5.1 Exponent Rules

Multiplication of Powers

Q1. 5^3 expanded is ____________________
5^4 expanded is ____________________

Thus, 5^3 x 5^4 =________________ x ____________________ =

Rule: To multiply powers with the ___________ ___________, you must _____________ the exponents together and _____________ the base the same.

In General: $\left(x^a\right)\left(x^b\right) = x^{a+b}$

Q2. Use this rule to simplify each of the following.

a) 8^{20} x 8^9 =

b) 5^7 x 5^{-2} =

c) 3^{-6} x 3^7 =

d) $(2^5)(2^4)(2^6)$ =

e) $(5^4)(5^4)(5^5)$ =

f) $(6^{-5})(6^4)(6^7)$ =

Divisions of Powers

Q3. 5^6 expanded is ______________________
5^4 expanded is ___________________________

Thus, $5^6 \div 5^4$ = _________________ =

Rule: To divide powers with the _____________ ___________, you must _____________ the exponents and _______________ the base the same.

In General: $\left(x^a\right) \div \left(x^b\right) = x^{a-b}$

Q4. Use the division rule to simplify each of the following.

a) $8^{20} \div 8^9$ =

b) $\frac{x^9}{x^6}$ =

c) $\frac{9^7}{9^{-3}}$ =

d) $3^{16} \div 3^7$ =

e) n) $c^5d^3 \div c^2d$ =

f) $\frac{a^{-3}b^8}{a^{-5}b^4}$ =

Power of a Power

Q5. Expand $(5^2)^3$

Rule: To raise a power to an exponent, ________________ the exponents and ________________ the base the same.

In General: $(x^a)^b = x^{ab}$

Q6. Use this rule to simplify the following.

a) $(2^3)^5 =$

b) $(3^3)^4 =$

c) $(x^2y^3)^2 =$

d) $(2m^4)^3 =$

When using the exponent rules to simplify more complicated expressions, use the following order:

1. Power to a power rule
2. Multiplication rule
3. Division rule

Q7. Simplify

a] $(2^3x^2\text{y})(3x^4y^2)^2$

b] $\dfrac{(a^2b^3)(a^5b^4)}{(a^4b)^{-3}}$

Negative and Zero Exponents

General Rule 1:

$$x^0 = \qquad x^{-a} =$$

General Rule 2:

$$\left(\frac{x}{y}\right)^{-m} = \left(\frac{y}{x}\right)^{m} = \frac{y^m}{x^m}$$

Q8. Write as a single positive exponent.

a) 2^{-1} b) $(-4)^{-3}$ c) b^{-4} d) $(6n)^{-2}$

Q9. Simplify or evaluate if possible.

a) $(-2)^{-4} + 4^{-2}$ b) $(2u^3v^{-2})^{-3}$ c) $\frac{a^2b^{-3}}{a^{-1}b^2}$ d) $\left(\frac{-27}{8}\right)^{-2}$

e) $\frac{3^{-1} + 2^{-2}}{3^{-2} - 2^{-1}}$

5.2 Rational Exponents

Warm-up:

Q1. Write as a power with a positive exponent:

a) 2^{-3}

b) $4^{3} \div 4^{5}$

c) $\left(15^{-2}\right)^{8}$

d) $\dfrac{(-2)^{7} \bullet (-2)^{5}}{\left[(-2)^{3}\right]^{5}}$

Fractional exponents can be written as roots (or radicals). EX. $\sqrt[3]{4^2} = 4^{\frac{2}{3}}$

Radical Form Exponential form

Q2. Evaluate each <u>square</u> root: a) $\sqrt{36}$ b) $\sqrt{121}$

=

Q3. Evaluate each <u>cube</u> root: a) $\sqrt[3]{8}$ b) $\sqrt[3]{27}$

=

Q4. Evaluate each <u>fourth</u> root: a) $\sqrt[4]{16}$ b) $\sqrt[4]{81}$

=

In question 2 a) we say that the __________root of ______ is ______. (6 x 6)

In question 3 a) we say that the __________root of ______ is ______. (2 x 2 x 2)

In question 4 a) we say that the __________root of ______ is ______. (2 x 2 x 2 x 2)

Q5. Evaluating Radicals

Evaluate, if possible. If not possible, explain why.

a) $\sqrt[3]{64}$ b) $\sqrt[4]{625}$ c) $\sqrt[3]{-8}$ d) $\sqrt[4]{-16}$

Q6. Evaluate a Power with a Rational Exponent on the Form

$$x^{\frac{m}{n}} = \sqrt[n]{x}^{\,m}$$

Write in radical form. Then, evaluate, if possible. If not possible, explain why.

a) $64^{\frac{1}{3}}$

b) $(-32)^{\frac{1}{5}}$

c) $(-36)^{\frac{1}{2}}$

d) $27^{\frac{2}{3}}$

e) $16^{\frac{5}{4}}$

Q7. Evaluate.

a) $169^{\frac{1}{2}}$

b) $\left(\frac{243}{32}\right)^{\frac{1}{5}}$

c) $81^{\frac{3}{4}}$

d) $\left(\frac{1}{25}\right)^{\frac{-3}{2}}$

e) $(-27)^{\frac{-4}{3}}$

Q8. Simplify. Express your answers using only positive exponents.

a) $\dfrac{(x^{\frac{2}{3}})(x^{\frac{2}{3}})}{x^{\frac{1}{3}}}$

b) $\left(y^{\frac{1}{4}}\right)^{2}$ x $\left(y^{\frac{-1}{3}}\right)^{2}$

c) $\left(5x^{\frac{1}{2}}\right)^{2}$ x $\left(4x^{\frac{-1}{2}}\right)$

5.3 Solving Exponential Equations (Using Common Bases)

Warm-Up

Q1. Write as a single power.

a) $(3^2)(3^5)$ b) $(3^4)^5$ c) $(3^2)(3^x)$

d) $(3^{4x})(3^{5x})$ e) $(3^{4x+2})(3^{5x-7})$ f) $(3^{4x+2})^5$

Solving Exponential Equations

Q2. Determine the value(s) for x that make Left Side = Right Side

a) $7^x = 7^{-2}$ b) $3^{x-1} = 3^{-2}$ c) $2^{2x-3} = 2^5$

Property: If two powers with the same base are equal, then the exponents are equal.

To solve an exponential equation:

- rewrite each side of the equation as a single power with the same base (using exponent laws)
- equate the exponents (set the exponents equal to each other)
- solve the equation made with the exponents (this could be linear, quadratic, etc.)

Q3. For the questions below, rewrite each power so the bases are equal. Then, solve the equation.

a) $2^x = 32$ b) $5^{x+1} = 1$ c) $5^x = \frac{1}{125}$

d) $9^{7x-2} = 3^{2x+8}$ e) $\left(\frac{1}{9}\right)^{x-3} = 3^{x+4}$ f) $27^x = 9$

Q4. Solve and check.

a) $2^x + 7 = 15$

b) $8^{3x} = 32^{x+20}$

c) $3^x - 3^{x+1} = -18$

d) $\frac{1}{3}\left(9^x\right) = 3^{x^2}$

5.4 Properties of the Exponential Function $f(x) = b^x$

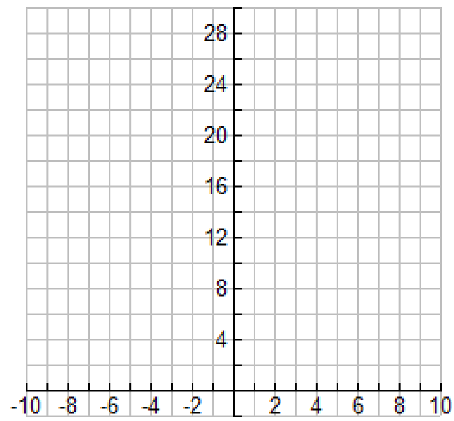
In this lesson we will learn the properties of basic exponential functions, and explore how the value of the base (b) changes the shape of an exponential function.

Q1. a) Graph the following exponential functions on the same Cartesian Plane:

i) $f(x) = 2^x$ ii) $g(x) = 3^x$

x	f(x)
-3	
-2	
-1	
0	
1	
2	
3	

x	g(x)
-3	
-2	
-1	
0	
1	
2	
3	



Domain

Range

Asymptote(s)

b) As x increases, what happens to the value of y?

c) Exponential Functions with a base greater than 1 (b>1) are called:

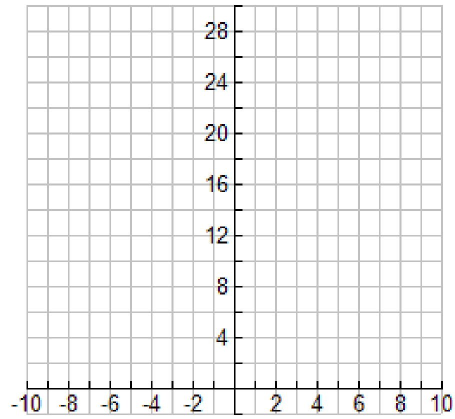
d) Describe what happens to an increasing exponential function as the base gets larger.

Q2. a) Graph the following exponential functions on the same Cartesian Plane:

i) $f(x) = \left(\frac{1}{2}\right)^x$ ii) $g(x) = \left(\frac{1}{3}\right)^x$

x	f(x)
-3	
-2	
-1	
0	
1	
2	
3	

x	g(x)
-3	
-2	
-1	
0	
1	
2	
3	



Domain

Range

Asymptote(s)

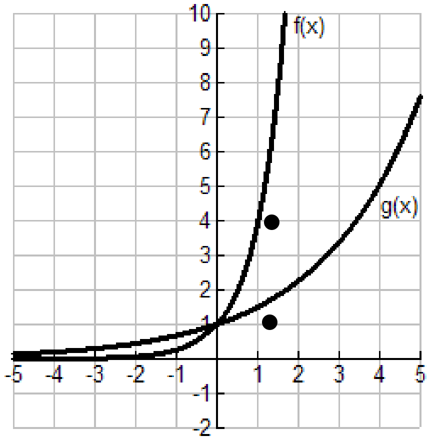
b) As x increases, what happens to the value of y?

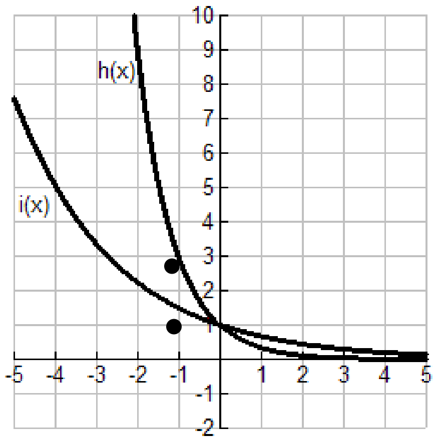
c) Exponential Functions with a base between 0 and 1 ($0 < b < 1$) are called:

d) Describe what happens to a decreasing exponential function as the base gets larger.

Q3. Determine an equation in the form $f(x) = b^x$ that represents the following exponential functions.

Problem: How do you determine the value of the base?





5.5 Transformations of Exponential Functions: $f(x) = a\left(b^{k(x-p)}\right) + q$

Warm-Up:

Complete the table of values for the function $f(x) = 2^x$, and sketch the graph on the grid provided. State the Domain, Range and Asymptote for the function f(x).

x	2^x
-1	
0	
1	
2	

Domain: ____________________

Range: ____________________

Asymptote: __________________

In this lesson we will explore how the transformations learned in the last unit affect the properties of the exponential functions.

Let's investigate each transformation individually to see how the properties of the exponential functions are affected. Let's use the function $f(x) = 2^x$ sketched above as our basic function.

For each of the six transformations on the following pages:

- Rewrite each transformation from the function notation to an explicit equation.
- Sketch the transformed function using a mapping rule.
- State the properties of the transformed function.

Function	Sketch Graph	Domain	Range	y–intercept	Asymptote
$f(x)=\frac{1}{2}(2^x)$					
$f(x)=2^{2x}$					
Function	Sketch Graph	Domain	Range	y–intercept	Asymptote

Function	Sketch Graph	Domain	Range	y–intercept	Asymptote
$f(x) = 2^x - 2$					
$f(x) = 2^{x-2}$					

$f(x) = 2^{-x}$					
$f(x) = -(2^x)$					

Summary of Transformations

The graph of $f(x) = b^x$ can be transformed to the graph of $f(x) = a(b^{k(x-p)}) + q$ where:

- ‘a’ represents a ______________________________

- ‘k’ represents a ______________________________

- ‘p’ represents a ______________________________

- ‘q’ represents a ______________________________

Mapping Rule

$f(x) = b^x \quad \Rightarrow \quad f(x) = a\left(b^{k(x-p)}\right) + q$

$(x, y) \quad \Rightarrow \quad (\qquad , \qquad)$

When graphing exponential functions using transformations:

1. determine the y-intercept and 3 other points (2 increasing and 1 decreasing value) on the basic exponential function (look at the base & reflections to determine if it’s increasing / decreasing).
2. use mapping rules to determine the position of the asymptote (look at the vertical shift)
3. use mapping rules to transform the 4 points

Practice Questions

For each of the following functions:

- start with the basic function $f(x) = 3^x$.
- identify the transformations and write a mapping rule
- sketch its graph with key points shown
- state the main properties (domain, range, y-intercept, asymptote, increasing/decreasing)

a) $f(x) = 3^{x+2} - 3$

b) $f(x) = -3^{-x}$

5.6 Exponential Growth and Decay Problems (Part 1)

There are many real-world situations that can be modeled with exponential functions.
They fall into two categories: exponential growth, and exponential decay.

Exponential growth/decay occurs when quantities increase/decrease at a rate proportional to the quantity present. That is, the current value can be multiplied by a fixed rate to get to the next value. These types of situations occur in some financial investments, the size of populations over time, the decay of radioactive chemicals, growth of bacteria cultures, and even the body's ability to break down many drugs.

Exponential growth/decay problems have the general form $y = a(b^x)$, where:

- **a** is the initial value (at time 0),
- **b** is the rate of growth/decay per period,
- **x** is the number of periods.

Example 1: The population of a city is currently 100 000 and increases by 5% of the population every year. You want to find a function representing the population of the city a number of years from now.

a) How many people would live in the city in 1 year's time?

b) What is the ratio between the population in 1 year's time, and the current population?

c) How many people would live in the city in 2 year's time?

d) What is the ratio between the population in 2 year's, and the population in 1 year?

e) What do you notice?

f) Construct a function representing the number of people, y, living in the city after x years.

Example 2: An antique vase was purchased in the year 2000 for $8000. If the vase increases in value by 6% per year, what is its estimated value in the year 2040, to the nearest thousand dollars?

In some situations, a value may be decreasing over time. What is the 'b' value?

Example 3: A car purchased for $15000 decreases in value by 15% per year. What is the function that represents the value of the car after ***x*** years?

a) Fill in the blank: If a car's value decreases by 15%, then ______% of the value remains.

b) In one year, the car will cost:

c) In two years the car will cost:

d) in *x* years, the car will cost:

Example 4: A ball is released from 150cm above ground. With each bounce off the ground it reaches a new maximum height that is 20% lower than the previous height. How high will the ball reach after it has bounced 5 times?

Homework for 5.6

1) The population of a city is 810 000 and is increasing by approximately 4% per year. What is the population's size in four years?

2) A painting purchased in the year 1990 for $10 000 is increasing in value by 8% per year. What is it's value in the year 2010?

3) A river is stocked with 5000 salmon. The population of salmon is increasing by 7% per year.
   a. Write an expression of the size of the population in t years
   b. What will the size of the population be in 3 years? 15 years?

4) A house was bought 6 years ago for $175 000. If real-estate values have been increasing at 4% per year since then, what is the current value of the house?

5) In the early 1990's, the Canadian dollar was declining in value due to inflation at a rate of 8.3% per year. If the situation continued, what would $1 be worth after 5 years?

6) To determine whether a pancreas is functioning normally, a tracer dye is injected. A normally functioning pancreas secretes 4% of the dye each minute. A doctor injects 0.5g of the dye, and checks twenty minutes later. He finds that 0.35g of the dye remains. If the pancreas were functioning normally, how much dye should he have found?

7) An element with initial mass 100g is decaying at the rate of 12% per hour. How much is left:
   a. After 10 hours?
   b. After 1 day?

Solutions:

1) 947585

2) $46 610

3)a) $5000(1.07)^t$
   b) 6125, 13795

4)$221430.82

5)$0.65

6)0.22g

7)a) 27.9g
   b) 4.7g

5.7 Exponential Growth and Decay Problems (Part 2)

Some situations that are modelled by exponential functions are not described using percentage increases or decreases per year. Instead, they are described using proportional change for a specified time period. That is, we are given a ratio for change, and told how long it takes for that change to happen. The most common of these types of problems are doubling-time and half-life.

A convenient measure of some exponential growth problems is a doubling-time (the length of time it takes for a quantity to double in size). Note that the quantity is not doubling every year. Doubling problems can also be thought of as a 100% increase per period.

Example 5: The population of the world was 6 billion in 1999. The population is growing exponentially and doubles every 35 years.

a) Estimate the world population in the year 2050.

i) Determine the number of years from 1999 to 2050.

ii) How many doubling periods does this equal?

iii) Determine the population of the world in 2050.

b) Create an exponential function where *P(x)* is the population of the world, and *x* is the number of years that have passed since 1999.

c) When will the population be 24 billion?

i) How many years must pass to get this population size?

ii) What year is it?

The term "half-life" is used to describe the amount of time needed for an exponentially decreasing substance to decay to half of the original quantity. That means, half of the original material will be left after one "half-life" period. This can also be thought of as a 50% decrease per period.

Example 6: The half-life of focusin in the blood stream is approximately 1.5 days. Suppose someone took focusin 9 days ago.

a) How many half-life periods have passed since this person used focusin?

b) What percentage of the initial amount of focusin in the blood stream would remain?

In some problems, you may not be given the rate of growth from one period to the next. However, if you know enough information, you can determine the rate of growth.

Example 7: A population grows exponentially at a constant rate of increase.
In 1980 the population was 24 000. In 1990 the population was 29 000.

a) Determine an expression for the population *P(x)*, where *x* is the # of years since 1980.

b) Use your expression to estimate the population in the year 2020.

Example 8: Your neighbour just sold his house for $400 000. He claims that since he bought the house 50 years ago, it has doubled in value every 10 years. What was the purchase price of the house?

Homework for 5.7

1) A population is doubling in size every 30 years. If the population was 1 million in the year 2000, what would the population be in the year 2020?

2) A research assistant made 160mg of a radioactive substance, and found that there was only 20mg left after 45 hours.
   a. What is the half-life of this substance?
   b. Write a function to represent the amount of substance left after *x* hours.
   c. If the researcher knew he needed 100mg of the same substance 12 hours after he was able to make a fresh batch, how much should he initially make?

3) A town has population 8400 in the year 1990. Fifteen years later, it grew to 12500. Determine the average annual growth rate, as a percentage to 1 decimal place.

4) The population of a city was 125000 in the year 1930, but half a million in the year 1998.
   a. Determine a function to represent the population of the city, *x* years after 1930.
   b. Use your function to determine the population of the city in the year 2020.

5) The half-life of medicine A in the blood stream is 1.5 hours. Aaron takes some of medicine A at 2:00pm, and it is absorbed into the blood stream. Suppose the initial amount in Aaron's blood stream was 80mg/mL.
   a. What quantity of medicine A remains in his blood stream after 210 minutes?
   b. When will the amount of medicine A in Aaron's blood stream be equal to 5mg/mL?

Solutions

1) 1.58 million

2) a) 15 hours.
   b) $160(0.5)^{\frac{x}{15}}$
   c) 174.1 mg

3) 2.69% increase per year

4) a) $125000(1.0206)^{x}$
   b) 782986 people

5) a) 15.87mg/mL
   b) 8:00pm.