Applications of Vectors Unit - Notes

Tentative TEST date Mon. Mar. 4

Big idea/Learning Goals

Some of the topics in this unit overlap with physics. If you never took physics, refer to this page for key ideas you need know:

- speed is rate of change of distance
- acceleration is rate of change of velocity/speed
- the gravitational acceleration due to gravity on earth is $9.8\ m/s^2$
- Newton's first law of motion states that if an object has no net force, then its acceleration is zero. (it is at rest or moving in a straight line with constant speed)
- Newton's second law of motion states a formula: $F = ma$

Corrections for the textbook answers:

Success Criteria

☐ I understand the new topics for this unit if I can do the practice questions in the textbook/handouts

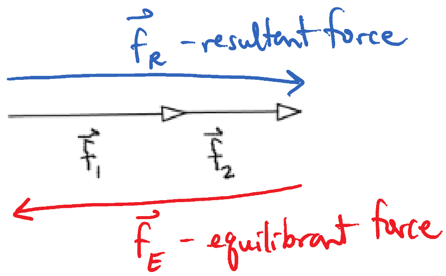
Date	pg	Topics	# of quest. done? You may be asked to show them
	2-3	Forces 7.1	
	4-5	Velocity 7.2	
	6-7	Dot Product (Geometric) 7.3	
	8-9	Dot Product (Algebraic) 7.4	
M 25	10-11	Scalar and Vector Projections 7.5	
T 26	12-14	Cross Product 7.6	
W 27	15-16	Applications of Dot and Cross Products 7.7	
Th 28		Review	

Reflect – previous TEST mark __________, Overall mark now__________.

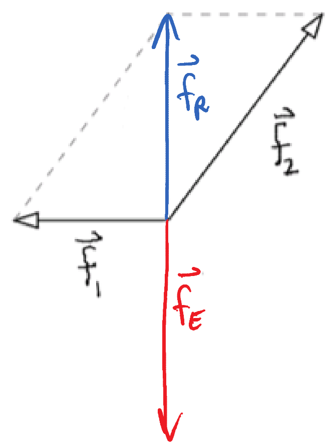
Force

1. When objects are at rest or move at a constant velocity in a straight line, then the forces that act upon the object cancel each other out. In other words, there is no NET force. The counteracting force is called the **equilibrant force**. Draw the resultant force and the corresponding equilibrant force for the following forces:

Collinear Forces at equilibrium

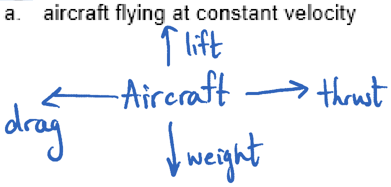


Coplanar Forces at equilibrium

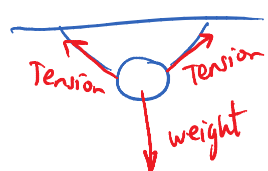


2. Describe the forces that act on the object to keep it in a state of equilibrium.

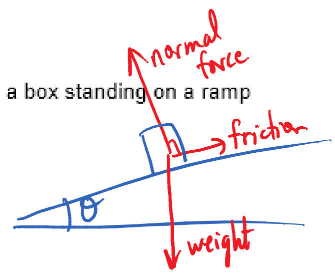
a. aircraft flying at constant velocity



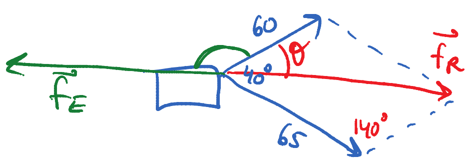
b. an object hanging on two wires



c. a box standing on a ramp



3. Jake and Maria are towing their friends on a toboggan. Jake is exerting a force of 65N and Maria a force of 60 N. Since they are walking side by side, the ropes pull to either side of the toboggan at 40° to each other.
   a. Find the resultant force pulling the toboggan forward from a stop. mag + dir!
   b. Soon the toboggan is travelling at a constant speed. Find the equilibrant force and explain what it represents.



ⓐ $|\vec{f}_R|^2 = 65^2 + 60^2 - 2(65)(60)\cos 140°$

$|\vec{f}_R| \doteq 117.5\text{ N}$

$\frac{\sin\theta}{65} = \frac{\sin 140°}{117.5}$

$\theta = 21°$

$\vec{f}_R = 117.5\text{ N}$ [21° off of 60N force]

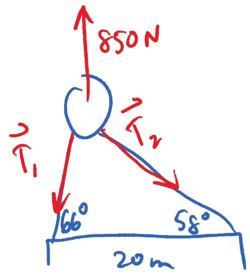
180 − 21°

ⓑ $\vec{f}_E = 117.5\text{ N}$ [159° off of 60N force]

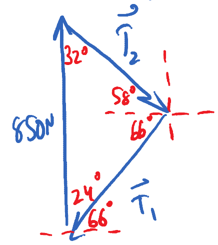
Newtons = $kg \cdot m/s^2$

4. A large balloon is tethered to the top of a building by two wires attached at points 20m apart. If the buoyant force on the balloon is 850N, and the two wires make angles or 58° and 66° with the horizontal, find the tension in each of the wires.

Real Life Pic



Free Body Diagram



$$\frac{|\vec{T}_2|}{\sin 24°} = \frac{850}{\sin 124°}$$

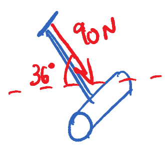
$|\vec{T}_2| \doteq 417\text{ N}$

$$\frac{|\vec{T}_1|}{\sin 32} = \frac{850}{\sin 124°}$$

$|\vec{T}_1| = 543\text{ N}$

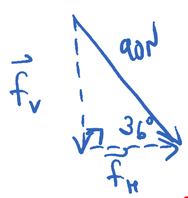
∴ the magnitudes of the tension in the wires are 417N and 543 N.

5. A lawn mower is pushed with a force of 90N directed along the handle, which makes an angle of 36° with the ground.
   a. Determine the horizontal and vertical components of the force on the mower.
   b. Describe the physical meaning of each component.



ⓐ $|\vec{f}_V| = 90\sin 36° = 53\text{N}$

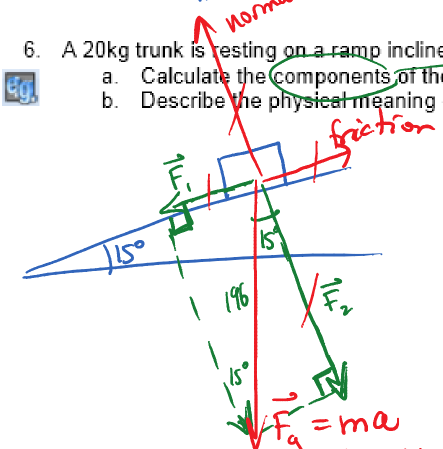
$|\vec{f}_H| = 90\cos 36° = 73\text{N}$



ⓑ $\vec{f}_H$ accelerates the lawn mower horizontally (may be counteracted by friction if go at constant speed)

$\vec{f}_V$ wasted force, counteracted by the normal force of the ground.

6. A 20kg trunk is resting on a ramp inclined at an angle of 15°.
   a. Calculate the components (magnitudes) of the force of gravity on the trunk that are parallel and perpendicular to the ramp.
   b. Describe the physical meaning of each component.



$\vec{F}_g = ma$
$= (20\text{kg})(9.8\text{ m/s}^2)$
$= 196\text{ N}$

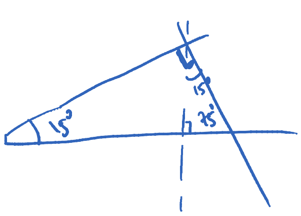
ⓐ $|\vec{F}_1| = 196\sin 15° \doteq 51\text{ N}$

SOH CAH TOA

$\sin 15° = \frac{|\vec{F}_1|}{196}$

$\cos 15° = \frac{|\vec{F}_2|}{196}$

$|\vec{F}_2| = 196\cos 15° \doteq 189\text{ N}$



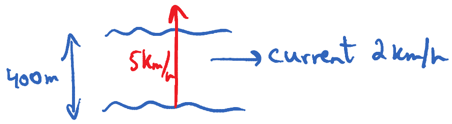
ⓑ $\vec{F}_1$ force is counteracted by frictional force

$\vec{F}_2$ is counteracted by the normal force (ramp pushes back on the box)

3

Velocity

1. Josh can paddle at a speed of 5 km/h in still water. He wishes to cross a river 400m wide that has a current of 2km/h.
   a. If he steers the canoe in a direction perpendicular to the current, determine the resultant velocity. Find the point on the opposite bank where the canoe touches.
   b. If he wishes to travel straight across the river, determine the direction he must head and the time it will take him to cross the river.



(a) $|\vec{r}| = \sqrt{5^2+2^2}$

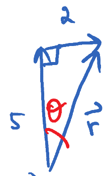
$= \sqrt{29} \sim 5.4$ km/h

$\theta = \tan^{-1}\left(\frac{2}{5}\right)$

$\theta \doteq 21.8°$

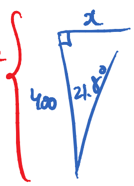
$\therefore \vec{r} = \sqrt{29}$ km/h [21.8° off of his original direction]

free Body diagram (just vectors)



speeds

distance

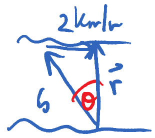


$\tan 21.8° = \frac{x}{400}$

$160\text{m} = x$

$\therefore$ Josh touches at 160m downstream from his starting pt. on the opposite side

(b)



$|\vec{r}| = \sqrt{5^2-2^2}$

$= \sqrt{21} \sim 4.6$ km/h

$\sin\theta = \frac{2}{5}$

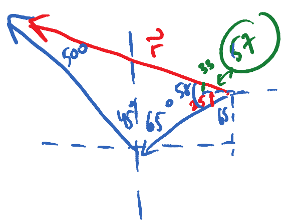
$\theta = 23.6°$

$T = \frac{D}{V} = \frac{0.4\text{ km}}{\sqrt{21}} \sim 0.087$ hr or 5.2 min

$\therefore$ he must head 23.6° up the stream off of the perpendicular to the current.

2. An airplane heading northwest at 500km/h encounters a wind of 120km/h from 65° east of north. Determine the resultant ground velocity of the plane.

N65°E



$|\vec{r}|^2 = 500^2 + 120^2 - 2(500)(120)\cos 110°$

$\boxed{|\vec{r}| = 553}$

$\frac{\sin 110}{553} = \frac{\sin\theta}{500}$

$\boxed{\theta = 58.2°}$

$\vec{r} = 553$ km/h [N56.8°W]

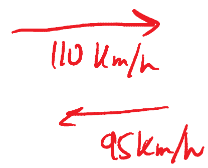
[W33.2°N]

$\vec{v}_{rel} = 95[W] - 110[E]$
$= 95[W] - -110[W] = 205[W]$

3. A car travelling east at 110km/h passes a truck going in the opposite direction at 95km/h.
   a. What is the velocity of the truck relative to the car?
   b. The truck turns onto a side road and heads northwest at the same speed. Now what is the velocity of the truck relative to the car?

Relative Velocity
relative velocity of B to A $\boxed{\vec{V}_{rel} = \vec{V}_B - \vec{V}_A}$

(a) $|\vec{V}_{rel}| = |\vec{V}_{truck}| - |\vec{V}_{car}|$
$= 95 - (-110)$
$= 205 \text{ km/h}$

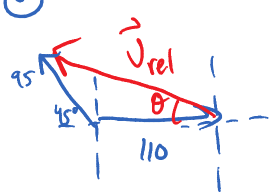


choose a positive direction as the truck
$\leftarrow +$

Since the vectors are collinear
$|\vec{V}_{truck} - \vec{V}_{car}| = |\vec{V}_{truck}| - |\vec{V}_{car}|$

p.6 Unit booklet

(b)

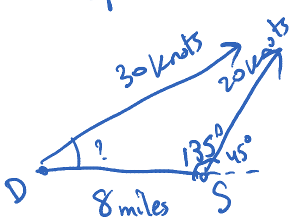


$|\vec{v}_{rel}|^2 = |\vec{v}_{truck} - \vec{v}_{car}|^2 = 95^2 + 110^2 - 2(95)(110)\cos 135°$
$|\vec{v}_{rel}| \doteq 189.5$

$\frac{\sin\theta}{95} = \frac{\sin 135°}{189.5}$ $\theta \doteq 20.8°$ $\therefore \vec{v}_{rel} = 189.5 \text{ km/h } [N69.2°W]$

4. A destroyer detects a submarine 8 nautical miles due east travelling northeast at 20knots. If the destroyer has a top speed of 30knots at what heading should it travel to intercept the submarine?

speed distance



$\frac{\sin\theta}{20} = \frac{\sin 135°}{30}$

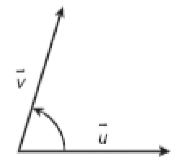
$\theta = 28.1°$

$\therefore$ should travel in direction

N 61.9° E

Two ways to multiply vectors Dot product $\vec{u}\cdot\vec{v}$ = scalar
Cross product $\vec{u}\times\vec{v}$ = vector

Dot Product (Geometric)



Dot Product with Geometric Vectors

$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$ ← angle between tails $0° \le \theta \le 180°$

1. Find the dot product of $\vec{u}\bullet\vec{v}$ for each of the following where θ is the angle between vectors.
   a. $|\vec{u}| = 7, |\vec{v}| = 12, \theta = 60°$
   b. $|\vec{a}| = 20, |\vec{b}| = 3, \theta = \frac{5\pi}{6}$
2. For above question a. find $\vec{v}\bullet\vec{u}$. What is property you can conclude from this?
3. Find $\vec{a}\bullet\vec{a}$ and $\vec{b}\bullet\vec{b}$. What can conclude from this?
4. Find $\vec{u}\bullet\vec{0}$ and $\vec{v}\bullet\vec{0}$. What can conclude from this?
5. Prove that two non-zero vectors $\vec{u}$ and $\vec{v}$ are perpendicular if and only if $\vec{u}\bullet\vec{v} = 0$. must prove both ways.
6. Explain why $(\vec{a}\bullet\vec{b})\bullet\vec{c} \ne \vec{a}\bullet(\vec{c}\bullet\vec{b})$

1ⓐ $\vec{u}\cdot\vec{v} = (7)(12)\cos 60° = 42$

ⓑ try yourself
leave exact answer $-30\sqrt{3}$

2. $\vec{v}\cdot\vec{u} = (12)(7)\cos 60° = 42$
$\therefore \vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
ie. dot product is COMMUTATIVE

3. $\vec{a}\cdot\vec{a} = (20)(20)\cos 0°$ ← angle between $\vec{a}$ and itself.
$= 400$
$\vec{b}\cdot\vec{b} = 9$ ie. $\vec{a}\cdot\vec{a} = |\vec{a}|^2$

4. $\vec{u}\cdot\vec{0} = (7)(0)\cos\theta = 0$
$\vec{v}\cdot\vec{0} = (12)(0)\cos\theta = 0$ ← always zero scalar #.

5. proof ⇒
assume $\vec{u} \perp \vec{v}$ then $\theta = 90°$
then $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos 90°$
$= |\vec{u}||\vec{v}|(0)$
$= 0$

proof ⇐
assume $\vec{u}\cdot\vec{v} = 0$
then $|\vec{u}||\vec{v}|\cos\theta = 0$
$\cos\theta = 0$ ← since $\vec{u}, \vec{v}$ are non zero vectors
$\theta = 90°$
ie. $\vec{u}$ and $\vec{v}$ have an angle of 90° between.

Dot Product Properties

1. $\vec{u}\bullet\vec{v} = 0$ iff $\vec{u} \perp \vec{v}$
2. $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ commutative
3. $\vec{u}\cdot\vec{u} = |\vec{u}|^2$
4. $\vec{u}\cdot\vec{0} = 0$ (scalar)
5. $\vec{u}\cdot(\vec{a}+\vec{b}) = \vec{u}\cdot\vec{a} + \vec{u}\cdot\vec{b}$ distributive
6. $k(\vec{u}\cdot\vec{v}) = (k\vec{u})\cdot\vec{v} = \vec{u}\cdot(k\vec{v})$

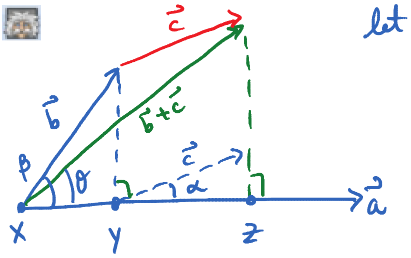
6. $\underbrace{(\vec{a}\cdot\vec{b})}_{\text{scalar \#}}\cdot\vec{c} \ne \vec{a}\cdot(\vec{c}\cdot\vec{b})$ ← ie. NOT ASSOCIATIVE!
can't do scalar # dot product with vector only regular multiplication.

Sorry ∵ MAY need extra page - if you write big to attach to these notes.

LS RS

LS RS

7. Prove the following distributive property: $\vec{a} \bullet (\vec{b}+\vec{c}) = \vec{a} \bullet \vec{b} + \vec{a} \bullet \vec{c}$



let α be angle between $\vec{a}$ and $\vec{c}$

β " " " $\vec{a}$ and $\vec{b}$

θ " " " $\vec{a}$ and $\vec{b}+\vec{c}$

RS: $\vec{a}\bullet\vec{b} + \vec{a}\bullet\vec{c}$

$= \underbrace{|\vec{a}||\vec{b}|\cos\beta}_{XY} + \underbrace{|\vec{a}||\vec{c}|\cos\alpha}_{YZ}$ by CAH rearranged

split vector $\vec{a}$ with points X, Y, Z so that it shows the altitude of the head of vectors $\vec{b}$ and $\vec{b}+\vec{c}$

$= |\vec{a}|\,XY + |\vec{a}|\,YZ$

$= |\vec{a}|(XY+YZ) = |\vec{a}|\,XZ = |\vec{a}||\vec{b}+\vec{c}|\cos\theta = \vec{a}\bullet(\vec{b}+\vec{c}) = LS$ DONE ☺

again by CAH

8. Three vectors $\vec{x}, \vec{y}$ and $\vec{z}$ satisfy $\vec{x}+\vec{y}+\vec{z}=\vec{0}$. Calculate the value of $\vec{x}\bullet\vec{y}+\vec{y}\bullet\vec{z}+\vec{z}\bullet\vec{x}$, if $|\vec{x}|=2, |\vec{y}|=3$ and $|\vec{z}|=4$

sub in three times

let A represent this scalar

then $A = \vec{x}\bullet(-\vec{x}-\vec{z}) + \vec{y}\bullet(-\vec{y}-\vec{x}) + \vec{z}\bullet(-\vec{y}-\vec{z})$

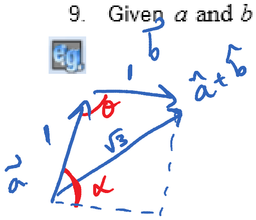
$A = -\vec{x}\bullet\vec{x} - \vec{x}\bullet\vec{z} - \vec{y}\bullet\vec{y} - \vec{x}\bullet\vec{y} - \vec{y}\bullet\vec{z} - \vec{z}\bullet\vec{z}$

$A + \underbrace{\vec{x}\bullet\vec{y} + \vec{x}\bullet\vec{z} + \vec{y}\bullet\vec{z}}_{A} = -(|\vec{x}|^2 + |\vec{y}|^2 + |\vec{z}|^2)$

$\therefore 2A = -(2^2+3^2+4^2)$

$A = -\frac{29}{2}$

9. Given $\hat{a}$ and $\hat{b}$ unit vectors, if $|\hat{a}+\hat{b}| = \sqrt{3}$ find $(2\hat{a}-5\hat{b})\bullet(\hat{b}+3\hat{a})$



α is tail to tail

$\cos\theta = \frac{1^2+1^2-(\sqrt{3})^2}{2(1)(1)}$

$\theta = 120°$

$\therefore \alpha = 60°$

FOIL

$= 2\hat{a}\bullet\hat{b} + 6\hat{a}\bullet\hat{a} - 5\hat{b}\bullet\hat{b} - 15\hat{b}\bullet\hat{a}$

$= -13\hat{a}\bullet\hat{b} + 6|\hat{a}|^2 - 5|\hat{b}|^2$

$= -13|\hat{a}||\hat{b}|\cos\alpha + 6(1)^2 - 5(1)^2$

$= -13(1)(1)\cos 60° + 1$

$= -13\left(\frac{1}{2}\right) + 1 = -\frac{11}{2}$

10. Two vectors $2\vec{a}+\vec{b}$ and $\vec{a}-3\vec{b}$ are perpendicular. Find the angle between $\vec{a}$ and $\vec{b}$, if $|\vec{a}| = 2|\vec{b}|$.

try yourself. Hint $(2\vec{a}+\vec{b})\bullet(\vec{a}-3\vec{b}) = 0$ since perpendicular

$2\vec{a}\bullet\vec{a} - 6\vec{a}\bullet\vec{b} + \vec{b}\bullet\vec{a} - 3\vec{b}\bullet\vec{b} = 0$

$2|\vec{a}|^2 - 5\vec{a}\bullet\vec{b} - 3|\vec{b}|^2 = 0$

$2|\vec{a}|^2 - 5|\vec{a}||\vec{b}|\cos\theta - 3|\vec{b}|^2 = 0$

between $\vec{a}$ and $\vec{b}$

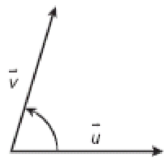
$2(2|\vec{b}|)^2 - 5(2|\vec{b}|)|\vec{b}|\cos\theta - 3|\vec{b}|^2 = 0$

$-10|\vec{b}|^2\cos\theta = -5|\vec{b}|^2$

$\cos\theta = \frac{1}{2}$

ANS: $\theta = 60°$

Dot Product (Algebraic)



Dot Product with Algebraic Vectors

$\vec{u} = (a_1, b_1, c_1)$ $\quad \vec{v} = (a_2, b_2, c_2)$

$\vec{u} \bullet \vec{v} = a_1a_2 + b_1b_2 + c_1c_2$

1. Find $(3\vec{a} + \vec{b}) \bullet (2\vec{b} - 4\vec{a})$, if $\vec{a} = -\hat{\imath} - 3\hat{\jmath} + \hat{k}$ and $\vec{b} = 2\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$.
2. Find the angle between the following vectors $\vec{u} = (-3, 1, 2)$ and $\vec{v} = (5, -4, -1)$
3. Given $\vec{a} = (2, 3, 7)$ and $\vec{b} = (-4, y, -14)$,
   a. for what value of y are the vectors collinear?
   b. for what value of y are the vectors perpendicular?

1.) $(3\vec{a} + \vec{b}) \bullet (2\vec{b} - 4\vec{a})$

$\vec{a} = (-1, -3, 1) \rightarrow 3\vec{a} = (-3, -9, 3)$

$\vec{b} = (2, 4, -5) \rightarrow 2\vec{b} = (4, 8, -10)$

$4\vec{a} = (-4, -12, 4)$

$= (-1, -5, -2) \bullet (8, 20, -14)$

$= (-1)(8) + -5(20) + -2(-14)$

$= -8 - 100 + 28$

$= -80$

2.) $\vec{u} \bullet \vec{v} = |\vec{u}||\vec{v}|\cos\theta$

$(-3, 1, 2) \bullet (5, -4, -1) = \sqrt{3^2 + 1^2 + 2^2}\sqrt{5^2 + 4^2 + 1^2}\cos\theta$

$-3(5) + 1(-4) + 2(-1) = \sqrt{14}\sqrt{42}\cos\theta$

$-21 = \sqrt{588}\cos\theta$

$\frac{-21}{\sqrt{588}} = \cos\theta$

$150^\circ = \theta$

3.) a) $y = -6$

$\vec{a}$ collinear to $\vec{b}$ if $\vec{a} = k\vec{b}$

$(2, 3, 7) = k(-4, y, -14)$

$2 = -4k$ $\quad 3 = ky$

$-\frac{1}{2} = k$ $\quad 3 = -\frac{1}{2}y$

$-6 = y$

3(b) $\vec{a} \perp \vec{b}$ iff $\vec{a} \bullet \vec{b} = 0$

$(2, 3, 7) \bullet (-4, y, -14) = 0$

$-8 + 3y - 98 = 0$

$3y = 106$

$y = \frac{106}{3}$

4. Find any vector $\vec{w}$ that is perpendicular to both $\vec{u} = 3\hat{j} + 4\hat{k}$ and $\vec{v} = 2\hat{i}$.

5. The vectors $\vec{a} = 3\hat{i} - 4\hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - 6\hat{k}$ are the diagonals of a parallelogram. Show that this parallelogram is a rhombus, and determine the lengths of the sides and the angles between the sides.

6. Find a unit vector that is parallel to the xy-plane and perpendicular to the vector $4\hat{i} - 3\hat{j} + \hat{k}$.

4. $\vec{w} = (a, b, c)$ $\quad \vec{u} = (0, 3, 4)$ $\quad \vec{v} = (2, 0, 0)$

$\vec{w} \perp \vec{u}$ then $\vec{w} \cdot \vec{u} = 0$ and $\vec{w} \cdot \vec{v} = 0 \leftarrow \vec{w} \perp \vec{v}$

$(a, b, c) \cdot (0, 3, 4) = 0$ $\qquad (a, b, c) \cdot (2, 0, 0) = 0$

$\cancel{0a} + 3b + 4c = 0$ $\qquad 2a + \cancel{0b} + \cancel{0c} = 0$

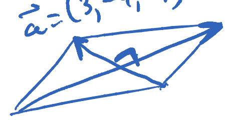
let $b = 2$ anything you wish. $\qquad \boxed{a = 0}$

then $3(2) + 4c = 0$

$c = \frac{-3}{2}$ $\qquad \therefore \vec{w} = (0, 2, -3/2)$

many other answers depends if you choose another value for b (or c)

5. $\vec{a} = (3, -4, -1)$ $\quad \vec{b} = (2, 3, -6)$



rhombus - sides are equal
- diagonals will meet at 90°

if $\vec{a} \cdot \vec{b} = 0$ then it will be a rhombus

$(3, -4, -1) \cdot (2, 3, -6) \stackrel{?}{=} 0$

$6 - 12 + 6$

0 ✓

6. let $\hat{v} = (a, b, c)$ be a unit vector

$|\hat{v}| = 1$

$\sqrt{a^2 + b^2 + \cancel{c^2}} = 1$ $\quad 0$

square.

$c = 0$ since vector is parallel to xy plane.

$\hat{v} \cdot (4, -3, 1) = 0$ for them to be $\perp$

$(a, b, 0) \cdot (4, -3, 1) = 0$

$4a - 3b = 0$ $\qquad a = \frac{3}{4}b$

$a^2 + b^2 = 1$

$\left(\frac{3}{4}b\right)^2 + b^2 = 1$

$\frac{9}{16}b^2 + \frac{1}{1}b^2 = 1$

$\frac{25}{16}b^2 = 1$

$b^2 = \frac{16}{25}$

$b = \pm\frac{4}{5}$ $\qquad a = \pm\frac{3}{5}$

$\therefore \hat{v} = \left(+\frac{3}{5}, \frac{4}{5}, 0\right)$

or

$\hat{v} = \left(-3/5, -\frac{4}{5}, 0\right)$