Derivative Unit - Notes

Tentative TEST date Fri. Mar. 13

Big idea/Learning Goals

This unit is a continuation of the introduction to calculus. You will be introduced to rates of change as they apply to first monomial functions, then polynomials and simple rational functions, and finally compositions of functions. You will develop an understanding of the relationship between the numeric, graphical and algebraic representations of the derivative and the original function. This knowledge will later be used to sketch complex functions by identifying the turning points and points where the function changes concavity from Concave Up to Concave Down.

Corrections for the textbook answers:

IoRC = slope = $m = \boxed{\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$ ← long way to find derivative of a function.

Success Criteria

☐ I understand the new topics for this unit if I can do the practice questions in the textbook/handouts

Date	pg	Topics	# of quest. done? You may be asked to show them
	2-4	The Derivative Sketches - Handouts online	
	5-7	The Derivative Function for Monomials 2.1	
	8-9	Derivatives of Polynomials 2.2	
	10-11	Product Rule 2.3	
	12-13	Quotient Rule 2.4	
Skip	14-16	Implicit Differentiation (AP Calculus) - Handout online - Appendix of textbook **p 561-564**	
	17-19	Chain Rule for Composed Functions 2.5	
		Review	

Reflect – previous TEST mark ___________, Overall mark now___________.

The Derivative Sketches

→process of finding the derivative

1. Differentiation was developed by Sir Isaac Newton and Gottfried Leibniz in 17th century. The output of the differentiation operation is called the __derivative__. It can be used to calculate the __slope of the tangent line__ at __ANY__ point.

2. Prime notation (Newton)
   $f(x) \to f'(x)$ "f prime at x"
   $y \to y'$
   $g(x) \to g'(x)$

3. Leibniz notation
   $y = \dots \to \frac{dy}{dx}$ "dee y by dee x"
   $f(x) \to \frac{df}{dx}$ $\frac{d \text{ dependent variable}}{d \text{ independent variable}}$

The First Principles Definition of the Derivative of f(x) is

$$\underbrace{f'(x) \text{ OR } \frac{df}{dx}}_{\text{derivative of } f(x)} = \boxed{\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$$

Simplify to an equation.

4. Find the derivative using 1st principles of $f(x) = x^2$ ← Quadratic degree = 2
   a. Use the derivative to calculate quickly the slopes of tangents at $x = -3, 0, 1$
   b. Graph the function and the tangents found above to show graphical representations of the derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$f(x+h) = (x+h)^2$
$f(x) = x^2$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= \lim_{h\to 0} 2x + h \qquad \text{sub } h = 0$$

$$= 2x + 0$$

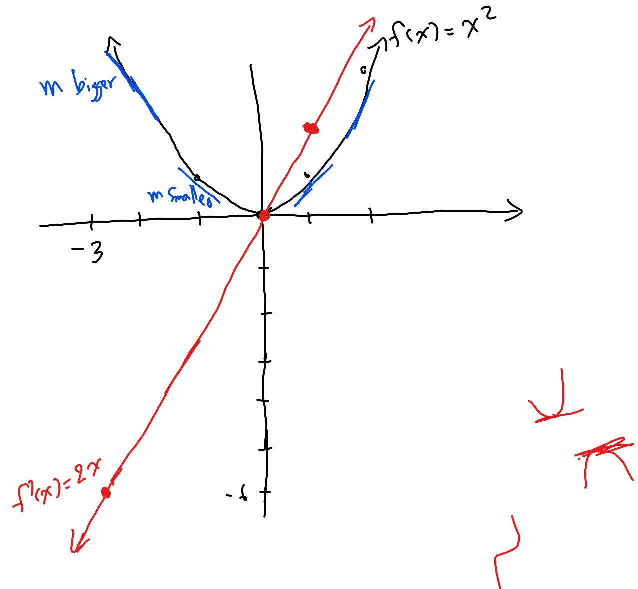
$$\boxed{f'(x) = 2x}$$

← Equation to find the derivative at any point

$\boxed{f'(x) = 2x}$ ← degree = 1
← linear function

a) at $x = -3$, $f'(-3) = 2(-3) = -6 = m_T$
at $x = 0$ $m_T = f'(0) = 2(0) = 0$
at $x = 1$ $m_T = f'(1) = 2(1) = 2$

b)

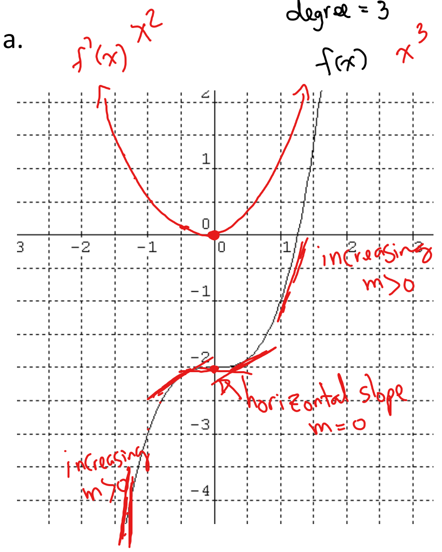


2. Sketch the derivative for each of the following functions (draw sketches on grids below).

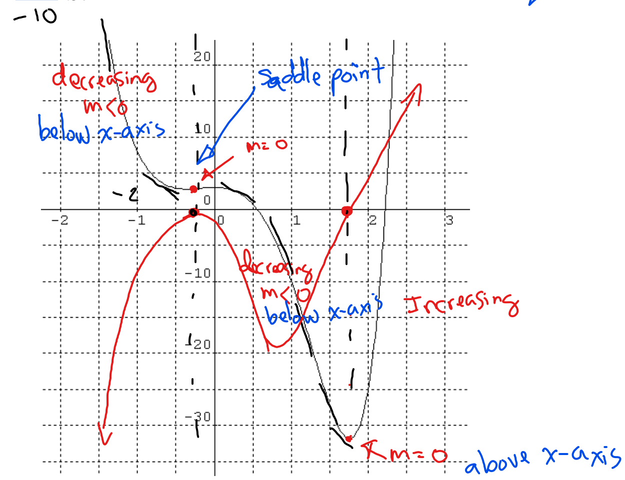
$f'(x)$
degree = 2

Key:
1) At turning or Saddle point slope = 0
2) Increasing graph ⇒ slope > 0 (above the x-axis)
3) Decreasing graph ⇒ slope < 0 (below the x-axis)
4) Derivatives of polynomial always have 1 degree less.

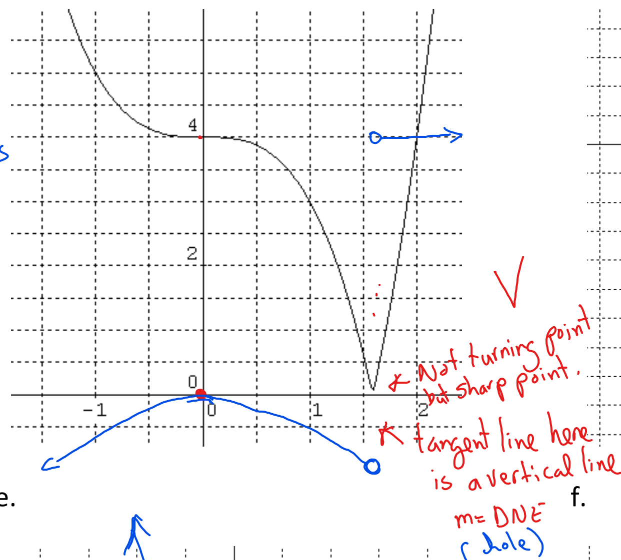
eg. a.



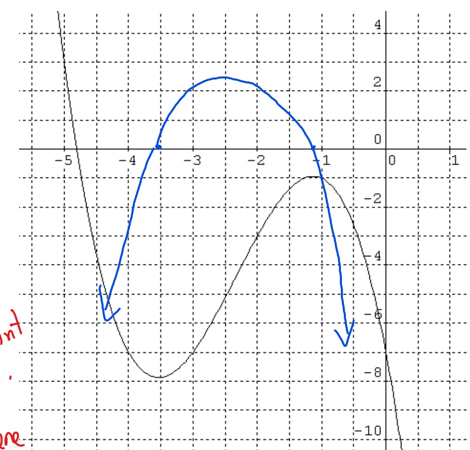
b.



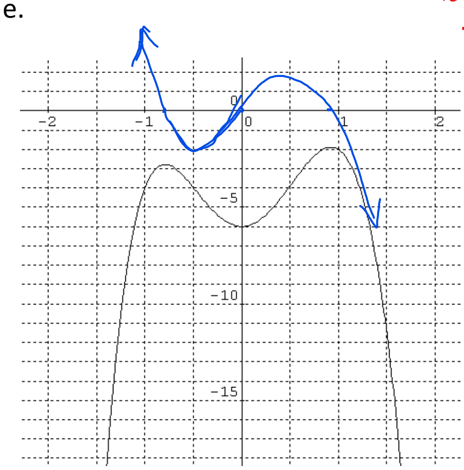
c.



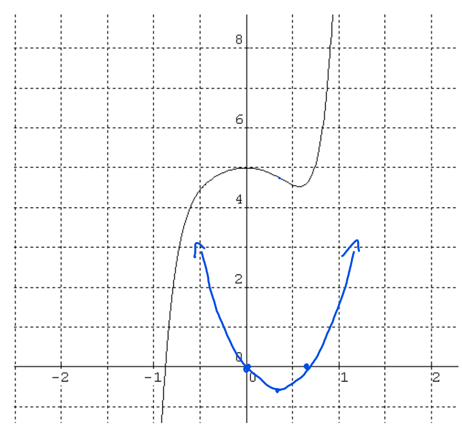
d.



e.



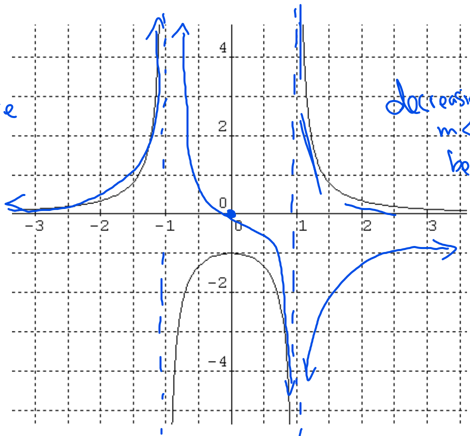
f.



derivatives is undefined where f(x) is undefined.

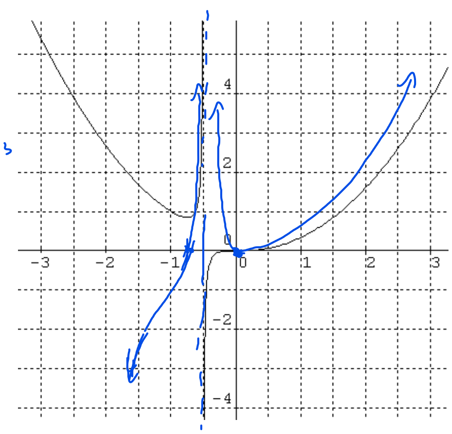
* V.A stay where they are
* sharp point → hole
* H.A stay where they are.

g. Rational function

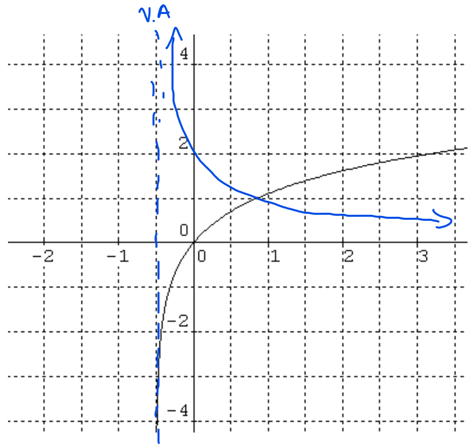


decreasing
m<0
below x-axis

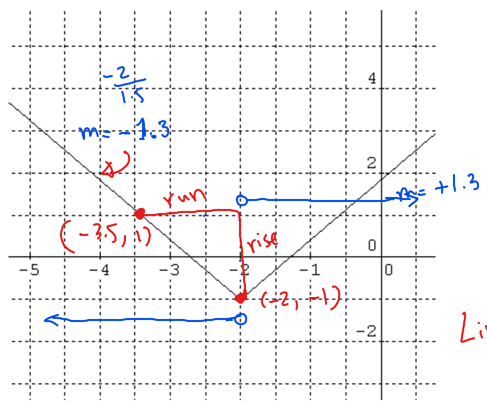
h.

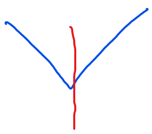


i.



j. $m = \frac{rise}{run} = \frac{\Delta y}{\Delta x}$

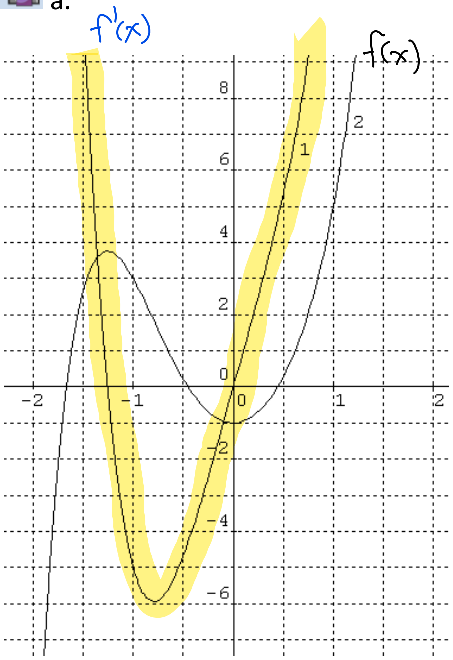




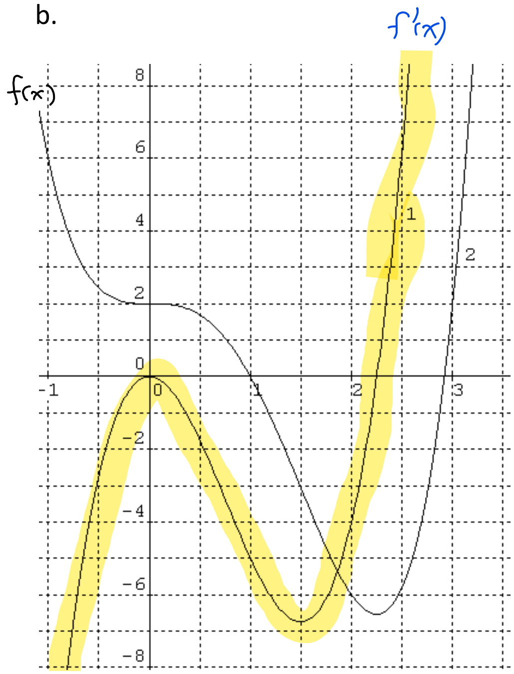
Linear function → Horizontal derivative

3. The graph of a function and its derivative is drawn in each of the following grids. State which is the function and which is the derivative. Explain your reasoning for each.

a.



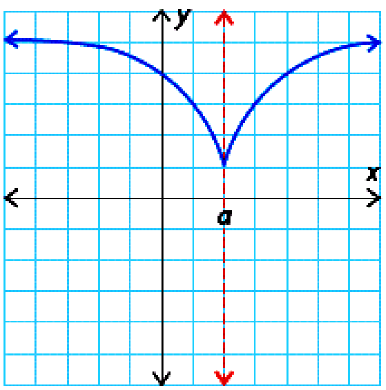
b.



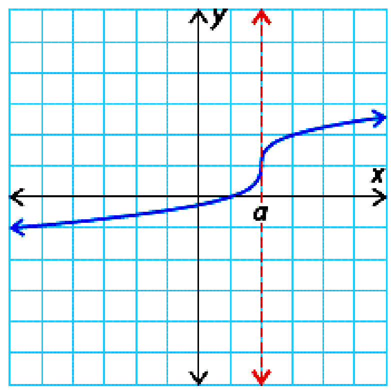
The Derivative Function for Monomials

Existence of Derivatives

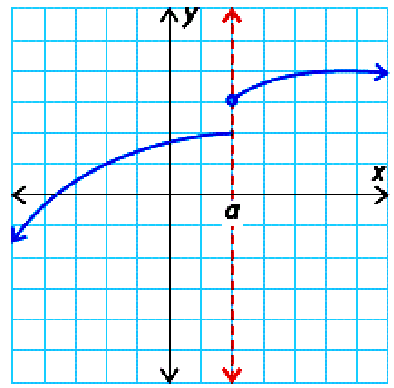
A function *f* is said to be differentiable at *a* if *f'(x)* exists. At points where *f* is not differentiable, we say that the <u>derivative does not exist</u>. Three common ways for a derivative to fail to exist are shown.



Cusp



Vertical Tangent



Discontinuity

1. Prove that the function $f(x)=|x|$ is not differentiable at x=0, but differentiable at x=3.

eg.

Prove $f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$ = DNE

$=\lim_{h\to 0}\frac{|0+h|-|0|}{h}$

$=\lim_{h\to 0}\frac{|h|}{h}$

Left side

$\lim_{h\to 0^-}\frac{-\cancel{h}}{\cancel{h}}$

$=\lim_{h\to 0^-} -1$

$=-1$ $\neq$

Right side

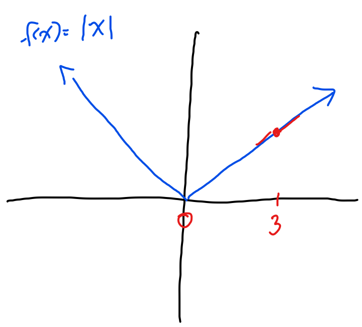
$\lim_{h\to 0^+}\frac{\cancel{h}}{\cancel{h}}$

$=\lim_{h\to 0^+} 1$

$=1$

$\therefore f'(0)=$ DNE

$|x|=\begin{cases}-x, & x<0\\ x, & x\geq 0\end{cases}$ ← x=3



$f'(3)=\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$

$=\lim_{h\to 0}\frac{\cancel{3}+h-\cancel{3}}{h}$

$=\lim_{h\to 0} 1$

$=1$

10 Min!

LOOKING FOR PATTERNS:

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

2. Determine the derivatives of the following, from first principles.

1. $f(x) = x^4$

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{(x+h)^4 - x^4}{h}$

Pascal triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$= \lim_{h \to 0} \frac{\cancel{x^4} + 4x^3\cancel{h} + 6x^2h^{\cancel{2}} + 4xh^{\cancel{3}2} + h^{\cancel{4}3} - \cancel{x^4}}{\cancel{h}}$

$= \lim_{h \to 0} 4x^3 + 6x^2h + 4xh^2 + h^3$

$= 4x^3 + 6x^2(0) + 4x(0^2) + 0^3$

$\boxed{f'(x) = 4x^3}$

2. $f(x) = \frac{1}{x^2}$

$f'(x) = \lim_{h \to 0} \frac{1}{h}\left[\frac{1}{(x+h)^2} - \frac{1}{x^2}\right]$

$= \lim_{h \to 0} \frac{1}{h}\left[\frac{x^2 - (x+h)^2}{x^2(x+h)^2}\right]$

$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\cancel{x^2} - (\cancel{x^2} + 2xh + h^2)}{x^2(x+h)^2}\right]$

$= \lim_{h \to 0} \frac{1}{\cancel{h}}\left[\frac{-2x\cancel{h} + h^{\cancel{2}}}{x^2(x+h)^2}\right]$

$= \lim_{h \to 0} \frac{-2x}{x^2(x+h)^2}$

$= \frac{-2x}{x^2(x+0)}$

$= \frac{-2\cancel{x}}{\cancel{x^4}3}$

$\boxed{f'(x) = -\frac{2}{x^3}}$

$f'(x) = -2x^{-3}$

3. $f(x) = \sqrt[3]{x}$

$f'(x) = \lim_{h \to 0} \frac{\sqrt[3]{x+h} - \sqrt[3]{x}}{h}$

let $u^3 = x+h$
$h = u^3 - x$
$u^3 = x + 0$
$u^3 = x$
$u \to \sqrt[3]{x}$

$= \lim_{u \to \sqrt[3]{x}} \frac{\sqrt[3]{u^3} - \sqrt[3]{x}}{u^3 - x}$

$= \lim_{u \to \sqrt[3]{x}} \frac{\cancel{u - \sqrt[3]{x}}}{(\cancel{u - \sqrt[3]{x}})(u^2 + u\sqrt[3]{x} + (\sqrt[3]{x})^2)}$

$= \lim_{u \to \sqrt[3]{x}} \frac{1}{u^2 + u\sqrt[3]{x} + \sqrt[3]{x}^2}$

$= \frac{1}{\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{x} + \sqrt[3]{x}^2}$

$= \frac{1}{3\sqrt[3]{x}^2}$

$f'(x) = \frac{1}{3x^{2/3}}$

$f'(x) = \frac{1}{3}x^{-2/3}$

4. $f(x) = \frac{4}{x} + 5x^2$

$f'(x) = \lim_{h \to 0} \frac{1}{h}\left[\frac{4}{x+h} + 5(x+h)^2 - \frac{4}{x} - 5x^2\right]$

$= \lim_{h \to 0} \frac{1}{h}\left[\frac{4x + 5x(x+h)^3 - 4(x+h) - 5x^2(x)(x+h)}{x(x+h)}\right]$

$= \lim_{h \to 0} \frac{1}{h} \; 4x - 5(x^4 + 3x^3h + 3x^2h^2 + xh^3) - 4x - 4h$

. . . .

$f'(x) = \frac{-4}{x^2} + 10x$

3. What is the pattern that you notice, describe it in words.

$\sqrt[2]{x} = x^{\frac{1}{2}}$

$\sqrt[3]{x} = x^{\frac{1}{3}}$

$\sqrt[4]{x} = x^{\frac{1}{4}}$

$(\sqrt[2]{x})^3 = x^{3/2}$

$\frac{2}{x^3} = 2x^{-3}$

$x^{-2} = \frac{1}{x^2}$

$f(x) = \frac{1}{x^2} = x^{-2}$

$f'(x) = -2x^{-2-1} = -2x^{-3} = \frac{-2}{x^3}$

ex: $f(x) = x^3$ $f'(x) = 3x^{3-1} = 3x^2$

$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ $f'(x) = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-2/3}$

$2x^{-3}$ $x^{2/5}$

4. Use the pattern to predict the derivatives of

$$f(x) = x^{57} - 6x^{14} - \frac{2}{x^3} + \sqrt[5]{x^2}$$

$f(x) = x^{57} - 6x^{14} - 2x^{-3} + x^{2/5}$

$f'(x) = 57x^{57-1} - 6(14)x^{14-1} - 2(-3)x^{-3-1} + \frac{2}{5}x^{2/5-1}$

$f'(x) = 57x^{56} - 84x^{13} + \frac{6}{x^4} + \frac{2}{5x^{3/5}}$

Power Rule

$f(x) = x^n$

$f'(x)$ or $\frac{df}{dx} = nx^{n-1}$

Try the following examples to see the importance of knowing what variable you're taking the derivative 'with respect to' and the need for Leibniz notation.

$f(x) = \#$

$f'(x) = 0$

$f(x) = 2x^0$

$f'(x) = 2(0)(x)^{0-1}$

$f'(x) = 0$

5. $f = \frac{a^3x^2}{z}$

a) $\frac{df}{dx}$ b) $\frac{df}{dy}$ c) $\frac{df}{da}$ d) $\frac{df}{dz}$

6. $f = x^2$ $p = t^3 + 2t$

a) $\frac{df}{dx}$ b) $\frac{df}{dt}$ c) $\frac{dp}{dx}$ d) $\frac{dp}{dt}$