Chapter 4: Factoring Algebraic Expressions:
Course: MPM2D – Principles of Mathematics

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Classification of Polynomials:

You classify a polynomial by its number of terms and its degree.

Degree of the Polynomial: - is the greatest degree of any of its terms. The degree of a term is the sum of the exponents on its variables.

Example:

2abc – is a monomial, because it has one term. It has 3 variables (a, b, c). Each variable has only an exponent of one. Exponent's sum is: $(a^1 + b^1 + c^1) = (1 + 1 + 1) = 3$. The sum of the exponents is of degree 3. This is called a **third-degree** polynomial.

$7x^2 + x$ – is a binomial, because it has two terms. The greatest power of this polynomial is 2 (which is x^2).

$7k^2m + 15k^3m^2 – 6km^2$ – is a trinomial, because it has 3 terms. The greatest exponent sum is for term $(15k^3m^2)$ – (3 for the power in k, 2 for the power in m) = (3 + 2) = 5. This is a fifth-degree polynomial.

Activity 8: Classification of Polynomials.

Question 1: Classify each polynomial in terms and degrees.

Polynomial	Number of Terms	Degree of Polynomial
-3y	1	first-degree
$5 + 6a^3$	2	third-degree
$6x^2 + x - 1$	3	second-degree
$8a^4b^4 – 6a^3b^2 + 2ab^2$	3	eight-degree
$5d^3e – 7e$	2	fourth-degree
$9 + 5y^5 – 4y^2 + y$	4	fifth-degree
$8a^3b^2 + 9a^2b – 6a^4b^2$	3	Sixth-degree
$10x^7y^2 – 3x^3y^3 + 5x^4y^4$	3	ninth-degree
$6abc – 5a^2bc^2 – 7abc^2$	3	fifth-degree

Question 2: Add and Subtract Polynomials:

To add, remove the brackets and then collect like terms.
To subtract, add the opposite polynomial.

Polynomial	Answer
$(2x^2 + 3x – 5) + (7x^2 + 6x – 2)$ $= 2x^2 + 3x – 5 + 7x^2 + 6x – 2$ $= 2x^2 + 7x^2 + 3x + 6x - 2$ $= 9x^2 + 9x – 7.$	$9x^2 + 9x – 7.$
$(4a^2 + 5ab – 9b^2) – (7a^2 – 6ab + 2b^2)$ $= 4a^2 + 5ab – 9b^2 – 7a^2 + 6ab - 2b^2$ $= 4a^{2\,-}\,7a^2 + 5ab + 6ab - 2b^2$ $= -3a^2 + 11ab – 11b^2$	$-3a^2 + 11ab – 11b^2$
Simply the following:	
$(5x + 7) + (2x – 11)$ $= 5x + 2x + 7 – 11$ $= 7x - 4$	$7x - 4$
$(3b – 8) – (6b – 7)$ $=$	
$(5x^2 + 6x + 8) + (2x^2 + 5x – 9)$	
$(9y^3 – 7y^2 + 6) – (3y^3 – 5y^2 + 8)$	
$(7a^2 + 3a – 4) + (8a^2 – 2a – 15)$	
$(2c^2 – 3c + 1) – (-c^2 - 3c – 5)$	
$(7x^2 + 3xy – 2y^2) + (8x^2 – xy – y^2)$	
$(4g^2 + gh – 7h^2) – (g^2 -2gh + 3h^2)$	
$(5ab^2 + 7a – b) + (3ab^2 – 5a + 6b)$	
$(3cd^2 + 2c + 9d) – (2cd^2 + 2c – d)$	
$(2x + 8) – (6x – 7) + (5x – 1)$	
$(5a2 – b) + (6b – 2a^2) – (b^2 + 7a^2)$	

Review: Product of a Monomial and Polynomial using Distributive Property:

Distributive Property
$2(x + 3)$ $= 2(x) + 2(3)$ $= 2x + 6$
$-a(3a + 5)$ $= -a(3a) + (-a)(5)$ $= -3a^2 -5a$
$2x(x + 1)$ $= 2x(x) + 2x(1)$ $= 2x^2 + 2x$
$3(x + 2)$ $= 3(x) + 3(2)$ $= 3x + 6$
$4(x + 2)$ $= 4(x) + 2(4)$ $= 4x + 8$
$x(x + 3)$ $= x(x) + 3x$ $= x^2 + 3x$
$4x(x + 4)$ $= 4x(x) + 4(4)$ $= 4x^2 + 16x$

Multiply these polynomials	Answer
(x + 1)(x + 2)	$X^2 + 3x + 2$
(x + 2)(x + 4)	
(x + 3)(2x + 1)	
(2x + 3)(x + 1)	
(x + 3)(x + 8)	
(2x + 5)(x + 4)	
(4x + 7)(3x + 1)	
(x + 2)(x + 5)	
(x – 2)(x + 4)	
(3x + 7)(x – 5)	
Expand and simply	
-2(4x – 5)(7x – 6)	
2(x + 7)(x – 3) – (4x + 3)(2x – 1)	
Use distributive Property to find the binomial product	
(k – 3)(k – 5)	
(y – 3)(y – 4)	
(x – 2)(x - 4)	
(q – 4)(q – 2)	
(j – 7)(j – 1)	
(p – 9)(p – 3)	
(z – 7x)(z – 8x)	
(b – 3c)(b – 11c)	

Factoring: Determine the Greatest Common Factor (GCF)

12	1, 12, 6, 2, 3, 4	1, 2, 3, 4, 6, 12,
18	1, 18, 2, 9, 3, 6	1, 2, 3, 6, 18
Greatest Common Factor is: 6 for (12, 18)		
10		
24		
Greatest Common Factor is:		
16		
32		
Greatest Common Factor is:		
8		
14		
Greatest Common Factor is:		
28		
40		
Greatest Common Factor is:		
Find the Greatest Common Factors for: 6 and 9		
25 and 15		
24 and 16		
20 and 28		
36 and 15		
32 and 40		

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Topic: Factors of Polynomials:

Special Products: Expand and Simply

	Expanded	Simplified
$(x + 2)^2$	**$(x + 2)(x + 2) = x^2$** $+ 2x + 2x + 2^2$	$x^2 + 4x + 4$
$(x – 6)^2$	**$(x – 6)(x – 6) = x^2$** $– 6x – 6x + 6^2$	**x^2** $– 12x + 36$
$(x – 4)^2$	**$(x – 4)(x – 4) = x^2$** $– 4x – 4x + 4^2$	**X^2** $– 8x + 16$
$(2x + 5)^2$	**$(2x + 5)(2x + 5) = 4x^2$** $+ 10x + 10x + 5^2$	**$4x^2$** $+ 20x + 25$
$(3x – 1)^2$	**$(3x – 1)(3x – 1) = 9x^2$** $– 3x – 3x + 1^2$	**$9x^2$** $– 6x + 1$
$(2x – 5y)^2$	**$(2x – 5y)(2x – 5y) = 4x^2$** $– 10xy – 10yx + (-5y)^2$ $=$ **$4x^2$** $– 10xy - 10yx + 25y^2$ $=$ **$4x^2$** $– 20xy + 25y^2$	**$4x^2$** $– 20xy + 25y^2$

Activity 9: Expand and Simply

	Expanded	Simplified
$(x + 3)^2$		
$(x + 2)^2$		
$(x – 6)^2$		
$(x – 4)^2$		
$(2x + 5)^2$		
$(3x – 1)^2$		
$(2x – 5y)^2$		
$(4x – y)^2$		
$(a + b)^2$		
$(a - b)^2$		
$(3a + 2)^2$		
$(5m – 3)^2$		
$(4 + 2b)^2$		
$(7 – 3z)^2$		
$(2x + 3y)^2$		

Topic 2: Product of a Sum and a Difference of Two Terms:

Example: Expand and Simply

	Expanded	Simplified
$(x + 3)(x – 3)$	$X^2 - 3x + 3x - 9$	$x^2 - 9$
$(2y + 5)(2y – 5)$	$4y^2 – 10y + 10y - 25$	$4y^2 - 25$
$(x – 4)(x + 4)$	$X^2 + 4x – 4x - 16$	$X^2 - 16$
$(3k – 7)(3k + 7)$	$9k^2 + 21k – 21k - 49$	$9k^2 - 49$

In general:

$(a + b)(a – b)$
$= a^2 – ab + ba – b^2$
$= a^2 – b^2$

This is referred to difference of squares.

$a^2 – b^2 = (a + b)(a – b)$

Factor These terms below:

Difference of Squares	Factor the squares	Proof Expand the terms.
$x^2 - 4$	$x^2 – 2^2 = (x + 2)(x – 2)$	$x^2 + 2x – 2x – 4 = x^2 - 4$
$x^2 - 9$		
$4x^2 - 1$	$(2x)^2 - 1^2 = (2x + 1)(2x – 1)$	
$9x^2 - 16$		
$4x^2 – 9y^2$		
$9m^2 – 4n^2$		

Topic 3: Perfect Square trinomials (3 terms)

Example: Expand and Simply

	Expanded	Simplified
$(x + 3)^2$	$(x + 3)(x + 3)$ $= x^2 + 3x + 3x + 9$ $= x^2 + 6x + 9$	
$(x + 2)^2$	$(x + 2)(x + 2)$ $= x^2 + 2x + 2x + 4$ $= x^2 + 4x + 4$	
$(x + 4)$		
Generic Form **$(a + b)^2$**	$(a + b)(a + b)$ $= a^2 + ab + ba + b^2$ $= a^2 + 2ab + b^2$	$= a^2 + (2)ab + b^2$ $=$ **$a^2 + 2ab + b^2$**
$(x – 6)^2$	$(x – 6)(x – 6)$ $= x^2 – 6x – 6x + 36$ $= x^2 – (2)(6x) + 36$ $= x^2 – 12x + 36$	
$(2x – 4)^2$	$(2x – 4)(2x – 4)$ $=4x^2 - 8x – 8x + 16$ $=4x^2 – 16x + 16$ Quickly: $=(2x)^2 – (2)(2x)(4) + (-4)(-4)$ $=4x^2 – 16x + 16$	
Generic Form **$(a - b)^2$**	**$(a -b)(a -b)$** $= a^2 – ab – ba + b^2$ $= a^2 – 2ab + b^2$	**$= a^2 - (2)ab + b^2$** **$= a^2 - 2ab + b^2$**

In general: Perfect Squares (Trinomials)

$(a + b)^2$
$= a^2 + 2ab + b^2$

$(a - b)^2$
$= a^2 - 2ab + b^2$

Topic 4: Factoring Quadratic Expression in form of $x^2 + bx + c$, a = 1

Quadratic questions: (3 forms)

1. $y = a(x - h)^2 + k$
2. $y = a(x - r)(x - s)$
3. **$y = ax^2 + bx + c$** 🡐 studying now, where a = 1

<table>
<tr><td></td><td>Expanded</td><td>Equate</td></tr>
<tr><td>General Form:
$y = ax^2 + bx + c$, a = 1
$y = x^2 + bx + c$</td><td></td><td></td></tr>
<tr><td>$y = a(x + r)(x + s)$, a = 1
$y = (x + r)(x + s)$</td><td>$= x^2 + rx + sx + rs$
$= x^2 + (r + s)x + rs$</td><td>$x^2 + bx + c$ = $x^2 + (r + s)x + rs$

b = (r + s)
c = (r x s)</td></tr>
<tr><td colspan="3">In general: Transition for the form 2 to 3 above.
$x^2 + bx + c$ = $x^2 + (r + s)x + rs$

b = (r + s)
c = (r x s)

Examples of factor r, s, where r, s are only integers.</td></tr>
<tr><td><u>Factor, if possible</u>

$x^2 + 7x + 12$</td><td>b = 7, c = 12

$(x + r)(x + s)$
$= (x + 3)(x + 4)$
$= x^2 + 4x + 3x + 12$
$= x^2 + 7x + 12$</td><td><table><tr><td>Factors
(r, s)</td><td>Product c
c= r x s</td><td>Sum
b = r+s</td></tr><tr><td>1, 12</td><td>12</td><td>13</td></tr><tr><td>2, 6</td><td>12</td><td>8</td></tr><tr><td>3, 4</td><td>12</td><td>7</td></tr></table></td></tr>
<tr><td>$x^2 + 4x + 6$</td><td></td><td></td></tr>
<tr><td>$x^2 - 29x + 28$</td><td></td><td></td></tr>
<tr><td>$x^2 + 3x - 28$</td><td></td><td></td></tr>
</table>

<table>
<tr>
<td>$x^2 - 4x - 21$</td>
<td>b = -4, c = -21

r, s = (3, -7)
b = (r + s)
= (x + 3)(x – 7)
= (x – 7) (x + 3)

Proof:
= $x^2 - 7x + 3x - 21$
= $x^2 - 4x - 21$</td>
<td>
<table>
<tr><td>Factors
(r, s)</td><td>Product (c)
c= r x s</td><td>Sum (b)
b = r+s</td></tr>
<tr><td>-1, 21</td><td>-21</td><td>20</td></tr>
<tr><td>-3, 7</td><td>-21</td><td>4</td></tr>
<tr><td>1, -21</td><td>-21</td><td>20</td></tr>
<tr><td>3, -7</td><td>-21</td><td>-4</td></tr>
</table>
</td>
</tr>
</table>

Date: Monday, February 12th, 2024

Goal:

Topic 5: Factoring Quadratic Expression in form of $ax^2 + bx + c$, $a \neq 1$, but a, b, c are integers

Example:

<table>
<tr><th>Example</th><th>Factors: Products and sums</th><th></th></tr>
<tr>
<td>General form:
$ax^2 + bx + c$

$3x^2 + 8x + 4$

a = 3,
b = 8,
c = 4

need to find two integers whose product is a x c = 12, whose sum of two factors is 8</td>
<td>
<table>
<tr><th>Factors of (a x c), factors of 12</th><th>Product</th><th>Sum (b)</th></tr>
<tr><td>1, 12</td><td>12</td><td>13</td></tr>
<tr><td>2, 6</td><td>12</td><td>8</td></tr>
<tr><td>3, 4</td><td>12</td><td>7</td></tr>
</table>
</td>
<td>$= 3x^2 + 8x + 4$
$= 3x^2 + 6x + 2x + 4$
$= 3x(x + 2) + 2(x + 2)$
$= (x + 2)(3x + 2)$
$= (3x + 2)(x + 2)$

Expand to prove:
$=(3x + 2)(x + 2)$
$=3x^2 + 6x + 2x + 4$
$=3x^2 + 8x + 4$</td>
</tr>
<tr>
<td>$6x^2 - 5x + 1$

a = 6
b = -5
c = 1</td>
<td>
<table>
<tr><th>Factors (a x c) = 6</th><th>Product</th><th>Sum</th></tr>
<tr><td>1, 6</td><td>6</td><td>7</td></tr>
<tr><td>2, 3</td><td>6</td><td>5</td></tr>
<tr><td>-1, -6</td><td>6</td><td>-7</td></tr>
<tr><td>-2, -3</td><td>6</td><td>-5</td></tr>
</table>
</td>
<td>$=6x^2 - 5x + 1$
$=6x^2 - 2x - 3x + 1$
$=2x(3x - 1) - (3x - 1)$
$=(3x -1)(2x -1)$

Expand to prove:
$=(3x -1)(2x -1)$
$=6x^2 - 3x - 2x + 1$
$=6x^2 - 5x + 1$</td>
</tr>
<tr>
<td>$16x^2 + 26x - 12$

Remove common factor first (GCF) = 2
$=2(8x^2 + 13x - 6)$

a=8
b=13
c= -6

a x c = 8 x (-6) = -48</td>
<td>
<table>
<tr><th>Factors (a x c) = -48</th><th>Product</th><th>Sum b = 13</th></tr>
<tr><td>-1, 48</td><td>-48</td><td>47</td></tr>
<tr><td>-2, 24</td><td>-48</td><td>22</td></tr>
<tr><td>-3, 16</td><td>-48</td><td>13</td></tr>
<tr><td>-4, 12</td><td>-48</td><td>8</td></tr>
<tr><td>-6, 8</td><td>-48</td><td>2</td></tr>
<tr><td>1, -48</td><td>-48</td><td>-47</td></tr>
<tr><td>2, -24</td><td>-48</td><td>-22</td></tr>
<tr><td>3, -16</td><td>-48</td><td>-13</td></tr>
<tr><td>6, -8</td><td>-48</td><td>-2</td></tr>
<tr><td></td><td></td><td></td></tr>
</table>
</td>
<td>$=2(8x^2 + 13x - 6)$
$=2[8x^2 + 16x - 3x - 6]$
$=2[8x^2 - 3x + 16x - 6]$
$=2[x(8x - 3) + 2(8x - 3)]$
$=2[(8x - 3)(x + 2)]$

Expand to prove:
$=2[(8x - 3)(x + 2)]$
$=2[8x^2 + 16x - 3x - 6]$
$=2[8x^2 + 13x - 6]$</td>
</tr>
</table>

Activity 10: Factor these parabolic equations (if possible)

$2x^2 + 5x + 3$		
$3x^2 + 7x + 4$		
$6x^2 + 5x + 1$		
$6x^2 + 11x + 1$		
$2x^2 + 7x + 5$		
$6y^2 + 19y + 8$		
$12q^2 + 17q + 6$		

Summary/Key Concepts:

1. Always look for a common factor first when factoring a trinomial
2. To factor $ax^2 + bx + c$, find two integers whose product is a x c, and whose sum is b. Then break up the middle term and factor by grouping.
3. Note: Not all quadratic expressions of the form $y = ax^2 + bx + c$, can be factored over the integers.

Grand Summary of all parabolic forms.

Quadratic Expression	$x^2 + bx + c$	$a^2 – b^2$	$a^2 + 2ab + b^2$ $a^2 - 2ab + b^2$	$ax^2 + bx + c$
Factoring Technique	Find two integers, r and s, with a product of c and a sum of b. Then write: $x^2 + bx + c$ as $(x + r)(x + 2)$	Use the difference of squares pattern. $a^2 – b^2$ $= (a – b)(a + b)$	Use a perfect square trinomial pattern. $a^2 + 2ab + b^2$ $= (a + b)^2$ $a^2 - 2ab + b^2$ $= (a - b)^2$	Find two integers with a product of a x c, and a sum of b. Then break up the middle term and factor by grouping
Example;	$x^2 + 11x + 18$ b = 11, c = 18, The two integers are: 9 and 2. $x^2 + 11x + 18$ $=(x + 9)(x + 2)$	$100x^2 – 9$ $= (10x)^2 – (3)^2$ $= (10x – 3)(10x + 3)$	$x^2 + 6x + 9$ $= x^2 + (2)(3)x + 9$ $= (x + 3)^2$ $25x^2 -40x + 16$ $=(5x)^2 -2(5x)(4) + 4^2$ $=(5x – 4)^2$	$6x^2 -11x – 7$ a = 6, c= -7 a x c = -42, b = -11 The two integers are: 3, -14 $= 6x^2 -11x – 7$ $= 6x^2 +3x -14x – 7$ $=3x(2x + 1) -7(2x +1)$ $=(2x + 1)(3x -7)$