2}{3}, b = \frac{3}{4}, c = \frac{1}{2}$

19. $\left(4, -\frac{7}{4}, \frac{7}{2}\right)$

20. $\left(-\frac{5}{3}, \frac{8}{3}, \frac{4}{3}\right)$

21. **a.** $\vec{r} = \left(\frac{45}{4}, 0, -\frac{21}{4}\right) + t(11, 2, -5), t \in \mathbf{R};$
$\vec{r} = \left(-\frac{37}{2}, 0, \frac{15}{2}\right) + t(11, 2, -5), t \in \mathbf{R};$
$\vec{r} = (7, 0, -1) + t(11, 2, -5),$ $t \in \mathbf{R}; z = -1 - 5t, t \in \mathbf{R}$
b. All three lines of intersection found in part a. have direction vector $(11, 2, -5)$, and so they are all parallel. Since no pair of normal vectors for these three planes is parallel, no pair of these planes is coincident.

22. $\left(\frac{1}{2}, 1, \frac{1}{3}\right), \left(\frac{1}{2}, 1, -\frac{1}{3}\right), \left(\frac{1}{2}, -1, \frac{1}{3}\right),$ $\left(\frac{1}{2}, -1, -\frac{1}{3}\right), \left(-\frac{1}{2}, 1, \frac{1}{3}\right),$ $\left(\frac{1}{2}, -1, -\frac{1}{3}\right) \left(-\frac{1}{2}, 1, -\frac{1}{3}\right)$, and $\left(-\frac{1}{2}, -1, \frac{1}{3}\right)$

23. $y = \frac{7}{6}x^2 - \frac{3}{2}x - \frac{2}{3}$

24. $\left(\frac{29}{7}, \frac{4}{7}, -\frac{33}{7}\right)$

25. $A = 5, B = 2, C = -4$

26. **a.** $\vec{r} = (-1, -4, -6) + t(-5, -4, -3), t \in \mathbf{R}$
b. $\left(\frac{13}{2}, 2, -\frac{3}{2}\right)$
c. about 33.26 units^2

27. $6x - 8y + 9z - 115 = 0$

Chapter 9 Test, p. 556

1. **a.** $(3, -1, -5)$
b. $3 - (-1) + (-5) + 1 = 0$
$3 + 1 - 5 + 1 = 0$
$0 = 0$

2. **a.** $\frac{13}{12}$ or 1.08
b. $\frac{40}{3}$ or 13.33

3. **a.** $x = \frac{4t}{5}, y = 1 - \frac{t}{5}, z = t, t \in \mathbf{R}$
b. $(4, 0, 5)$

4. **a.** $(1, -5, 4)$
b. The three planes intersect at the point $(1, -5, 4)$.

5. **a.** $x = -\frac{1}{2} - \frac{t}{4}, y = \frac{3t}{4} + \frac{1}{2}, z = t,$ $t \in \mathbf{R}$
b. The three planes intersect at this line.

6. **a.** $m = -1, n = -3$
b. $x = -1, y = 1 - t, z = t, t \in \mathbf{R}$

7. 10.20

Cumulative Review of Vectors, pp. 557–560

1. **a.** about 111.0°
b. scalar projection: $-\frac{14}{13}$, vector projection: $\left(-\frac{52}{169}, \frac{56}{169}, -\frac{168}{169}\right)$
c. scalar projection: $-\frac{14}{3}$, vector projection: $\left(-\frac{28}{9}, \frac{14}{9}, \frac{28}{9}\right)$

2. **a.** $x = 8 + 4t, y = t, z = -3 - 3t,$ $t \in \mathbf{R}$
b. about 51.9°

3. **a.** $\frac{1}{2}$
b. 3
c. $\frac{3}{2}$

4. **a.** $-7\vec{i} - 19\vec{j} - 14\vec{k}$
b. 18

5. x-axis: about 42.0°, y-axis: about 111.8°, z-axis: about 123.9°

6. **a.** $(-7, -5, -1)$
b. $(-42, -30, -6)$
c. about 8.66 square units
d. 0

7. $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

8. **a.** vector equation: Answers may vary.
$\vec{r} = (2, -3, 1) + t(-1, 5, 2), t \in \mathbf{R};$
parametric equation:
$x = 2 - t, y = -3 + 5t,$
$z = 1 + 2t, t \in \mathbf{R}$
b. If the x-coordinate of a point on the line is 4, then $2 - t = 4$, or $t = -2$. At $t = -2$, the point on the line is $(2, -3, 1) - 2(-1, 5, 2) = (4, -13, -3)$. Hence, $C(4, -13, -3)$ is a point on the line.

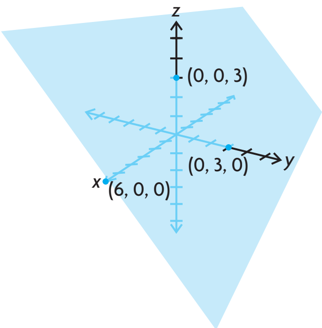
9. The direction vector of the first line is $(-1, 5, 2)$ and of the second line is $(1, -5, -2) = -(-1, 5, 2)$. So they are collinear and hence parallel.
The lines coincide if and only if for any point on the first line and second line, the vector connecting the two points is a multiple of the direction vector for the lines. $(2, 0, 9)$ is a point on the first line and $(3, -5, 10)$ is a point on the second line.
$(2, 0, 9) - (3, -5, 10) = (-1, 5, -1)$ $\neq k(-1, 5, 2)$ for $k \in \mathbf{R}$. Hence, the lines are parallel and distinct.

10. vector equation:
$\vec{r} = (0, 0, 4) + t(0, 1, 1), t \in \mathbf{R}$;
parametric equation: $x = 0, y = t,$
$z = 4 + t, t \in \mathbf{R}$

11. -13

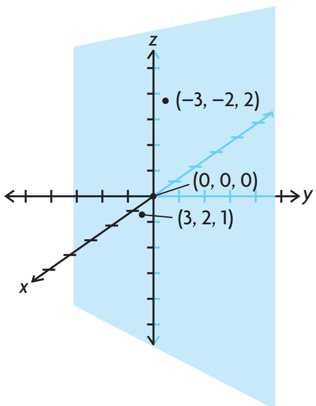
12. $\left(\frac{3}{2}, -\frac{31}{6}, \frac{13}{6}\right)$

13. **a.**



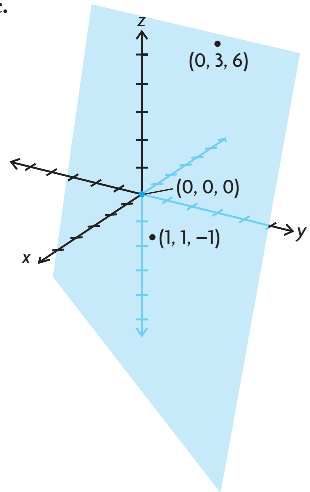
Answers may vary. For example, $(0, 3, -3)$ and $(6, 0, -3)$.

b.



Answers may vary. For example, $(-3, -2, 2)$ and $(3, 2, 1)$.

c.



Answers may vary. For example, $(0, 3, 6)$ and $(1, 1, -1)$.

14. $(-7, 10, 20)$

15. $\vec{q} = (1, 0, 2) + t(-11, 7, 2), t \in \mathbf{R}$

16. **a.** $12x - 9y - 6z + 24 = 0$
b. about 1.49 units

17. **a.** $3x - 5y + 4z - 7 = 0$
b. $x - y + 12z - 27 = 0$
c. $z - 3 = 0$
d. $x + 2z + 1 = 0$

18. 336.80 km/h, N 12.1° W

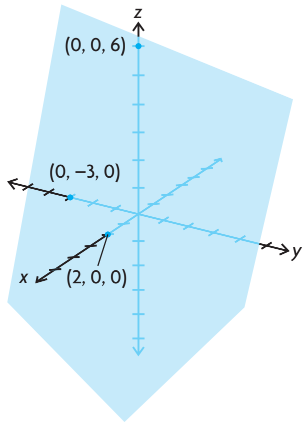
19. **a.** $\vec{r} = (0, 0, 6) + s(1, 0, -3) + t(0, 1, 2), s, t \in \mathbf{R}$. To verify, find the Cartesian equation corresponding to the above vector equation and see if it is equivalent to the Cartesian equation given in the problem. A normal vector to this plane is the cross product of the two directional vectors.

$\vec{n} = (1, 0, -3) \times (0, 1, 2)$

$= (0(2) - (-3)(1), -3(0) - 1(2), 1(1) - 0(0))$
$= (3, -2, 1)$

So the plane has the form $3x + 2y + z + D = 0$, for some constant D. To find D, we know that $(0, 0, 6)$ is a point on the plane, so $3(0) - 2(0) + (6) + D = 0$. So, $6 + D = 0$, or $D = -6$. So, the Cartesian equation for the plane is $3x - 2y + z - 6 = 0$. Since this is the same as the initial Cartesian equation, the vector equation for the plane is correct.

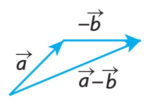
b.



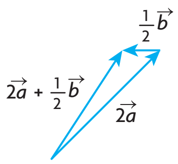
20. **a.** 16°
b. The two planes are perpendicular if and only if their normal vectors are also perpendicular. A normal vector for the first plane is $(2, -3, 1)$ and a normal vector for the second plane is $(4, -3, -17)$. The two vectors are perpendicular if and only if their dot product is zero.
$(2, -3, 1) \cdot (4, -3, -17)$
$= 2(4) - 3(-3) + 1(-17)$
$= 0$
Hence, the normal vectors are perpendicular. Thus, the planes are perpendicular.
c. The two planes are parallel if and only if their normal vectors are also parallel. A normal vector for the first plane is $(2, -3, 2)$ and a normal vector for the second plane is $(2, -3, 2)$. Since both normal vectors are the same, the planes are parallel. Since $2(0) - 3(-1) + 2(0) - 3 = 0$, the point $(0, -1, 0)$ is on the second plane. Yet since $2(0) - 3(-1) + 2(0) - 1 = 2 \neq 0$, $(0, -1, 0)$ is not on the first plane. Thus, the two planes are parallel but not coincident.

21. resultant: about 56.79 N, 37.6° from the 25 N force toward the 40 N force, equilibrant: about 56.79 N, 142.4° from the 25 N force away from the 40 N force

22. **a.**



b.



23. **a.** $\left(\frac{6}{7}, \frac{2}{7}, -\frac{3}{7}\right)$

b. $\left(-\frac{6}{7}, -\frac{2}{7}, \frac{3}{7}\right)$

24. **a.** $\overrightarrow{OC} = (8, 9)$,
$\overrightarrow{BD} = (10, -5)$

b. about 74.9°

c. about 85.6°

25. **a.** $x = t, y = -1 + t, z = 1, t \in \mathbf{R}$

b. $(1, 2, -3)$

c. $x = 1, y = t, z = -3 + t, t \in \mathbf{R}$

d. $x = 1 + 3s + t, y = t, z = s,$
$s, t \in \mathbf{R}$

26. **a.** yes; $x = 0, y = -1 + t, z = t, t \in \mathbf{R}$

b. no

c. yes;
$x = 2 - 2t, y = t, z = 3t, t \in \mathbf{R}$

27. 30°

28. **a.** $-\frac{3}{2}$

b. 84

29. $\vec{r} = t(-1, 3, 1), t \in \mathbf{R}$,
$-x + 3y + z - 11 = 0$

30. $(-1, 1, 0)$

31. **a.** 0.8 km

b. 12 min

32. **a.** Answers may vary.
$\vec{r} = (6, 3, 4) + t(4, 4, 1), t \in \mathbf{R}$

b. The line found in part a will lie in the plane $x - 2y + 4z - 16 = 0$ if and only if both points $A(2, -1, 3)$ and $B(6, 3, 4)$ lie in this plane. We verify this by substituting these points into the equation of the plane, and checking for consistency.
For A:
$2 - 2(-1) + 4(3) - 16 = 0$
For B:
$6 - 2(3) + 4(4) - 16 = 0$
Since both points lie on the plane, so does the line found in part a.

33. 20 km/h at N 53.1° E

34. parallel: 1960 N,
perpendicular: about 3394.82 N

35. **a.** True; all non-parallel pairs of lines intersect in exactly one point in $\mathbf{R}^2$. However, this is not the case for lines in $\mathbf{R}^3$ (skew lines provide a counterexample).

b. True; all non-parallel pairs of planes intersect in a line in $\mathbf{R}^3$.

c. True; the line $x = y = z$ has direction vector $(1, 1, 1)$, which is not perpendicular to the normal vector $(1, -2, 2)$ to the plane $x - 2y + 2z = k$, k is any constant. Since these vectors are not perpendicular, the line is not parallel to the plane, and so they will intersect in exactly one point.

d. False; a direction vector for the line $\frac{x}{2} = y - 1 = \frac{z + 1}{2}$ is $(2, 1, 2)$. A direction vector for the line $\frac{x - 1}{-4} = \frac{y - 1}{-2} = \frac{z + 1}{-2}$ is $(-4, -2, -2)$, or $(2, 1, 1)$ (which is parallel to $(-4, -2, -2)$). Since $(2, 1, 2)$ and $(2, 1, 1)$ are obviously not parallel, these two lines are not parallel.

36. **a.** A direction vector for
L_1: $x = 2, \frac{y - 2}{3} = z$ is $(0, 3, 1)$,
and a direction vector for
L_2: $x = y + k = \frac{z + 14}{k}$ is $(1, 1, k)$.
But $(0, 3, 1)$ is not a nonzero scalar multiple of $(1, 1, k)$ for any k, since the first component of $(0, 3, 1)$ is 0. This means that the direction vectors for L_1 and L_2 are never parallel, which means that these lines are never parallel for any k.

b. 6; $(2, -4, -2)$

Calculus Appendix

Implicit Differentiation, p. 564

1. The chain rule states that if y is a composite function, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$. To differentiate an equation implicitly, first differentiate both sides of the equation with respect to x, using the chain rule for terms involving y, then solve for $\frac{dy}{dx}$.

2. **a.** $-\frac{x}{y}$

b. $\frac{x^2}{5y}$

c. $\frac{-y^2}{2xy + y^2}$

d. $\frac{9x}{16y}$

e. $-\frac{13x}{48y}$

f. $-\frac{2x}{2y + 5}$

3. **a.** $y = \frac{2}{3}x - \frac{13}{3}$

b. $y = \frac{2}{3}(x + 8) + 3$

c. $y = -\frac{3\sqrt{3}}{5}x - 3$

d. $y = \frac{11}{10}(x + 11) - 4$

4. $(0, 1)$

5. **a.** 1

b. $\left(\frac{3}{\sqrt{5}}, \sqrt{5}\right)$ and $\left(-\frac{3}{\sqrt{5}}, -\sqrt{5}\right)$

6. -10

7. $7x - y - 11 = 0$

8. $y = \frac{1}{2}x - \frac{3}{2}$

9. **a.** $\frac{4}{(x + y)^2} - 1$

b. $4\sqrt{x + y} - 1$

10. **a.** $\frac{3x^2 - 8xy}{4x^2 - 3}$

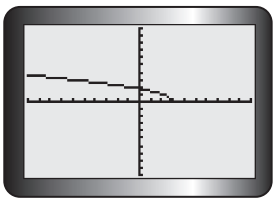
b. $y = \frac{x^3}{4x^2 - 3}; \frac{4x^4 - 9x^2}{(4x^2 - 3)^2}$

c. $\frac{dy}{dx} = \frac{3x^2 - 8xy}{4x^2 - 3}$

$y = \frac{x^3}{4x^2 - 3}$

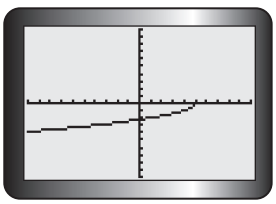
$$\frac{dy}{dx} = \frac{3x^2 - 8x\left(\frac{x^3}{4x^2 - 3}\right)}{4x^2 - 3}$$
$$= \frac{3x^2 - (4x^2 - 3) - 8x^4}{(4x^2 - 3)^2}$$
$$= \frac{12x^4 - 9x^2 - 8x^4}{(4x^2 - 3)^2}$$
$$= \frac{4x^4 - 9x^2}{(4x^2 - 3)^2}$$

11. **a.**



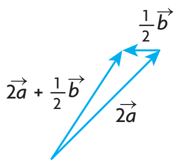
one tangent

b.



one tangent

b.



23. **a.** $\left(\frac{6}{7}, \frac{2}{7}, -\frac{3}{7}\right)$

b. $\left(-\frac{6}{7}, -\frac{2}{7}, \frac{3}{7}\right)$

24. **a.** $\overrightarrow{OC} = (8, 9)$,
$\overrightarrow{BD} = (10, -5)$

b. about 74.9°

c. about 85.6°

25. **a.** $x = t, y = -1 + t, z = 1, t \in \mathbf{R}$

b. $(1, 2, -3)$

c. $x = 1, y = t, z = -3 + t, t \in \mathbf{R}$

d. $x = 1 + 3s + t, y = t, z = s,$
$s, t \in \mathbf{R}$

26. **a.** yes; $x = 0, y = -1 + t, z = t, t \in \mathbf{R}$

b. no

c. yes;
$x = 2 - 2t, y = t, z = 3t, t \in \mathbf{R}$

27. 30°

28. **a.** $-\frac{3}{2}$

b. 84

29. $\vec{r} = t(-1, 3, 1), t \in \mathbf{R}$,
$-x + 3y + z - 11 = 0$

30. $(-1, 1, 0)$

31. **a.** 0.8 km

b. 12 min

32. **a.** Answers may vary.
$\vec{r} = (6, 3, 4) + t(4, 4, 1), t \in \mathbf{R}$

b. The line found in part a will lie in the plane $x - 2y + 4z - 16 = 0$ if and only if both points $A(2, -1, 3)$ and $B(6, 3, 4)$ lie in this plane. We verify this by substituting these points into the equation of the plane, and checking for consistency.
For A:
$2 - 2(-1) + 4(3) - 16 = 0$
For B:
$6 - 2(3) + 4(4) - 16 = 0$
Since both points lie on the plane, so does the line found in part a.

33. 20 km/h at N 53.1° E

34. parallel: 1960 N,
perpendicular: about 3394.82 N

35. **a.** True; all non-parallel pairs of lines intersect in exactly one point in $\mathbf{R}^2$. However, this is not the case for lines in $\mathbf{R}^3$ (skew lines provide a counterexample).

b. True; all non-parallel pairs of planes intersect in a line in $\mathbf{R}^3$.

c. True; the line $x = y = z$ has direction vector $(1, 1, 1)$, which is not perpendicular to the normal vector $(1, -2, 2)$ to the plane $x - 2y + 2z = k$, k is any constant. Since these vectors are not perpendicular, the line is not parallel to the plane, and so they will intersect in exactly one point.

d. False; a direction vector for the line $\frac{x}{2} = y - 1 = \frac{z + 1}{2}$ is $(2, 1, 2)$. A direction vector for the line $\frac{x - 1}{-4} = \frac{y - 1}{-2} = \frac{z + 1}{-2}$ is $(-4, -2, -2)$, or $(2, 1, 1)$ (which is parallel to $(-4, -2, -2)$). Since $(2, 1, 2)$ and $(2, 1, 1)$ are obviously not parallel, these two lines are not parallel.

36. **a.** A direction vector for
L_1: $x = 2, \frac{y - 2}{3} = z$ is $(0, 3, 1)$,
and a direction vector for
L_2: $x = y + k = \frac{z + 14}{k}$ is $(1, 1, k)$.
But $(0, 3, 1)$ is not a nonzero scalar multiple of $(1, 1, k)$ for any k, since the first component of $(0, 3, 1)$ is 0. This means that the direction vectors for L_1 and L_2 are never parallel, which means that these lines are never parallel for any k.

b. 6; $(2, -4, -2)$

Calculus Appendix

Implicit Differentiation, p. 564

1. The chain rule states that if y is a composite function, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$. To differentiate an equation implicitly, first differentiate both sides of the equation with respect to x, using the chain rule for terms involving y, then solve for $\frac{dy}{dx}$.

2. **a.** $-\frac{x}{y}$

b. $\frac{x^2}{5y}$

c. $\frac{-y^2}{2xy + y^2}$

d. $\frac{9x}{16y}$

e. $-\frac{13x}{48y}$

f. $-\frac{2x}{2y + 5}$

3. **a.** $y = \frac{2}{3}x - \frac{13}{3}$

b. $y = \frac{2}{3}(x + 8) + 3$

c. $y = -\frac{3\sqrt{3}}{5}x - 3$

d. $y = \frac{11}{10}(x + 11) - 4$

4. $(0, 1)$

5. **a.** 1

b. $\left(\frac{3}{\sqrt{5}}, \sqrt{5}\right)$ and $\left(-\frac{3}{\sqrt{5}}, -\sqrt{5}\right)$

6. -10

7. $7x - y - 11 = 0$

8. $y = \frac{1}{2}x - \frac{3}{2}$

9. **a.** $\frac{4}{(x + y)^2} - 1$

b. $4\sqrt{x + y} - 1$

10. **a.** $\frac{3x^2 - 8xy}{4x^2 - 3}$

b. $y = \frac{x^3}{4x^2 - 3}$; $\frac{4x^4 - 9x^2}{(4x^2 - 3)^2}$

c. $\frac{dy}{dx} = \frac{3x^2 - 8xy}{4x^2 - 3}$

$$y = \frac{x^3}{4x^2 - 3}$$

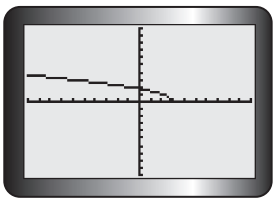
$$\frac{dy}{dx} = \frac{3x^2 - 8x\left(\frac{x^3}{4x^2 - 3}\right)}{4x^2 - 3}$$

$$= \frac{3x^2 - (4x^2 - 3) - 8x^4}{(4x^2 - 3)^2}$$

$$= \frac{12x^4 - 9x^2 - 8x^4}{(4x^2 - 3)^2}$$

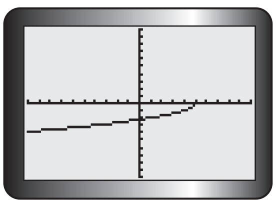
$$= \frac{4x^4 - 9x^2}{(4x^2 - 3)^2}$$

11. **a.**



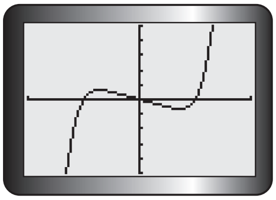
one tangent

b.



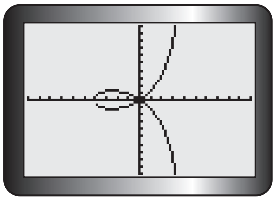
one tangent

c.



one tangent

d.



two tangents

12. $\frac{1}{2}\left(\frac{x}{y}\right)^{-\frac{1}{2}}\frac{1y - \frac{dy}{dx}x}{y^2}$

$+ \frac{1}{2}\left(\frac{y}{x}\right)^{-\frac{1}{2}}\frac{\frac{dy}{dx}x - y}{x^2} = 0$

$\frac{y^{\frac{1}{2}}}{2x^{\frac{1}{2}}}\frac{1y - \frac{dy}{dx}x}{y^2} + \frac{x^{\frac{1}{2}}}{2y^{\frac{1}{2}}}\frac{\frac{dy}{dx}x - y}{x^2} = 0$

Multiply by $2x^2y^2$:

$x^{\frac{3}{2}}y^{\frac{1}{2}}\left(y - x\frac{dy}{dx}\right) + x^{\frac{1}{2}}y^{\frac{3}{2}}\left(\frac{dy}{dx}x - y\right) = 0$

$x^{\frac{3}{2}}y^{\frac{3}{2}} - x^{\frac{5}{2}}y^{\frac{1}{2}}\frac{dy}{dx} + x^{\frac{3}{2}}y^{\frac{3}{2}}\frac{dy}{dx} - x^{\frac{1}{2}}y^{\frac{5}{2}} = 0$

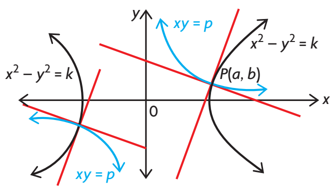
$\frac{dy}{dx}\left(x^{\frac{3}{2}}y^{\frac{3}{2}} - x^{\frac{5}{2}}y^{\frac{1}{2}}\right) = x^{\frac{1}{2}}y^{\frac{5}{2}} - x^{\frac{3}{2}}y^{\frac{3}{2}}$

$\frac{dy}{dx} = \frac{x^{\frac{1}{2}}y^{\frac{3}{2}}(y - x)}{x^{\frac{3}{2}}y^{\frac{1}{2}}(y - x)}$

$\frac{dy}{dx} = \frac{y}{x}$, as required.

13. $2x - 3y + 10 = 0$ and $x = 4$

14.



Let $P(a, b)$ be the point of intersection where $a \neq 0$ and $b \neq 0$.

For $x^2 - y^2 = k$,

$2x - 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{x}{y}$

At $P(a, b)$,

$\frac{dy}{dx} = \frac{a}{b}$

For $xy = P$,

$1 \cdot y + \frac{dy}{dx}x = P$

$\frac{dy}{dx} = -\frac{y}{x}$

At $P(a, b)$,

$\frac{dy}{dx} = -\frac{b}{a}$

At point $P(a, b)$, the slope of the tangent line of $xy = P$ is the negative reciprocal of the slope of the tangent line of $x^2 - y^2 = k$. Therefore, the tangent lines intersect at right angles, and thus, the two curves intersect orthogonally for all values of the constants k and P.

15. $\frac{1}{2}x^{\frac{1}{2}} + \frac{1}{2}y^{\frac{1}{2}}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$

Let $P(a, b)$ be the point of tangency.

$\frac{dy}{dx} - \frac{\sqrt{b}}{\sqrt{a}}$

Equation on tangent line l and P is

$\frac{y - b}{x - a} = -\frac{\sqrt{b}}{\sqrt{a}}.$

x-intercept is found when $y = 0$.

$\frac{-b}{x - a} = -\frac{\sqrt{b}}{\sqrt{a}}$

$-b\sqrt{a} = -\sqrt{b}x + a\sqrt{b}$

$x = \frac{a\sqrt{b} + b\sqrt{a}}{\sqrt{b}}$

Therefore, the x-intercept is $\frac{a\sqrt{b} + b\sqrt{a}}{\sqrt{b}}$.

For the y-intercept, let $x = 0$,

$\frac{y - b}{-a} = -\frac{\sqrt{b}}{\sqrt{a}}.$

y-intercept is $\frac{a\sqrt{b}}{\sqrt{a}} + b$.

The sum of the intercepts is

$\frac{a\sqrt{b} + b\sqrt{a}}{\sqrt{b}} + \frac{a\sqrt{b} + b\sqrt{a}}{\sqrt{a}}$

$= \frac{a^{\frac{3}{2}}b^{\frac{1}{2}} + 2ab + b^{\frac{3}{2}}a^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}}$

$= \frac{a^{\frac{1}{2}}b^{\frac{1}{2}}(a + 2\sqrt{a}\sqrt{b} + b)}{a^{\frac{1}{2}}b^{\frac{1}{2}}}$

$= a + 2\sqrt{a}\sqrt{b} + b$

$= (a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$

Since $P(a, b)$ is on the curve, then $\sqrt{a} + \sqrt{b} = \sqrt{k}$, or $a^{\frac{1}{2}} + b^{\frac{1}{2}} = k^{\frac{1}{2}}$.

Therefore, the sum of the intercepts is $(k^{\frac{1}{2}})^2 = k$, as required.

16. $(x + 2)^2 + (y - 5)^5 = 18$ and $(x - 4)^2 + (y + 1)^2 = 18$

Related Rates, pp. 569–570

1. a. $\frac{dA}{dt} = 4\ \text{m/s}^2$

b. $\frac{dS}{dt} = -3\ \text{m}^2/\text{min}$

c. $\frac{ds}{dt} = 70$ km/h, when $t = 0.25$

d. $\frac{dx}{dt} = \frac{dy}{dt}$

e. $\frac{d\theta}{dt} = \frac{\pi}{10}$ rad/s

2. a. decreasing at 5.9 °C/s

b. about 0.58 m

c. Solve $T''(x) = 0$.

3. area increasing at 100 cm^2/s; perimeter increasing at 20 cm/s

4. a. increasing at 300 cm^3/s

b. increasing at 336 cm^2/s

5. increasing at 40 cm^2/s

6. a. $\frac{5}{6\pi}$ km/h

b. $\frac{5}{3\pi}$ m/s

7. $\frac{1}{\pi}$ km/h

8. 4 m/s

9. 8 m/min

10. 214 m/s

11. $5\sqrt{13}$ km/h

12. a. $\frac{1}{72\pi}$ cm/s

b. $\frac{2}{49\pi}$ cm/s or about 0.01 cm/s

c. $\frac{1}{8\pi}$ cm/s or about 0.04 cm/s

13. $\frac{50}{\pi}$ cm/min; 94.25 min (or about 1.5 h)

14. Answers may vary. For example:

a. The diameter of a right-circular cone is expanding at a rate of 4 cm/min. Its height remains constant at 10 cm. Find its radius when the volume is increasing at a rate of $80\pi\ \text{cm}^3/\text{min}$.

b. Water is being poured into a right-circular tank at the rate of $12\pi\ \text{m}^3/\text{min}$. Its height is 4 m and its radius is 1 m. At what rate is the water level rising?

c. The volume of a right-circular cone is expanding because its radius is increasing at 12 cm/min and its height is increasing at 6 cm/min. Find the rate at which its volume is changing when its radius is 20 cm and its height is 40 cm.

15. $0.145\pi\ \text{m}^3/\text{year}$

16. $\dfrac{2}{\pi}$ cm/min

17. $\dfrac{\sqrt{3}}{4}$ m/min

18. 144 m/min

19. 62.8 km/h

20. a. $\dfrac{4}{5\pi}$ cm/s

b. $\dfrac{8}{25\pi}$ cm/s

21. a. $x^2 + y^2 = \left(\dfrac{l}{2}\right)^2$

b. $\dfrac{y^2}{k^2} + \dfrac{y^2}{(l - k)^2} = 1$

The Natural Logarithm and its Derivative, p. 575

1. A natural logarithm has base e; a common logarithm has base 10.

2. Since $e = \lim_{h\to 0}(1 + h)^{\frac{1}{h}}$, let $h = \frac{1}{n}$.
Therefore,
$e = \lim_{\frac{1}{n}\to 0}\left(1 + \dfrac{1}{n}\right)^n$.
But as $\dfrac{1}{n} \to 0$, $n \to \infty$.
Therefore, $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$.
If $n = 100$, $e \doteq \left(1 + \dfrac{1}{100}\right)^{100}$
$= 1.01^{100}$
$\doteq 2.704\ 81$
Try $n = 100\ 000$, etc.

3. a. $\dfrac{5}{5x + 8}$

b. $\dfrac{2x}{x^2 + 1}$

c. $\dfrac{15}{t}$

d. $\dfrac{1}{2(x + 1)}$

e. $\dfrac{3t^2 - 4t}{t^3 - 2t^2 + 5}$

f. $\dfrac{2z + 3}{2(z^2 + 3z)}$

4. a. $\ln x + 1$

b. 1

c. $e^t \ln t + \dfrac{e^t}{t}$

d. $\dfrac{-ze^{-z}}{e^{-z} + ze^{-z}}$

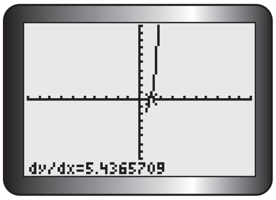
e. $\dfrac{te^t \ln t - e^t}{t(\ln t)^2}$

f. $\dfrac{1}{2}e^{\sqrt{u}}\left(\dfrac{1}{2}e^{\sqrt{u}} \ln u + \dfrac{1}{u}\right)$

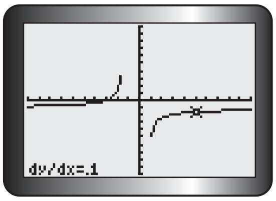
5. a. $2e$

b. 0.1

c.



The value shown is approximately $2e$, which matches the calculation in part a.



This value matches the calculation in part b.

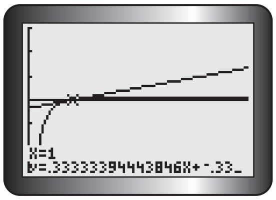
6. a. $x = 0$

b. no solution

c. $x = 0, \pm\sqrt{e - 1}$

7. a. $x - 3y - 1 = 0$

b.

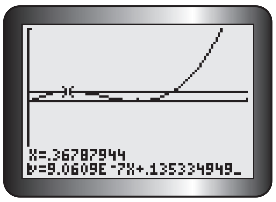


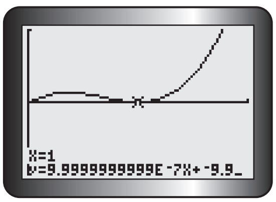
c. The equation on the calculator is in a different form, but is equivalent to the equation in part a.

8. $x - 2y + (2 \ln 2 - 4) = 0$

9. a. $\left(\dfrac{1}{e}, \dfrac{1}{e^2}\right)$ and $(1, 0)$

b.





c. The solution in part a is more precise and efficient.

10. $y = -\frac{1}{2}x + \ln 2$

11. a. 90 km/h

b. $\dfrac{-90}{3t + 1}$

c. about -12.8 km/h/s

d. 6.36 s

12. $\dfrac{1}{2}$

13. a. $\dfrac{1}{x \ln x}$

b. The function's domain is $\{x \in \mathbf{R} | x > 1\}$.
The domain of the derivative is $\{x \in \mathbf{R} | x > 0 \text{ and } x \neq 1\}$.

The Derivatives of General Logarithmic Functions, p. 578

1. a. $\dfrac{1}{x \ln 5}$

b. $\dfrac{1}{x \ln 3}$

c. $\dfrac{2}{x \ln 4}$

d. $\dfrac{-3}{x \ln 7}$

e. $\dfrac{-1}{x \ln 10}$

f. $\dfrac{3}{x \ln 6}$

2. a. $\dfrac{1}{(x + 2) \ln 3}$

b. $\dfrac{1}{x \ln 8}$

c. $\dfrac{-6}{(2x + 3) \ln 3}$

d. $\dfrac{-2}{(5 - 2x) \ln 10}$

e. $\dfrac{2}{(2x + 6)\ln 8} = \dfrac{1}{(x + 3)\ln 8}$

f. $\dfrac{2x + 1}{(x^2 + x + 1)\ln 7}$

3. a. $\dfrac{5}{52 \ln 2}$

b. $\dfrac{1}{8 \log_2(8)(\ln 3)(\ln 2)}$

4. a. $\dfrac{2}{(1 - x^2)\ln 10}$

b. $\dfrac{2x + 3}{2(x^2 + 3x)\ln(2)}$

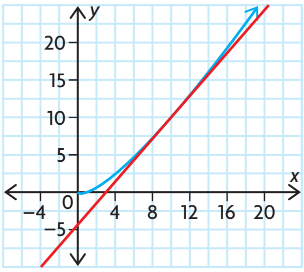
c. $\dfrac{2 \ln 5 - \ln 4}{\ln 3}$

d. $\dfrac{x \ln 3(3^x)(\ln x) + 3^x}{x \ln 3}$

e. $\frac{\ln x + 1}{\ln 2}$

f. $\frac{4x + 1 - x\ln(3x^2)}{2x\ln 5(x+1)^{\frac{3}{2}}}$

5. $y = 1.434x - 4.343$



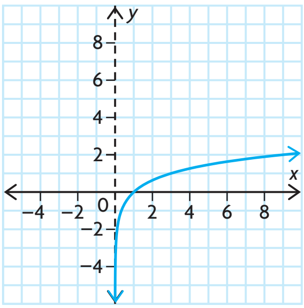
6. $y = \log_a kx$

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)\ln(a)} = \frac{k}{kx\ln(a)} = \frac{1}{x\ln(a)}$$

7. $y = 49.1x - 235.5$

8. Since the derivative is positive at $t = 15$, the distance is increasing at that point.

9. a. $y = 0.1x + 1.1$

b. 

vertical asymptote at $x = 0$

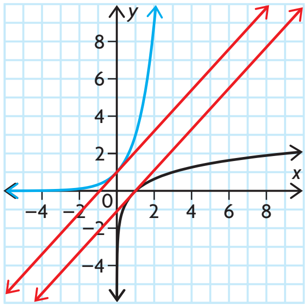
c. The tangent line will intersect this asymptote because it is defined for $x = 0$.

10. $D = \{x \in \mathbf{R} | x < -2 \text{ or } x > 2\}$; critical number at $x = 0$, $x = 2$, and $x = -2$; function is decreasing for $x < -2$ and increasing for $x > 2$

11. a. point of inflection at $x = 0$

b. $x = 0$ is a possible point of inflection. Since the graph is always concave up, there is no point of inflection.

12. The slope of $y = \log_3 x$ at $(1, 0)$ is $\frac{1}{\ln 3}$. Since $\ln 3 > 1$, the slope of $y = 3^x$ at $(0, 1)$ is greater than the slope of $y = \log_3 x$ at $(1, 0)$.



Logarithmic Differentiation, p. 582

1. a. $\sqrt{10}x^{\sqrt{10}-1}$

b. $15\sqrt{2}\,x^{3\sqrt{2}-1}$

c. $\pi t^{\pi-1}$

d. $ex^{e-1} + e^x$

2. a. $\frac{2x^{\ln x}\ln x}{x}$

b. $\frac{(x+1)(x-3)^3}{(x+2)^3} \times \left(\frac{1}{x+1} + \frac{2}{x-3} - \frac{3}{x+2}\right)$

c. $\left(x^{\sqrt{x}}\right)\frac{\ln x + 2}{2\sqrt{x}}$

d. $\left(\frac{1}{t}\right)^t\left(\ln\frac{1}{t} - 1\right)$

3. a. $2e^e$

b. $e^2 + e \cdot 2^{e-1}$

c. $-\frac{4}{27}$

4. $y = 32(2\ln 2 + 1)(x - 128\ln 2 - 48)$

5. $-\frac{11}{36}$

6. $(e, e^{\frac{1}{e}})$

7. $(1, 1)$ and $(2, 4 + 4\ln 2)$

8. $\frac{32(\ln 4 + 1)^2}{\ln 4 + 2}$

9. $\frac{1}{8}$

10. $\left(\frac{x\sin x}{x^2 - 1}\right)^2 \times \left(\frac{2(\sin x + x\cos x)}{x\sin x} - \frac{4x}{x^2 - 1}\right)$

11. $x^{\cos x}\left(\sin x\ln x + \frac{\cos x}{x}\right)$

12. $y = x$

13. a. $v(t) = t^{\frac{1}{t}}\left(\frac{1 - \ln t}{t^2}\right)$,

$a(t) = \frac{t^{\frac{1}{t}}}{t^4}[1 - 2\ln t + (\ln t)^2 + 2t\ln t - 3t]$

b. $t = e$; $a(e) = -e^{\frac{1}{e} - 3}$

14. Using a calculator, $e^\pi \doteq 23.14$ and $\pi^e \doteq 22.46$. So, $e^\pi > \pi^e$.

Vector Appendix

Gaussian Elimination, pp. 588–590

1. a. $\left[\begin{array}{rrr|r} 1 & 2 & -1 & -1 \\ -1 & 3 & -2 & -1 \\ 0 & 3 & -2 & -3 \end{array}\right]$

b. $\left[\begin{array}{rrr|r} 2 & 0 & -1 & 1 \\ 0 & 2 & -1 & 16 \\ -3 & 1 & 0 & 10 \end{array}\right]$

c. $\left[\begin{array}{rrr|r} 2 & -1 & -1 & -2 \\ 1 & -1 & 4 & -1 \\ -1 & -1 & 0 & 13 \end{array}\right]$

2. Answers may vary. For example:

$\left[\begin{array}{rr|r} 1 & 1.5 & 0 \\ 0 & -5.5 & 1 \end{array}\right]$

$\left[\begin{array}{rr|r} 2 & 3 & 0 \\ 0 & -5.5 & 1 \end{array}\right]$

3. Answers may vary. For example:

$\left[\begin{array}{rrr|r} 2 & 1 & 6 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -37 & 4 \end{array}\right]$

4. a. Answers may vary. For example:

$\left[\begin{array}{rrr|r} 1 & 0 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -36 & 16 \end{array}\right]$

b. $x = -\frac{22}{9}, y = -\frac{8}{9}, z = -\frac{4}{9}$

5. a. $x - 2y = -1$
$2x - 3y = 1$
$2x - y = 0$

b. $-2x - z = 0$
$x - 2y = 4$
$y + 2z = -3$

c. $-z = 0$
$x = -2$
$y + z = 0$

6. a. $x = -\frac{9}{2}, y = -3$

b. $x = 13, y = 9, z = -6$

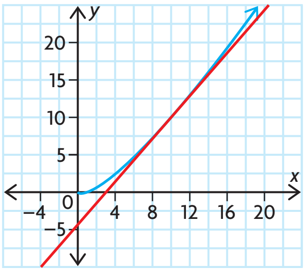
c. no solution

d. $x = -\frac{9}{4}, y = -4, z = -5$

e. $\dfrac{\ln x + 1}{\ln 2}$

f. $\dfrac{4x + 1 - x\ln(3x^2)}{2x \ln 5(x + 1)^{\frac{3}{2}}}$

5. $y = 1.434x - 4.343$



6. $y = \log_a kx$

$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)\ln(a)}$

$= \dfrac{k}{kx\ln(a)}$

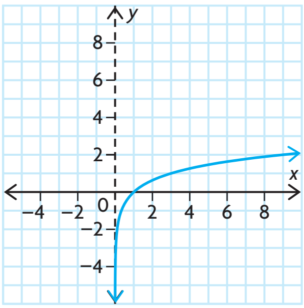
$= \dfrac{1}{x\ln(a)}$

7. $y = 49.1x - 235.5$

8. Since the derivative is positive at $t = 15$, the distance is increasing at that point.

9. a. $y = 0.1x + 1.1$

b.



vertical asymptote at $x = 0$

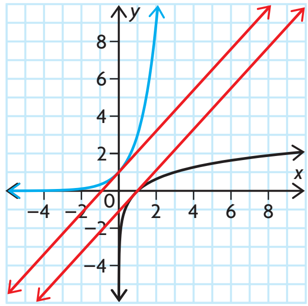
c. The tangent line will intersect this asymptote because it is defined for $x = 0$.

10. $D = \{x \in \mathbf{R} | x < -2 \text{ or } x > 2\}$; critical number at $x = 0$, $x = 2$, and $x = -2$; function is decreasing for $x < -2$ and increasing for $x > 2$

11. a. point of inflection at $x = 0$

b. $x = 0$ is a possible point of inflection. Since the graph is always concave up, there is no point of inflection.

12. The slope of $y = \log_3 x$ at $(1, 0)$ is $\frac{1}{\ln 3}$. Since $\ln 3 > 1$, the slope of $y = 3^x$ at $(0, 1)$ is greater than the slope of $y = \log_3 x$ at $(1, 0)$.



Logarithmic Differentiation, p. 582

1. a. $\sqrt{10}x^{\sqrt{10}-1}$

b. $15\sqrt{2}\,x^{3\sqrt{2}-1}$

c. $\pi t^{\pi - 1}$

d. $ex^{e-1} + e^x$

2. a. $\dfrac{2x^{\ln x}\ln x}{x}$

b. $\dfrac{(x + 1)(x - 3)^3}{(x + 2)^3} \times \left(\dfrac{1}{x + 1} + \dfrac{2}{x - 3} - \dfrac{3}{x + 2}\right)$

c. $\left(x^{\sqrt{x}}\right)\dfrac{\ln x + 2}{2\sqrt{x}}$

d. $\left(\dfrac{1}{t}\right)^t\left(\ln\dfrac{1}{t} - 1\right)$

3. a. $2e^e$

b. $e^2 + e \cdot 2^{e-1}$

c. $-\dfrac{4}{27}$

4. $y = 32(2\ln 2 + 1)(x - 128\ln 2 - 48)$

5. $-\dfrac{11}{36}$

6. $(e, e^{\frac{1}{e}})$

7. $(1, 1)$ and $(2, 4 + 4\ln 2)$

8. $\dfrac{32(\ln 4 + 1)^2}{\ln 4 + 2}$

9. $\dfrac{1}{8}$

10. $\left(\dfrac{x\sin x}{x^2 - 1}\right)^2 \times \left(\dfrac{2(\sin x + x\cos x)}{x\sin x} - \dfrac{4x}{x^2 - 1}\right)$

11. $x^{\cos x}\left(\sin x \ln x + \dfrac{\cos x}{x}\right)$

12. $y = x$

13. a. $v(t) = t^{\frac{1}{t}}\left(\dfrac{1 - \ln t}{t^2}\right)$,

$a(t) = \dfrac{t^{\frac{1}{t}}}{t^4}[1 - 2\ln t + (\ln t)^2 + 2t\ln t - 3t]$

b. $t = e$; $a(e) = -e^{\frac{1}{e} - 3}$

14. Using a calculator, $e^\pi \doteq 23.14$ and $\pi^e \doteq 22.46$. So, $e^\pi > \pi^e$.

Vector Appendix

Gaussian Elimination, pp. 588–590

1. a. $\left[\begin{array}{rrr|r} 1 & 2 & -1 & -1 \\ -1 & 3 & -2 & -1 \\ 0 & 3 & -2 & -3 \end{array}\right]$

b. $\left[\begin{array}{rrr|r} 2 & 0 & -1 & 1 \\ 0 & 2 & -1 & 16 \\ -3 & 1 & 0 & 10 \end{array}\right]$

c. $\left[\begin{array}{rrr|r} 2 & -1 & -1 & -2 \\ 1 & -1 & 4 & -1 \\ -1 & -1 & 0 & 13 \end{array}\right]$

2. Answers may vary. For example:

$\left[\begin{array}{rr|r} 1 & 1.5 & 0 \\ 0 & -5.5 & 1 \end{array}\right]$

$\left[\begin{array}{rr|r} 2 & 3 & 0 \\ 0 & -5.5 & 1 \end{array}\right]$

3. Answers may vary. For example:

$\left[\begin{array}{rrr|r} 2 & 1 & 6 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -37 & 4 \end{array}\right]$

4. a. Answers may vary. For example:

$\left[\begin{array}{rrr|r} 1 & 0 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -36 & 16 \end{array}\right]$

b. $x = -\dfrac{22}{9}, y = -\dfrac{8}{9}, z = -\dfrac{4}{9}$

5. a. $x - 2y = -1$
$2x - 3y = 1$
$2x - y = 0$

b. $-2x - z = 0$
$x - 2y = 4$
$y + 2z = -3$

c. $-z = 0$
$x = -2$
$y + z = 0$

6. a. $x = -\dfrac{9}{2}, y = -3$

b. $x = 13, y = 9, z = -6$

c. no solution

d. $x = -\dfrac{9}{4}, y = -4, z = -5$

e. $x = 2 - 3t + s, y = s,$
$z = t, s, t \in \mathbf{R}$

f. $x = 4, y = 8, z = -2$

7. a. It satisfies both properties of a matrix in row-echelon form.

1. All rows that consist entirely of zeros must be written at the bottom of the matrix.
2. In any two successive rows not consisting entirely of zeros, the first nonzero number in the lower row must occur further to the right that the first nonzero number in the row directly above.

b. A solution does not exist to this system, because the second row has no variables, but is still equal to a nonzero number, which is not possible.

c. Answers may vary. For example:

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 3 \\ -2 & 2 & 2 & 3 \\ -1 & 1 & 1 & 3 \end{array}\right]$$

8. a. no; Answers may vary. For example:

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

b. no; Answers may vary. For example:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 1 & -10 & 11 \\ 0 & 0 & 3 & 6 \end{array}\right]$$

c. no; Answers may vary. For example:

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

d. yes

9. a. i. $x = -\frac{5}{2}, y = 0, z = \frac{1}{2}$

ii. $x = -7, y = 31, z = 2$

iii. $x = 2t - 6, y = t, z = -6,$
$t \in \mathbf{R}$

iv. $x = -12 - 9t, y = -3 - 2t,$
$z = t, t \in \mathbf{R}$

b. i. The solution is the point at which the three planes meet.

ii. The solution is the point at which the three planes meet.

iii. The solution is the line at which the three planes meet.

iv. The solution is the line at which the three planes meet.

10. a. $x = -3, y = -4, z = 10$
The three planes meet at the point $(-3, -4, 10)$.

b. $x = -2t, y = t, z = t, t \in \mathbf{R}$
The three planes meet at this line.

c. $x = -1, y = 3t, z = t, t \in \mathbf{R}$
The three planes meet at this line.

d. $x = 0, y = 4, z = -2$
The three planes meet at the point $(0, 4, -2)$.

e. $x = -\frac{1}{2}, y = 2 - t, z = t, t \in \mathbf{R}$
The three planes meet at this line.

f. $x = 500, y = 1000, z = -1500$
The three planes meet at the point $(500, 1000, -1500)$.

11. $x = \frac{7a - 3c + 5b}{3},$
$y = \frac{3c - 4b - 5a}{3}, z = c - b - 2a$

12. $y = 2x^2 + 7x - 2$

13. $p = \frac{143}{9}, q = \frac{9}{121}, r = 33$

14. a. $a = -2$

b. $a = 1$

c. $a \neq -2$ or $a \neq 1$

Gauss-Jordan Method for Solving Systems of Equations, pp. 594–595

1. a. $\left[\begin{array}{cc|c} 1 & 0 & -7 \\ 0 & 1 & -2 \end{array}\right]$

b. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$

c. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

d. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -4 \end{array}\right]$

2. a. $(-7, -2)$

b. $(3, 2, 0)$

c. $(1, 1, 0)$

d. $(8, -1, -4)$

3. a. $x = -1, y = 10, z = 11$

b. $x = 3, y = 5, z = 7$

c. $x = 1, y = 2, z = -4$

d. $x = 0, y = 0, z = -1$

e. $x = -4, y = \frac{1}{3}, z = 0$

f. $x = \frac{1}{2}, y = \frac{1}{3}, z = \frac{1}{6}$

4. a. $x = -1, y = 2, z = 6$

b. $x = 38, y = 82, z = 14$

5. a. $k = 3$

b. $k \neq 3, k \in \mathbf{R}$

c. The matrix cannot be put in reduced row-echelon form.

6. a. Every homogeneous system has at least one solution, because (0, 0, 0) satisfies each equation.

b. $\left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

The reduced row-echelon form shows that the intersection of these planes is a line that goes through the point $(0, 0, 0)$.

$x = -\frac{2}{3}t, y = -\frac{1}{3}t, z = t, t \in \mathbf{R}$

7. (2, 3, 6)

Index

Q

R

S

T

U

V

X

Y

Z