# Compound Interest and Annuities

| | |
|---|---|
| **Textbook:** section 8.3 p. 493 – 497<br>section 8.4 p. 504 – 510<br>section 8.5 p. 513 – 519<br>section 8.6 p. 523 – 529 | **Homework:** p. 498 – 499 #3 bc, 4, 5, 7, 9, 10<br>p. 511 – 512 #3, 4, 5, 6, 8, 9, 10, 14<br>p. 520 – 522 #3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 18<br>p. 530 – 531 #3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14 |

Annuity

- a series of **equal** deposits or payments made at regular intervals
  - ⇒ for example:
    - ↳ mortgage payments, paying off a car loan, receiving a pension, depositing $50/month to save for a car, etc.
- a payment period is the time between successive payments
  - ⇒ the payment interval may be any length of time

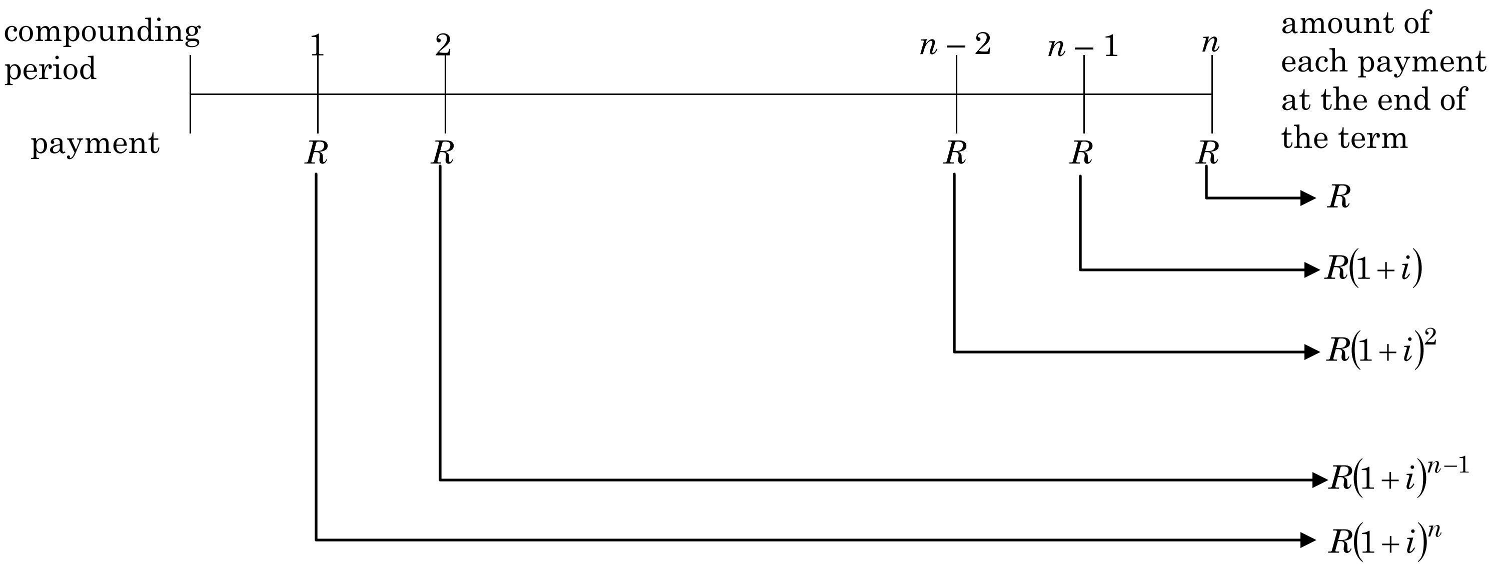
Future Value of an Annuity (FV)

- the sum of all the future values for each regular payment
- the value at the end of the term of an annuity
- the value of the set of payments at the end of the term (including the compound interest)

Present Value of an Annuity (PV)

- the amount of money that must be invested today at a given interest rate in order to make the regular payments
  - ⇒ also called Discounted Value of an Annuity
- it is the value at the beginning of the term of an annuity
- it is equal to the sum of all the present value of the payments

Determine the total value (future value) of \$$R$ invested at $i$% for $n$ compounding periods.




Amount of an annuity ($FV$) = sum of all the payments with their accumulated interest

↳ Geometric series with $a = R$, $r = 1 + i$

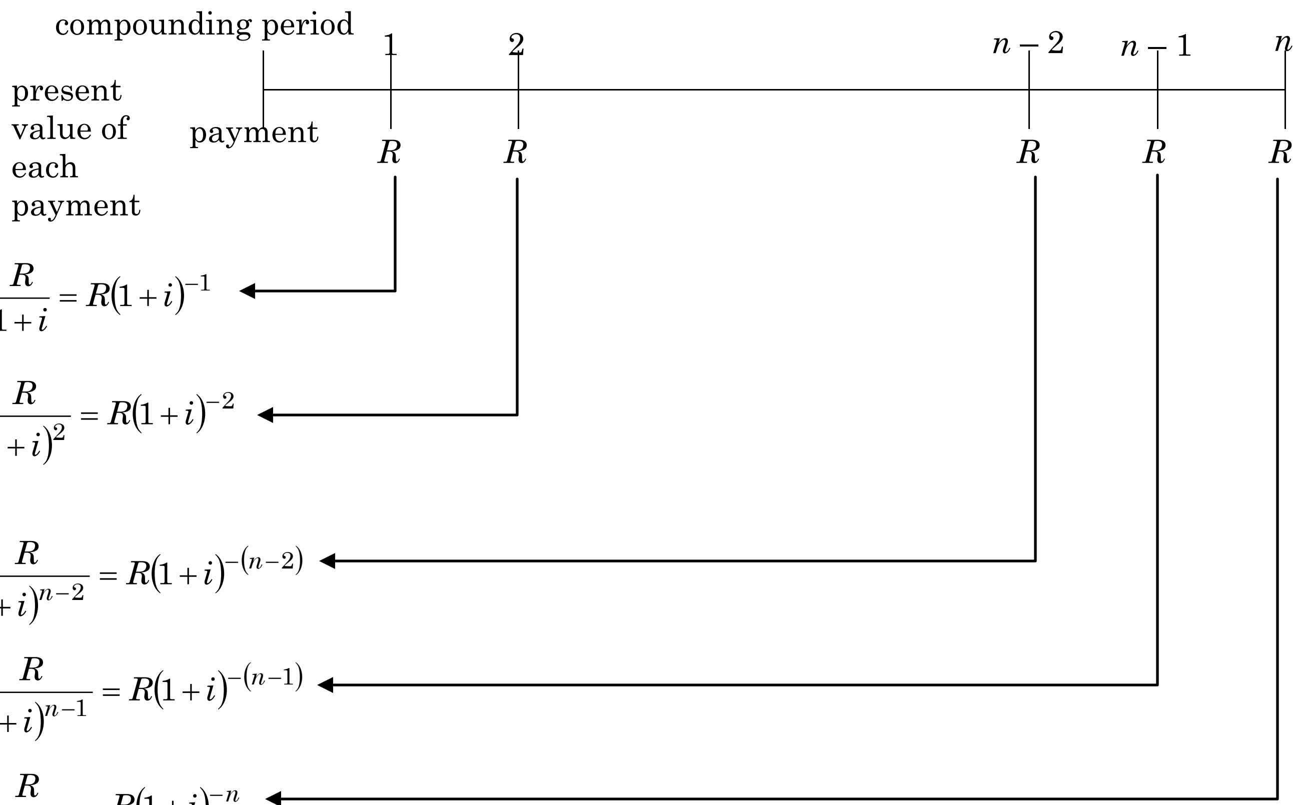
Future Value ($FV$) of an annuity is:

$$FV = \frac{a(r^n - 1)}{r - 1} = \frac{R\left[(1+i)^n - 1\right]}{1 + i - 1} = \frac{R\left[(1+i)^n - 1\right]}{i}$$

where

- $FV$ is the future value of an ordinary annuity
- $R$ is the regular payment
- $i$ is the interest rate per compounding period
- $n$ is the number of compounding periods

Determine the present value of $\$R$ invested at $i\%$ for $n$ compounding periods.




$$\frac{R}{1+i} = R(1+i)^{-1}$$

$$\frac{R}{(1+i)^2} = R(1+i)^{-2}$$

$$\frac{R}{(1+i)^{n-2}} = R(1+i)^{-(n-2)}$$

$$\frac{R}{(1+i)^{n-1}} = R(1+i)^{-(n-1)}$$

$$\frac{R}{(1+i)^{n}} = R(1+i)^{-n}$$

Present value of an annuity ($PV$) = sum of all the present values of the payments

↳ Geometric series with $a = R(1+i)^{-1}$, $r = (1+i)^{-1}$

Present Value ($PV$) of an annuity is:

$$\begin{aligned} PV &= \frac{a(r^n - 1)}{r - 1} \\ &= \frac{R(1+i)^{-1}\left[(1+i)^{-n} - 1\right]}{(1+i)^{-1} - 1} \\ &= \frac{R\left[(1+i)^{-n} - 1\right]}{(1+i)\left[(1+i)^{-1} - 1\right]} \\ &= \frac{R\left[(1+i)^{-n} - 1\right]}{1 - (1+i)} \\ &= \frac{R\left[(1+i)^{-n} - 1\right]}{1 - 1 - i} \\ &= \frac{R\left[(1+i)^{-n} - 1\right]}{-i} \\ &= \frac{R\left[1 - (1+i)^{-n}\right]}{i} \end{aligned}$$

where
- $PV$ is the present value of an ordinary annuity
- $R$ is the regular payment
- $i$ is the interest rate per compounding period
- $n$ is the number of compounding periods

Example 1: Determine the future value of 20 annual deposits of $1 000 if the deposits earn 8%/a compounded annually.

$$FV = \frac{R\left[(1+i)^n - 1\right]}{i}$$
$$= \frac{1000\left[(1+0.08)^{20} - 1\right]}{0.08}$$
$$\doteq 45761.96$$

$\therefore$ The future value of the annuity is approximately $45 761.96.

Example 2: Determine the monthly payments required to accumulate a future value of $10 000 in four years, if the payments earn 6.5%/a compounded monthly.

$$FV = \frac{R\left[(1+i)^n - 1\right]}{i}$$
$$10000 = \frac{R\left[\left(1+\frac{0.065}{12}\right)^{48} - 1\right]}{\frac{0.065}{12}}$$
$$10000\left(\frac{0.065}{12}\right) = R\left[\left(1+\frac{0.065}{12}\right)^{48} - 1\right]$$
$$R = \frac{10000\left(\frac{0.065}{12}\right)}{\left[\left(1+\frac{0.065}{12}\right)^{48} - 1\right]}$$
$$R \doteq 182.98$$

$\therefore$ The monthly payment is approximately $182.98.

Example 3: Lea has money in an account that pays 9%/a compounded annually. In a year, she will go to college. While she is there, the annuity must provide her with four equal annual payments of $5 000 for tuition. How much must be in Lea's account right now if the first payment starts in a year?

$$PV = \frac{R\left[1 - (1+i)^{-n}\right]}{i}$$
$$= \frac{5000\left[1 - (1+0.09)^{-4}\right]}{0.09}$$
$$\doteq 16198.60$$

$\therefore$ Lea will need to have approximately $16 198.60 in her account.

Example 4: Determine the missing values.

a) What rate of interest compounded quarterly will double a principal of $7 000 in 5 years?

| |
|---|
| N = **5** |
| I% = **0** |
| PV = **–7000** |
| PMT = **0** |
| FV = **14000** |
| P/Y = **1** |
| C/Y = **4** |
| PMT = **END** |

∴ The interest rate is approximately 14.11%.

---

b) What rate of interest compounded monthly is necessary for $1 250 to grow to $5 600 in 8 years?

| |
|---|
| N = **8** |
| I% = **0** |
| PV = **–1250** |
| PMT = **0** |
| FV = **5600** |
| P/Y = **1** |
| C/Y = **12** |
| PMT = **END** |

∴ The interest rate is approximately 18.89%.

---

c) How many years will it take $20 000 to grow to $26 000 at 7.5%/a, compounded quarterly?

| |
|---|
| N = **0** |
| I% = **7.5** |
| PV = **–20000** |
| PMT = **0** |
| FV = **26000** |
| P/Y = **1** |
| C/Y = **4** |
| PMT = **END** |

$\because N \doteq 3.5$

∴ It will take approximately 3 years and 6 months for the money to grow.

---

d) How much money should be invested to obtain $50 000 invested for 18 years at 6.3%/a compounded monthly?

| |
|---|
| N = **18** |
| I% = **6.3** |
| PV = **0** |
| PMT = **0** |
| FV = **50000** |
| P/Y = **1** |
| C/Y = **12** |
| PMT = **END** |

∴ Approximately $16 134.98 needs to be invested to obtain $50 000 in 18 years.

Example 5: a) Allison deposits \$100 at the end of each month in a savings account that earns 3%/a compounded monthly. What will her savings be at the end of two years?

- ⇒ since equal deposits, annuity question
- ⇒ since saving money, assume present value (PV) of the account (which is the starting balance) is 0
- ⇒ the savings at the end of two years is the future value (FV) of the annuity

N = **24**
I% = **3**
PV = **0**
PMT = **–100**
FV = **0**
P/Y = **12**
C/Y = **12**
PMT: **END**

∴ Allison will have approximately \$2 470.28 in her savings account.

b) How much interest did Allison earn?

$$\begin{aligned} I &= FV - N(PMT) \\ &\doteq 2470.28 - (24)(100) \\ &= 2470.28 - 2400 \\ &= 70.28 \end{aligned}$$

∴ Allison earned approximately \$70.28 in interest.

Example 6: a) Megan will receive an annuity from her parents to attend college. The annuity will allow Megan to withdraw \$2 700 at the end of every six months for four years. The annuity pays 6.5%/a compounded semi-annually. How much money should Megan's parents invest? The first withdrawal will be made six months from now when Megan starts college.

- ⇒ since equal withdrawals, annuity question
- ⇒ the present value (PV) of the account is the initial amount invested
- ⇒ since withdrawing money, assume the final balance of the account (FV) is 0

N = **8**
I% = **6.5**
PV = **0**
PMT = **2700**
FV = **0**
P/Y = **2**
C/Y = **2**
PMT: **END**

∴ Megan's parents will have to invest approximately \$18 754.87.

b) How much are they saving by using an annuity rather than giving Megan \$2 700 every six months for four years?

$$\begin{aligned} \text{amount} &= 2(4)(2700) \\ &= 21600 \end{aligned}$$

$$\begin{aligned} \text{savings} &\doteq 21600 - 18754.87 \\ &= 2845.13 \end{aligned}$$

∴ Megan's parents are saving approximately \$2 845.13, because of the interest on the annuity.

Example 7: Teagan's parents want to invest money for her university education. They open an account that earns 6.3%/a compounded monthly. They want to save $50 000 by Teagan's 18th birthday. How much will they need to invest on the day Teagan's born to achieve their goal?

N = **18**
I% = **6.3**
PV = **?**
PMT = **0**
FV = **50000**
P/Y = **1**
C/Y = **12**
PMT: **END**

$\therefore PV \doteq 16134.98$

$\therefore$ Teagan's parents will have to invest approximately $16 134.98 in order for her to have $50 000 on her 18th birthday.

---

Example 8: Bruce plans to save $2 000 in two years. What must his regular quarterly deposits be into his savings account that earns 6%/a compounded quarterly?

- ⇒ since regular deposit, annuity question
- ⇒ since saving money, assume present value (PV) of the account (which is the starting balance) is 0
- ⇒ the amount that Bruce wants to save up to is the future value (FV) of the annuity

N = **8**
I% = **6**
PV = **0**
PMT = **0**
FV = **2000**
P/Y = **4**
C/Y = **4**
PMT: **END**

$\because PMT \doteq 237.17$
$\therefore$ Bruce's quarterly deposits should be $237.17.

---

Example 9: Samantha bought a computer one year ago and, at that time, she made a down-payment of $1 000 cash. Today she will make the second and final payment of $745.89 (which includes the interest charge on the balance owing). She financed this purchase at 6.4%/a compounded quarterly. Determine the purchase price of the computer.

- ⇒ since not equal payments, non-annuity question
- ⇒ the final amount paid is the future value (FV) of the loan

N = **1**
I% = **6.4**
PV = **0**
PMT = **0**
FV = **–745.89**
P/Y = **1**
C/Y = **4**
PMT: **END**

$\because PV \doteq 700$

$\therefore$ The computer cost a total of $1 700.

Example 10: A small business has borrowed \$10 000 and must begin to repay it today. The payments are to be made at the end of every six months over a five year term. Interest on the loan is 9%/a compounded semi-annually. What will be the size of each regular payment? What was the total value of the loan? How much interest was paid on the loan?

- since we know the amount that the business started with, this is a **PRESENT VALUE** situation

⇒ since regular payments, annuity question
⇒ the amount of the loan is the present value (PV) of the annuity
⇒ since paying off a loan, assume the final amount owing after the last payment (FV) is 0

N = **10**
I% = **9**
PV = **10000**
PMT = **0**
FV = **0**
P/Y = **2**
C/Y = **2**
PMT: **END**

$\because PMT \doteq 1263.79$
$\therefore$The regular payments should be \$1 263.79.

$$\begin{aligned}\text{Total Amount} &= (PMT)(N)\\ &\doteq (1263.79)(10)\\ &= 12637.90\end{aligned}$$

$\therefore$ The total value of the loan is approximately \$12 637.90.

$$\begin{aligned}\text{Total Interest} &= \text{Total Amount} - PV\\ &\doteq 12637.90 - 10000\\ &= 2637.90\end{aligned}$$

$\therefore$ The amount of interest on the loan is approximately \$2 637.90.

---

Example 11: How much must I deposit at the end of every month in order to have saved \$20 000 at the end of four years? The first deposit will be in one month and the rate of interest is 9%/a compounded monthly.

⇒ since regular deposits, annuity question
⇒ since saving money, assume present value (PV) of the account (which is the starting balance) is 0
⇒ the final amount saved is the future value (FV) of the investment

N = **48**
I% = **9**
PV = **0**
PMT = **0**
FV = **20000**
P/Y = **12**
C/Y = **12**
PMT: **END**

$\because PMT \doteq 347.70$

$\therefore$ I must deposit \$347.70 each month.

Example 12: To provide an annual scholarship for 25 years, a donation of $50 000 is invested in an account for a scholarship that will start a year after the investment is made. If the money is invested at 5.5%/a compounded annually, how much is each scholarship?

- ⇒ since equal scholarships, annuity question
- ⇒ the initial amount invested is the present value (PV) of the annuity
- ⇒ since withdrawing money, assume the final balance of the account (FV) is 0

| |
|---|
| N = **25** |
| I% = **5.5** |
| PV = **–50000** |
| PMT = **0** |
| FV = **0** |
| P/Y = **1** |
| C/Y = **1** |
| PMT: **END** |

$\because PMT \doteq 3727.47$

$\therefore$ The scholarships are $3 727.47 each.

---

Example 13: $100 is deposited every week for 1.5 years at a rate of 6%/a compounded quarterly. Determine the amount of the annuity on the date of the last payment.

- ⇒ since equal deposits, annuity question
- ⇒ since saving money, assume present value (PV) of the account (which is the starting balance) is 0
- ⇒ the amount of the annuity is the future value (FV) of the annuity

| |
|---|
| N = **78 (1.5 • 52)** |
| I% = **6** |
| PV = **0** |
| PMT = **–100** |
| FV = **0** |
| P/Y = **52** |
| C/Y = **4** |
| PMT: **END** |

$\because FV \doteq 38154.33$

$\therefore$ The amount of the annuity is $8 154.33.