Properties of Limits

For any real number a, suppose that f and g both have limits that exist at $x = a$. **THINGS to try**

1. $\lim_{x\to a} k = k$, for any constant k ex: $\lim_{x\to 2} 5 = 5$ y=5 (horizontal line)
2. $\lim_{x\to a} x = a$ $y=x$  ex: $\lim_{x\to 2} x = 2$
3. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$ ex: $\lim_{x\to a}(x^2+x)$ $f(x)=x^2$ $g(x)=x$ $\therefore \lim_{x\to a} x^2 + \lim_{x\to a} x$
4. $\lim_{x\to a}[cf(x)] = c[\lim_{x\to a} f(x)]$, for any constant c
5. $\lim_{x\to a}[f(x)g(x)] = [\lim_{x\to a} f(x)][\lim_{x\to a} g(x)]$
6. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided that $\lim_{x\to a} g(x) \neq 0$
7. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$, for any rational number n

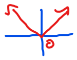
Evaluate the limits

1. $\lim_{x\to 5}\sqrt{\frac{x^2}{x-1}}$ $\frac{0}{0}$ or $\frac{\#}{0}$ $\frac{3}{0}$ ① plug $x=a$ & solve.

(use the properties above to explain why you can just substitute the value in if it is not indeterminate form.)

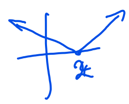
plug $x=5$

$\lim_{x\to 5}\sqrt{\frac{x^2}{x-1}} = \sqrt{\frac{5^2}{5-1}} = \sqrt{\frac{25}{4}} = \frac{5}{2}$



$|x|$ absolute function $|x| = \begin{cases} -x, & x<0 \\ x, & x>0 \end{cases}$

$|2(x-2)|$ 2 shift 2 units right



$|2x-4| = \begin{cases} -2x+4, & x<2 \text{ (left side)} \\ 2x-4, & x>2 \text{ (right side)} \end{cases}$

eg.

2. $\lim\limits_{x\to 2} \dfrac{|2x-4|}{x-2}$

Left limit:

$\lim\limits_{x\to 2^-} \dfrac{-2x+4}{x-2}$ ← factor & cancel

$= \cancel{\dfrac{-2(2)+4}{2-2}} = \dfrac{0}{0}$ indeterminate form.

$= \lim\limits_{x\to 2^-} \dfrac{-2(\cancel{x-2})}{\cancel{x-2}}$

$= \lim\limits_{x\to 2^-} -2$

$= -2$

Right limit:

$\lim\limits_{x\to 2^+} \dfrac{2x-4}{x-2}$

$= \lim\limits_{x\to 2^+} \dfrac{2(\cancel{x-2})}{\cancel{x-2}}$

$= \lim\limits_{x\to 2^+} 2$

$= 2$

$\therefore \lim\limits_{x\to 2} \dfrac{|2x-4|}{x-2} = \text{DNE}.$

3. $\lim\limits_{x\to 3} \dfrac{\frac{1}{3}-\frac{1}{x}}{x-3}$ } LCD

plug $x=3$ → $\boxed{\frac{0}{0}}$

$= \lim\limits_{x\to 3} \dfrac{\cancel{x-3}}{3x} \times \dfrac{1}{\cancel{x-3}}$

$= \lim\limits_{x\to 3} \dfrac{1}{3x}$ sub $x=3$

$= \dfrac{1}{3(3)}$

$= \dfrac{1}{9}$

4. $\lim\limits_{x\to 3} \sqrt{9-x^2}$

Sub $x = 3.0001$

$= \sqrt{9-(3.0001)^2}$

$= \sqrt{\text{negative \#}}$

$= \text{DNE}$

$\frac{0}{0}$

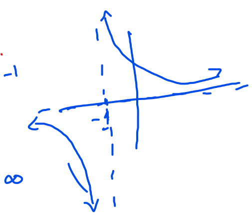
5. $\lim\limits_{x\to -1} \dfrac{x}{x^2+x}$ ← factor

sub $x=-1$ $\dfrac{-1}{1-1} = \dfrac{-1}{0}$

$= \lim\limits_{x\to -1} \dfrac{\cancel{x}}{\cancel{x}(x+1)}$

$= \lim\limits_{x\to -1} \dfrac{1}{x+1}$

V.A. ~~Sub~~ $x=-1$



$= \cancel{\dfrac{1}{-1+1}}$

$= \cancel{\dfrac{1}{0}}$

Sub $x=-1.00001$

$\lim\limits_{x\to -1^-} \dfrac{1}{x+1} = \dfrac{1}{-1.0001+1} = \dfrac{1}{-0.0001} = -\infty$

$\lim\limits_{x\to -1^+} \dfrac{1}{x+1} = \dfrac{1}{-0.9999+1} = \dfrac{1}{0.0001} = +\infty$

$\therefore \lim\limits_{x\to -1} \dfrac{x}{x^2+x} = \text{DNE}$

6. $\lim\limits_{x\to 0} \dfrac{\sqrt[2]{1+x}-1}{\sqrt[3]{1+x}-1}$ try to use a substitution that will get rid of both roots

let $u^6 = 1+x$

$x=0$ $u^6 = 1+0$, $u=1$

$\lim\limits_{u\to 1} \dfrac{\sqrt{u^6}-1}{\sqrt[3]{u^6}-1}$

Difference of Cube.

$a^3-b^3 = (a-b)(a^2+ab+b^2)$

difference of squares.

$a^2-b^2 = (a-b)(a+b)$

$= \lim\limits_{u\to 1} \dfrac{u^3-1}{u^2-1}$

$= \lim\limits_{u\to 1} \dfrac{(\cancel{u-1})(u^2+u+1)}{(\cancel{u-1})(u+1)}$

$= \lim\limits_{u\to 1} \dfrac{u^2+u+1}{u+1}$

Sub $u=1$

$= \dfrac{1+1+1}{1+1}$

$= \dfrac{3}{2}$

7. $\lim\limits_{x\to\infty} \dfrac{4-3x^3+2x}{8-x^2+7x^3}$

l.c. -3 l.c. 7

divide the highest Power of x into each term.

$= \lim\limits_{x\to\infty} \dfrac{\frac{4}{x^3} - \frac{3x^3}{x^3} + \frac{2x}{x^3}}{\frac{8}{x^3} - \frac{x^2}{x^3} + \frac{7x^3}{x^3}}$

Simplify.

$= \lim\limits_{x\to\infty} \dfrac{\frac{4}{x^3} - 3 + \frac{2}{x^2}}{\frac{8}{x^3} - \frac{1}{x} + 7}$

plug $x=\infty$

$= \dfrac{\cancel{\frac{4}{\infty^3}}^{0} - 3 + \cancel{\frac{2}{\infty^2}}^{0}}{\cancel{\frac{8}{\infty^3}}^{0} - \cancel{\frac{1}{\infty}}^{0} + 7}$

$= \dfrac{-3}{7}$ ⇒ H.A.

$(\sqrt{4+x}-2)(\sqrt{4+x}+2)$

$(a-b)(a+b) = a^2-b^2$

$\boxed{\text{Rationalize}}$ ~~numerator~~ → remove √ using difference of squares.

8. $\lim_{x\to 0}\dfrac{\sqrt{4+x}-2}{x} \times \dfrac{\sqrt{4+x}+2}{\sqrt{4+x}+2}$

$= \lim_{x\to 0}\dfrac{(\sqrt{4+x})^2-2^2}{x(\sqrt{4+x}+2)}$

$= \lim_{x\to 0}\dfrac{\cancel{4}+\cancel{x}-\cancel{4}}{\cancel{x}(\sqrt{4+x}+2)}$

$= \lim_{x\to 0}\dfrac{1}{\sqrt{4+x}+2}$ sub $x=0$

$= \dfrac{1}{\sqrt{4+0}+2}$

$= \dfrac{1}{4}$

9. $\lim_{x\to -3}\dfrac{(1+x)^2-4}{x+3}$ ← Expand & Simplify.

$= \lim_{x\to -3}\dfrac{1^2+2x+x^2-4}{x+3}$

$= \lim_{x\to -3}\dfrac{x^2+2x-3}{x+3}$ ← factor

$= \lim_{x\to -3}\dfrac{(x-1)\cancel{(x+3)}}{\cancel{x+3}}$

$= \lim_{x\to -3} x-1$ sub $x=-3$

$= -3-1$

$= -4$

H.A $y=0$

10. $\lim_{x\to -\infty}\dfrac{1-x}{5+x^2}-2$

$= \lim_{x\to -\infty}\dfrac{\frac{1}{x^2}-\frac{x}{x^2}}{\frac{5}{x^2}+\frac{x^2}{x^2}}-2$

$= \lim_{x\to -\infty}\dfrac{\frac{1}{x^2}-\frac{1}{x}}{\frac{5}{x^2}+1}-2$

$= \dfrac{\cancelto{0}{\frac{1}{(-\infty)^2}}-\cancelto{0}{\frac{1}{-\infty}}}{\cancelto{0}{\frac{5}{(-\infty)^2}}+1}-2$

$= \dfrac{0}{1}-2$

$= 0-2 = -2$

11. $\lim_{x\to 9}\dfrac{2x}{x^2-9}$ Sub $x=9$ & solve.

$= \dfrac{2(9)}{9^2-9}$

$= \dfrac{18}{81-9}$

$= \dfrac{18}{72}$

$= \dfrac{1}{4}$

12. $\lim_{x\to 8}\dfrac{2-\sqrt[3]{x}}{8-x}$ change variable let...

$= \lim_{x\to 8}\dfrac{2-x^{\frac{1}{3}}}{8-x}$

let $u=\sqrt[3]{x}$ or $x^{\frac{1}{3}}$

$u^3=x$ when $x=8$

$u=\sqrt[3]{8}$

$u=2$

$8-u^3=2^3-u^3$

$= \lim_{u\to 2}\dfrac{2-u}{8-u^3}$

$= \lim_{u\to 2}\dfrac{\cancel{2-u}}{\cancel{(2-u)}(2^2+2u+u^2)}$

$= \lim_{u\to 2}\dfrac{1}{4+2u+u^2}$

$= \dfrac{1}{4+2(2)+2^2}$

$= \dfrac{1}{12}$

13. $\lim_{x\to 4}\dfrac{4+x}{16-x^4}$

$= \dfrac{4+4}{16-4^4}$

$= \dfrac{8}{16-256}$

$= \dfrac{8}{-240}$

$= -\dfrac{1}{30}$

SUMMARY:

① Sub in what x approaches & solve.

② If you have $\frac{0}{0}$ (indeterminate form)
- ↳ factor
- ↳ Expand
- ↳ Simplify
- ↳ LCD
- ↳ Rationalize (numerator/denominator)
- ↳ Change in variable

③ If you have $\frac{\#}{0}$ or single square root
↳ try # close to the point on both side to show the limit will not exist.

④ If you have piecewise function or absolute function.
↳ you must do both sides limit separately + compare.

⑤ If you have $\lim_{x\to\pm\infty} f(x)$
↳ then divide by highest power.

Homework:
① Journal # 2+3
② Textbook Practice: Pg. 45-46 #4 af, 7 bce, 8, 9 cef, 10 ad