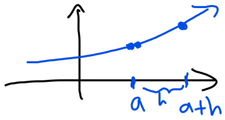
$m = \frac{f(a+h) - f(a)}{a+h-a}$



$\lim_{h \to 0}$

Rate of change = slope.

Slope of Secants and Tangents with Limits

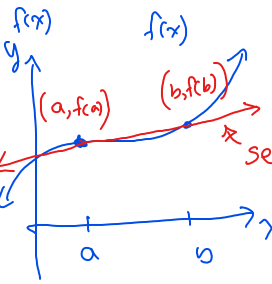
1. What is the slope equation for straight lines?

$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

2. How is rate of change of a straight line different from rate of change of a curve?

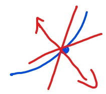
3. Recall from advanced functions, (two points)

a. the picture and formula of the **average** rate of change of a curve. $x \in [a, b]$ ($x_1 = a$, $x_2 = b$; $y_1 = ?$, $y_2 = ?$) $a, b \in \mathbb{R}$

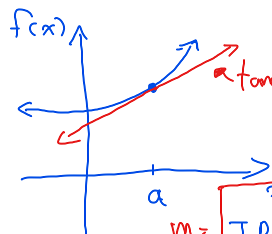


$$\boxed{ARoC = \frac{f(b) - f(a)}{b - a}}$$

Wrong!



b. the picture and formula of the **instantaneous** rate of change of a curve. $x = a$ (1 point)



$$m = \boxed{IRoC = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

instantaneous rate of change
slope of tangent line at $x = a$.

$m_T = ?$

4. Determine the slope of the tangent for rational function $f(x) = \frac{5+x}{x^2}$ at $x = 5$

$m_T = \lim_{h \to 0} \frac{f(5+h) - f(5)}{h}$ ✓

$= \lim_{h \to 0} \frac{1}{h}\left(f(5+h) - f(5)\right)$

$= \lim_{h \to 0} \frac{1}{h}\left[\frac{5+(5+h)}{(5+h)^2} - \frac{5+5}{5^2}\right]$ ✓

$= \lim_{h \to 0} \frac{1}{h}\left[\frac{10+h}{25+10h+h^2} - \frac{10}{25}\right]$ LCD

$= \lim_{h \to 0} \frac{1}{h}\left[\frac{25(10+h) - 10(25+10h+h^2)}{25(25+10h+h^2)}\right]$

$= \lim_{h \to 0} \frac{1}{\cancel{h}}\left[\frac{\cancel{250} + 25\cancel{h} - \cancel{250} - 100\cancel{h} - 10h^2}{25(25+10h+h^2)}\right]$ ✓

$= \lim_{h \to 0} \frac{25 - 100 - 10h}{25(25+10h+h^2)}$ ✓ Sub $h = 0$

$= \frac{25 - 100 - 0}{25(25+0+0)}$ ✓ $= \frac{-\cancel{75}^{3}}{\cancel{25}(25)} = -\frac{3}{25}$ ✓

① $f(5) = \frac{5+5}{5^2} = \frac{10}{25} = \frac{2}{5}$

② $f(5+h) = \frac{5+(5+h)}{(5+h)^2} = \frac{10+h}{25+10h+h^2}$

③ $f(5+h) - f(5) = \frac{10+h}{25+10h+h^2} - \frac{2}{5}$

$= \frac{5(10+h) - 2(25+10h+h^2)}{5(25+10h+h^2)}$

$= \frac{\cancel{50} + 5h - \cancel{50} - 20h - 2h^2}{5(25+10h+h^2)}$

$= \frac{-2h^2 - 15h}{5(25+10h+h^2)}$

④ $m = \lim_{h \to 0} \frac{1}{\cancel{h}}\left(\frac{-2h^{\cancel{2}} - 15\cancel{h}}{5(25+10h+h^2)}\right)$

$= \lim_{h \to 0} \frac{-2h - 15}{5(25+10h+h^2)}$

$= \frac{-2(0) - 15}{5(25+10(0)+0^2)}$

$= \frac{-15}{125}$

$= -\frac{3}{25}$

Alternate formula: use in University.

$m_T = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

$x = 5+h \quad g = ?$

5. Determine the slope of the tangent for square root function $g(x)=\sqrt{x-4}$ at $x=5$

$m = \lim_{h\to 0} \frac{g(5+h) - g(5)}{h}$ Rationalize

$g(5) = \sqrt{5-4} = 1$

$g(5+h) = \sqrt{5+h-4} = \sqrt{1+h}$

$= \lim_{h\to 0} \frac{\sqrt{1+h} - 1}{h} \times \frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}$

$= \lim_{h\to 0} \frac{1+h-1}{h(\sqrt{1+h}+1)}$

$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{1+h}+1)}$

$= \lim_{h\to 0} \frac{1}{\sqrt{1+h}+1}$ sub $h=0$

$= \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}$

if $m_T = -2 \Rightarrow m_\perp = \frac{1}{2}$

if $m_T = \frac{1}{2}$ $m_\perp = -\frac{2}{1}$

if $m_T = 3$ $m_\perp = -\frac{1}{3}$

Parallel

90°

-2

→ Tangent Perpendicular lines:

$m_\perp = -\frac{1}{m_T}$ (Negative Reciprocal)

6. Determine the equation of the line that is perpendicular to the tangent to $y = x^5$ at $x = -2$, and which passes through the tangent point.

↳ Cross at 90° $f(x)$ Point $(-2,-32)$

① find m_T using $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

② find $m_\perp$ (negative reciprocal to m_T)

③ $m_\perp$ + point into $y=mx+b$, find b.

Parallel lines:
↳ Same slopes.

tangent

$m_{//} = m_T$

Step 1: find m_T

$m_T = \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h}$

$= \lim_{h\to 0} \frac{(-2+h)^5 - (-2)^5}{h}$

use Pascal triangle to expand

1
1 1
n=2 1 2 1
n=3 1 3 3 1
1 4 6 4 1
n=5 $\boxed{1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1}$

$= \lim_{h\to 0} \frac{\cancel{(-2)^5} h^0 + 5(-2)^4 (h) + 10(-2)^3 (h^2) + 10(-2)^2 h^3 + 5(-2)(h^4) + h^5 - \cancel{(-2)^5}}{h}$

sub $h=0$

$= 5(-2)^4$

$= 80$

Step 2: find slope of perpendicular line, $m_\perp$

$m_\perp = -\frac{1}{80}$

Step 3: point $(-2,-32)$ + $m_\perp = -\frac{1}{80}$

$y = mx+b$

$-32 = -\frac{1}{80}(-2) + b$

$b = \frac{-1279}{40}$

$\boxed{\therefore \text{ Equation is } y = -\frac{1}{80}x - \frac{1279}{40}}$

1. An outdoor hot tub holds 2700L of water. When a valve at the bottom of the tub is opened, it takes 3h for the water to completely drain. The volume of water in the tub is modelled by the function $V(t) = \frac{1}{12}(180-t)^2$, where V is the volume of water in the hot tub, in litres, and t is the time, in minutes, that the valve is open.
   a. Determine the average rate of change of volume from 2 min to 10min.
   b. Determine the instantaneous rate of change of the volume of water at 60min. (Use the second version of the definition for rate here)

y L min $t = 60$

$\frac{V}{t} = \frac{L}{min}$

a) ARoC = ? $t \in [2, 10]$

$$ARoC = \frac{V(10) - V(2)}{10-2}$$

$$V(10) = \frac{1}{12}(180-10)^2 = \frac{7225}{3}$$

$$V(2) = \frac{1}{12}(180-2)^2 = \frac{7921}{3}$$

$$= \frac{\frac{7225}{3} - \frac{7921}{3}}{8}$$

$$= -29$$

negative ⇒ decreasing

What is the unit ⇒ L/min

∴ The average rate of change of Volume from 2min to 10 min is decreasing by 29L/min

b) $t = 60$

$$IRoC = \lim_{h \to 0} \frac{V(60+h) - V(60)}{h}$$

$$V(60) = \frac{1}{12}(180-60)^2 = \frac{1}{12}(120)^2$$

$$V(60+h) = \frac{1}{12}(180-60+h)^2 = \frac{1}{12}(120-h)^2$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{12}(120-h)^2 - \frac{120^2}{12}\right)$$

$$= \lim_{h \to 0} \frac{1}{12\cancel{h}}\left[\cancel{120^2} - 240\cancel{h} + h^2 - \cancel{120^2}\right]$$

$$= \lim_{h \to 0} \frac{-240+h}{12}$$

$$= \frac{-240+0}{12} = -20$$

∴ the instantaneous rate of change at 60min is decreasing by 20L/min.

Homework: Pg. 20 # 11, 16, 25
Pg. 30 # 8, 10, 11, 12, 13.
Journal Unit 5 #5

Test Monday, Mar. 9
- Limits
- Continuity
- slope of tangent
- Equation of line ⊥ or // to tangent
- Average and Instantaneous Rate of change.